§ 1. Introduction.—A discussion of the photoelectric effect for hydrogen-like atoms has been given by many authors. In the simplest case it is assumed that the wave-length of the incident radiation is large compared with the "radius of the atom." If \( z \) be a co-ordinate measured from the centre of the atom, the axes may be chosen so that the perturbing vector potential involves the factor \( \exp \left( \frac{2\pi i z}{\lambda} \right) \), where \( \lambda \) is the wave-length of the radiation. For very large wave-lengths we put this equal to unity, or we may use the first two terms of the expansion. For very short wave-lengths, however, the expansion is illusory, and we must use some other method. For \( \gamma \)-rays the "radius of the atom" is large compared with the wave-length of the rays. A knowledge of the wave-function near the nucleus is therefore necessary. Further, the photo-electrons emitted have velocities comparable to that of light. In view of these two circumstances it is necessary to use a relativistic theory of the atom, and in the following we shall endeavour to apply the theory of the Dirac electron.

In § 2 we develop a normalised solution of the equations for the hydrogen-like atom, when the total energy, \( E \), is greater than \( mc^2 \), the rest energy of an electron. Next we consider the perturbation theory, which gives the total number of electrons emitted and their resultant forward momentum, in terms of the matrix elements representing transitions from the ground state to states where the electron is free. The chief difficulty is the evaluation of these matrix elements, which is only carried through when the atomic number \( Z \) is small, and the wave-length of the incident light is such that the energy of one quantum is comparable to \( mc^2 \), corresponding to wave-lengths of the order \( 10^{-10} \) cm. The first restriction is very unsatisfactory, since the photoelectric effect is best observed in heavy atoms. The results obtained are, however, in qualitative agreement with the experimental results for heavy atoms, if we exclude the variation with atomic number.

§ 2. Solution of the Wave-equation for States of Positive Energy.—We may write the wave-equation of the electron* as

\[
\left[ \frac{E}{c} + \frac{e}{c} A_0 + \rho_1 \left( \sigma, p + \frac{e}{c} A \right) + \rho_2 mc \right] \psi = 0, \tag{1}
\]

where \( E \) is the energy of the system, \( A_0 \) and \( A \) the scalar and vector potentials, \( p \) the momentum vector \((p_x, p_y, p_z)\) and \( \sigma \) the vector \((\sigma_x, \sigma_y, \sigma_z)\). The quantities \( \sigma_x, \sigma_y, \sigma_z, p_1, (p_3) \) and \( \varphi_3 \) do not commute with each other, and may be conveniently represented by certain matrices of four rows and columns which obey the same non-commutability relations. In this representation the wave function \( \psi \) has four components. We have further

\[
\begin{align*}
p_x &= -\frac{i\hbar}{2\pi} \frac{\partial}{\partial x}, & p_y &= -\frac{i\hbar}{2\pi} \frac{\partial}{\partial y}, & p_z &= -\frac{i\hbar}{2\pi} \frac{\partial}{\partial z}.
\end{align*}
\]

Let us consider a hydrogen-like atom, that is an electron under the influence of a central charge \( Z e \). We have \( A = 0, A_0 = V = Z e/r \), and adopting the representation of the \( \sigma \)'s and \( p \)'s given by Dirac, we find equation (1) becomes on writing out in full:

\[
\begin{align*}
\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} + mc \right) \psi_1 + \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \psi_4 + \frac{\partial}{\partial z} \psi_3 &= 0, \\
\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} + mc \right) \psi_2 + \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \psi_3 - \frac{\partial}{\partial z} \psi_4 &= 0, \\
\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} - mc \right) \psi_3 + \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \psi_2 + \frac{\partial}{\partial z} \psi_1 &= 0, \\
\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} - mc \right) \psi_4 + \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \psi_1 - \frac{\partial}{\partial z} \psi_2 &= 0.
\end{align*}
\]

To solve these equations we transform to polar co-ordinates, \( \theta \) being the angle between the \( z \)-axis and the radius vector. The solutions are then of two types\

\[
\begin{align*}
\psi_1 &= -iF_k^u P_{k+1}^u, & \psi_2 &= -iF_k^u P_{k+1}^u, \\
\psi_3 &= (k + u + 1) G_k^u, & \psi_4 &= (-k + u) G_k^u, \\
\end{align*}
\]

and

\[
\begin{align*}
\psi_1 &= -i(k + u) F_{-k-1}^u P_k^u, & \psi_2 &= -i(-k + u + 1) F_{-k-1}^u P_k^u, \\
\psi_3 &= G_{-k-1}^u, & \psi_4 &= G_{-k-1}^u.
\end{align*}
\]

where \( F_k, G_k \) satisfy

\[
\begin{align*}
\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} + mc \right) F_k + \frac{dG_k}{dr} - \frac{k}{r} G_k &= 0, \\
-\frac{2\pi}{\hbar} \left( \frac{E + eV}{c} - mc \right) G_k + \frac{dF_k}{dr} + \frac{k + 2}{r} F_k &= 0.
\end{align*}
\]

and \( P_k^u \) is the associated Legendre function given by

\[
P_k^u = (k - u)! \sin^u \theta \left( \frac{d}{d \cos \theta} \right)^{k+u} \left( \cos^2 \theta - 1 \right)^k \frac{e^{i u \phi}}{2^k},
\]

\( u \) and \( k \) being any numbers such that the Legendre functions involved have a meaning (see Darwin, loc. cit.). We shall write
\[
\frac{2\pi}{\hbar} \left( mc + \frac{E}{c} \right) = A^2
\]
\[
\frac{2\pi}{\hbar} \left( mc - \frac{E}{c} \right) = B^2
\]
The equations (4) then become
\[
\begin{align*}
\left( A^2 + \frac{\gamma}{r} \right) F_k + \frac{dG_k}{dr} - \frac{k}{r} G_k &= 0 \\
\left( B^2 - \frac{\gamma}{r} \right) G_k + \frac{dF_k}{dr} + \frac{k + 2}{r} F_k &= 0
\end{align*}
\]
When \( E < mc^2 \), suitable solutions of (7) can be obtained in the form of polynomials for a set of discrete values of the energy. Any positive value of \( (E - mc^2) \) will yield a permissible solution representing a diverging wave for large \( r \), corresponding to a hyperbolic orbit on the older theory. For our purposes we require a normalised solution valid for all values of \( r \), and such a solution may be obtained in the form of a contour integral.

Equations (7) may be transformed into a standard form by writing
\[
\begin{align*}
\mathcal{F}_k &= AF_k - BG_k \\
\mathcal{G}_k &= AF_k + BG_k
\end{align*}
\]
and we obtain
\[
\begin{align*}
\frac{r}{d} \frac{d\mathcal{F}_k}{dr} &= \left\{ (ib - 1) + iar \right\} \mathcal{F}_k + \left\{ -ic - (k + 1) \right\} \mathcal{G}_k \\
\frac{r}{d} \frac{d\mathcal{G}_k}{dr} &= \left\{ ic - (k + 1) \right\} \mathcal{F}_k + \left\{ -ib - 1 - iar \right\} \mathcal{G}_k
\end{align*}
\]
where
\[
\begin{align*}
AB &= ia \\
\gamma \left( \frac{A}{2B} - \frac{B}{A} \right) &= -ib \\
\gamma \left( \frac{A}{2B} + \frac{B}{A} \right) &= -ic
\end{align*}
\]
and \( a, b, c \) are real and positive and \( b > c \). Since \( E > mc^2 \) we see from equations (6) that \( A \) is real and \( B \) is imaginary. \( \mathcal{F}_k \) and \( \mathcal{G}_k \) are therefore conjugate quantities.

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Let us now try to satisfy (9) by a set of Laplace integrals

$$\mathcal{F}_k = \int e^{r \xi} v_1 (\zeta) \, d\zeta$$
$$\mathcal{G}_k = \int e^{r \xi} v_2 (\zeta) \, d\zeta$$  \hspace{1cm} (11)

On substituting in the equations (9) we find that the conditions to be satisfied are

$$(- ib - 1) \int e^{r \xi} v_2 (\zeta) \, d\zeta + \{ i \omega - (k + 1) \} \int e^{r \xi} v_1 (\zeta) \, d\zeta = \int r e^{r \xi} (\zeta + ia) v_2 (\zeta) \, d\zeta$$

\[ = [e^{r \xi} (\zeta + ia) v_2 (\zeta)] - \int e^{r \xi} \left\{ v_2 (\zeta) + (\zeta + ia) \frac{dv_2 (\zeta)}{d\zeta} \right\} \, d\zeta \]

on integrating by parts, together with a conjugate condition for $v_1 (\zeta)$ and $v_2 (\zeta)$ interchanged. These conditions can be fulfilled if we take $v_1 (\zeta)$ and $v_2 (\zeta)$ to satisfy the Laplace transformed system

$$(\zeta - ib) \frac{dv_1}{d\zeta} = - ib v_1 + \{ (k + 1) + ic \} v_2$$

$$(\zeta + ia) \frac{dv_2}{d\zeta} = \{ (k + 1) - ic \} v_1 + ib v_2$$  \hspace{1cm} (12)

and choose the contours so that the terms

$$[e^{r \xi} (\zeta - ia) v_1 (\zeta)]$$

and

$$[e^{r \xi} (\zeta + ia) v_2 (\zeta)]$$

vanish identically in $r$. Since $r$ is necessarily positive, one solution of our equations is a contour starting from $- \infty$, encircling the point $\zeta = ia$ and returning to $- \infty$. A similar contour encircling $\zeta = - ia$ gives a second and independent solution, but in order to obtain a convenient form we shall take a combination of the two—a contour which encircles both points (see figure). We shall now find integrals for $v_1 (\zeta)$ and $v_2 (\zeta)$. Eliminating $v_1 (\zeta)$ from equations (12) we obtain

$$\left( \zeta^2 + a^2 \right) \frac{d^2 v_2}{d\zeta^2} + \left( \zeta - ia - 2ab \right) \frac{dv_2}{d\zeta} + \left( \gamma^2 - (k + 1)^2 \right) v_2 = 0$$  \hspace{1cm} (13)

using the fact that $\gamma^2 = b^2 - a^2$. This is of standard form, the solution being

$$v_2 (\zeta) = \int_C (t - ia)^{-(A+b+1)} (t + ia)^{-(A-b+2)} (t - \zeta)^{\lambda+1} \, dt,$$

where $\lambda = -1 \pm \sqrt{[(k + 1)^2 - \gamma^2]}$, and the contour $C$ is chosen so that

$$\int_C d \left[ (t - \zeta)^{\lambda} (t - ia)^{-(A+b)} (t + ia)^{-(A-b+1)} \right] = 0$$  \hspace{1cm} (14)
independently of $\xi$. If we choose $\lambda = -1 - \sqrt{(k+1)^2 - \gamma^2}$, then $\lambda$ and $\lambda + 1$ are negative for all $k \gg 0$, and we may take the contour $C$ from $-ia$ to $+ia$. We obtain $v_1(\zeta)$ in a similar manner. We then find

$$v_1(\zeta) = S \int_{-ia}^{+ia} \frac{(t - ia)^{-\lambda+ib+2}}{(t + ia)^{-\lambda-ib+1}} (t - \zeta)^{\lambda+1} dt,$$

$$v_2(\zeta) = T \int_{-ia}^{+ia} \frac{(t - ia)^{-\lambda+ib+1}}{(t + ia)^{-\lambda-ib+2}} (t - \zeta)^{\lambda+1} dt,$$

where $S$ and $T$ are some constants and $|\arg (t \pm ia)| \leq \pi$ at the lower limit to make the solution definite. By substitution in equations (12) we find

$$S = - \frac{(k + \lambda + 2) + i(b + c)}{(k + \lambda + 2) - i(b + c)}.$$

We now have

$$\mathcal{F}_k = S \int_{C'} e^{r\zeta} \int_{-ia}^{+ia} \frac{(t - ia)^{-\lambda+ib+2}}{(t + ia)^{-\lambda-ib+1}} (t - \zeta)^{\lambda+1} dt d\zeta,$$

$$\mathcal{G}_k = T \int_{C'} e^{r\zeta} \int_{-ia}^{+ia} \frac{(t - ia)^{-\lambda+ib+1}}{(t + ia)^{-\lambda-ib+2}} (t - \zeta)^{\lambda+1} dt d\zeta,$$

where the contour $C'$ is shown in the figure.

$\lambda + 1$ is negative, so that provided $(t - \zeta)$ is never equal to zero, we may interchange the order of integration. This is realised if we choose the contours as above, and we find on integrating with respect to $r$

$$\mathcal{G}_k = T (-)^{\lambda+1} 2i \sin [(\lambda + 2)\pi] \Gamma (\lambda + 2) r^{-\lambda+2}$$

$$\times \int_{-ia}^{+ia} e^r (t - ia)^{-\lambda+ib+1} (t + ia)^{-\lambda-ib+2} dt,$$

and a similar expression for $\mathcal{F}_k$. Now we are only interested in the ratios of

$$2 \sigma 2$$
$\mathcal{F}_k$ and $\mathcal{G}_k$, but we must choose the values of $S$ and $T$ so that the functions $\mathcal{F}_k$ and $\mathcal{G}_k$ are conjugate complex. We may take as a solution

$$\mathcal{F}_k = -[(k - s) + i(b + c)]r^s \int_{-ia}^{+ia} (t - ia)^{s-ib} (t + ia)^{s+ib+1} e^{rt} dt$$

$$\mathcal{G}_k = [(k - s) - i(b + c)]r^s \int_{-ia}^{+ia} (t - ia)^{s-ib+1} (t + ia)^{s+ib} e^{rt} dt$$

where

$$s = -(\lambda + 2) = \sqrt{(k + 1)^2 - \gamma^2} - 1$$

For the second form of solution, given by equations (17b), we must replace $k$ by $-k - 1$ in equations (7). We then find in a similar way

$$\mathcal{F}_{k-1} = -[(b - c) - i(k - 1 - s')]r^s \int_{-ia}^{+ia} (t - ia)^{s'-ib} (t + ia)^{s'+ib+1} e^{rt} dt$$

$$\mathcal{G}_{k-1} = [(b - c) + i(k - 1 - s')]r^s \int_{-ia}^{+ia} (t - ia)^{s'-ib+1} (t + ia)^{s'+ib} e^{rt} dt$$

where

$$s' = -(\lambda' + 2) = \sqrt{(k^2 - \gamma^2) - 1}$$

There are no solutions of this type for $k = 0$, so that the condition (14) can always be fulfilled.

For the purposes of normalisation, we shall require the forms of $\mathcal{F}_k$ and $\mathcal{G}_k$ when $r$ is large. To find these we deform the contour into three parts and put

$$\int_{-ia}^{+ia} f(r, t) dt = \int_{-ia}^{-ia-R} f(r, t) dt + \int_{-ia-R}^{+ia-R} f(r, t) dt + \int_{+ia-R}^{+ia} f(r, t) dt$$

On allowing $R$ to tend to $\infty$, the middle integral tends to zero, and we are left with the other two. In the first one we put $t = -ia - x$, and in the second $t = ia - x$. This gives

$$\int_{-ia}^{+ia} (t - ia)^{s-ib+1} (t + ia)^{s+ib} e^{rt} dt = -(e^{i\pi})^{s+ib} e^{-i\pi r} (2a)^{s-ib+1} e^{-i\pi r} x^{s-ib+1} e^{-\pi r} dx$$

$$\times \int_{0}^{\infty} x^{s+ib} \left(1 + \frac{x}{2ia}\right)^{s-ib+1} e^{-\pi r} dx$$

$$-(e^{-i\pi})^{s-ib+1} e^{i\pi r} (2a)^{s+ib} e^{i\pi r} (\frac{1}{2ia})^{s+ib} e^{-\pi r} dx.$$ 

When $r$ is large, the important part of the integral occurs when $x$ is small.
We may therefore expand in powers of $x$ and integrate term by term, obtaining, since the second integral gives $r^{-(ib+2)}$,

$$
G_k = i^{s+1}[(k - s) - i(b + c)] e^{-\frac{3\pi b}{2}(2a)^{s+1}} \Gamma(s + ib + 1)
\times e^{-iar} r^{-(ib+1)} \left(1 + O\left(\frac{1}{r}\right)\right), \tag{18}
$$

which gives asymptotic forms of $F_k$ and $G_k$.

$$
F_k = \frac{1}{A} \sqrt{[(k - s)^2 + (b + c)^2]} |\Gamma(s + ib + 1)|
\times e^{-\frac{3\pi b}{2}(2a)^{s+1}} \frac{1}{r} \cos(ar + b \log r + \delta_a)
$$

$$
G_k = \frac{1}{|B|} \sqrt{[(k - s)^2 + (b + c)^2]} |\Gamma(s + ib + 1)|
\times e^{-\frac{3\pi b}{2}(2a)^{s+1}} \frac{1}{r} \cos(ar + b \log r + \delta'_a)
$$

where $\delta_a$ and $\delta'_a$ are independent of $r$. We have corresponding asymptotic forms for $F_{k-1}$ and $G_{k-1}$.

We may write the normalising and orthogonality conditions as follows

$$
\int \Sigma \psi_r (E, k, u) \psi^*_r (E', k', u') d\text{vol}
= \begin{cases} 
\delta_{EE'} \delta_{kk'} \delta_{uu'} & \text{for discrete states} \\
\frac{1}{h} \delta(E - E') \delta_{kk'} \delta_{uu'} & \text{for continuous states}
\end{cases}, \tag{20}
$$

using the $\delta$ function, the integral being taken over all space. It is easily verified in the usual way that the solutions are orthogonal. We have further

$$
4\pi (k + u + 1)! (k - u)! \int_0^\infty (F_k(E) F^*_k(E') + G_k(E) G^*_k(E')) r^2 dr = \frac{1}{h} \delta(E - E')
$$

which gives for the solution (3A)

$$
4\pi (k + u + 1)! (k - u)! \int_0^\infty (F_k(E) F^*_k(E') + G_k(E) G^*_k(E')) r^2 dr = \frac{1}{h} \delta(E - E')
$$
as the normalisation condition.

We shall employ the method of normalisation used by Gaunt,† which is equivalent to taking a special form of the $\delta$ function. It is easily adapted to the case when $\psi$ has four components, and for convenience we shall express the normalising factor, $\xi(E, k, u)$, as the product of $\xi(E, k)$, the normalising

factor of the radial part of the wave function, and \( \xi(k, u) \), that of the angular part. We obtain for the solutions (3a)

\[
\xi(E, k) = \left( \frac{2\pi E}{\hbar c^2 a} \right)^i \frac{A|B|}{A^2 + |B|^2} K^{-1}
\]

with

\[
K = \frac{1}{2} \sqrt{[(k - s)^2 + (b + c)^2]} |\Gamma(s + ib + 1)| e^{-\frac{3\pi b}{2} (2a)^{s+1}}.
\]

and

\[
\xi(k, u) = \sqrt{[A\pi (k + u + 1)]} (k - u)!
\]

To obtain \( \xi(E, -k - 1) \) and \( \xi(-k - 1, u) \), the normalising factors for solutions of type (3b), we replace \( k \) by \(-k - 1\), \( s \) by \( s' \) and by \(-c\) in the above.

§ 3. Perturbation Theory.—Let us now consider the case of a hydrogen-like atom in the field of a train of plane waves travelling in a direction parallel to the \( z \)-axis and represented by

\[
A_y = b_0 \left[ e^{2\pi i \rho_1 (t-z/c)} + e^{-2\pi i \rho_2 (t-z/c)} \right],
\]

\[
A_x = A_z = 0, \quad A_0 = 0
\]

(23)

From (1), the wave equation is

\[
\left[ \frac{E}{c} + \frac{e}{c} A_0 + \rho_1 (\sigma, p) + \rho_3 mc \right] \psi = -\frac{e}{c} A_y \rho_1 \sigma_y \psi.
\]

(24)

If \( A_0 = Ze/r \), the equation obtained by equating the left-hand side to zero gives the equation for the undisturbed atom. We shall assume \( A_y \) to be small, and treat the right-hand side as a small perturbation. Put

\[
V = -\frac{e}{c} A_y \rho_1 \sigma_y
\]

(25)

and try as a solution

\[
\psi = \sum_{E', k', u'} a_{E', k', u'} \psi_{E', k', u'} e^{-2\pi i E't \hbar} + \sum_{k', u'} \int_{mc}^{\infty} a(E', k', u') \psi(E', k', u') \times e^{-2\pi i E't \hbar} dE'/\hbar,
\]

(26)

where \( \psi_{E', k', u'} \) and \( \psi(E', k', u') \) are solutions of the equation for the undisturbed atom, and the \( a \)'s are functions of the time. In this section we are assuming the \( \psi \)'s to be normalised, and labelling the final state with \( E', k', u' \), and the initial with \( E'', k'', u'' \). When \( A_y \) is small, \( \psi \) changes slowly, and if the time \( t \) be not too great we may replace \( \psi \) by \( \psi_{E', k', u'} \exp. (-2\pi i E''t / \hbar) \) on the
right-hand side of (24), which involves an error of the second order since $V$ is already small. Substituting (25) and (26) in equation (24) we have

$$\sum_{E^{'}, E^{'}, k', u'} \psi_{E^{'}, k'} \psi_{E^{'}, u'} e^{-2\pi i E' t \hbar} dE' / \hbar$$

$$= \frac{2\pi e_0}{\hbar} V |\psi_{E''}, k', u'\rangle e^{-2\pi i E' t \hbar},$$

whence we obtain

$$\hat{a} (E', k', u') = \frac{2\pi e_0}{\hbar} (E', k', u' \mid V \mid E'', k'', u'').$$

Putting

$$E' - E'' = h\nu,$$

and taking the time factor outside, we obtain on integration

$$\mathfrak{a} (E', k', u') = \frac{2\pi e_0}{\hbar} \left[ (E', k', u') - b_0 e_0 |\psi_{E''}, k', u'\rangle e^{2\pi i (\nu - \nu_0) t} - \frac{1}{2\pi i (\nu - \nu_0)} \right]$$

The first term is important in the region $\nu \approx \nu_0$, or $E' \approx E'' + h\nu_0$, and concerns the absorption of radiation. The second is only important when $\nu \approx -\nu_0$, or $E' \approx E'' - h\nu_0$, and concerns the stimulated emission. We shall therefore neglect the second term as we are considering the absorption of radiation leading to a transition from a state where $E < me^2$, to one where $E > me^2$, that is, the emission of an electron. Practically all the electrons are emitted with energies $E' \approx E'' + h\nu_0$, and the total number transferred to energies in this neighbourhood is given by

$$\sum_{k', u'} |\mathfrak{a} (E', k', u')|^2 dE' / \hbar (E' \approx E'' + h\nu_0).$$

Since the integrand is only important when $E' \approx E'' + h\nu_0$, we may use the first term of (30) and take the matrix element outside the integral. On writing

$$x = 2\pi (\nu - \nu_0),$$

we see that the limits of the integral are immaterial provided they enclose the region $x \approx 0$. We take then from $-\infty$ to $+\infty$, and the expression for the total number of electrons emitted becomes

$$\sum_{k', u'} \left\{ \frac{2\pi e_0}{\hbar} \right\}^2 |(E', k', u') - \varphi_1 \sigma_x e^{2\pi i \nu} |E''', k', u''\rangle|^2 \int_{-\infty}^{+\infty} \frac{2 (1 - \cos x) t \, dx}{x^2 \pi}$$

which is proportional to $t$, as it must be. For the validity of the above, $t$ must be large compared with the periods of the atomic oscillations, but not too
large, or else we cannot replace \( \psi \) by \( \psi_{E', k', \nu'} \exp \left( -\frac{2\pi i E' t}{\hbar} \right) \) in (27). Both these conditions can be realised provided the transition probability is small compared with unity. If we multiply (32) by \( h\nu_0 \) and divide by \( t \) we obtain the rate of absorption of energy. The intensity of the radiation is \( 2\pi \nu_0 \sigma b_0^2/c \), so that the rate of absorption from a beam of unit intensity is

\[
\frac{2\pi \sigma c^2}{h\nu_0} \sum_{k' u'} \left| \langle E', k', \nu' \mid -\sigma_1 \sigma_2 e^{2\pi i \nu \cdot \xi} \mid E'', k'', \nu'' \rangle \right|^2,
\]

which we shall define as the absorption coefficient.

\section*{§ 4. Space Distribution of Photo-electrons}

The distribution of the initial directions of the photo-electrons relative to the direction of the incident beam is usually called the "longitudinal distribution." In our case this is the dependence on \( \theta \), the angle between the direction of the wave and that of the velocity of the electron emitted. The "lateral distribution," which is relative to \( \phi \), is not of very great interest, and we shall not consider it, except to remark that it is uniform for a non-polarised beam.

For very long wave-lengths, experiment shows that the density is proportional to \( \sin^2 \theta \), whilst for shorter wave-lengths there is greater emission in a forward direction. If \( \lambda \), the wave-length, be not too small, the theory yields the following variation of the density \( J \) with \( \theta \),

\[
J \sim (1 - \varepsilon \cos \theta) \sin^2 \theta.
\]

The first calculations of the quantity \( \varepsilon \) were incorrect because insufficient distinction was drawn between the wave-function and its component parts representing converging and diverging waves. This is corrected in a paper by Sommerfeld and Schur* who obtain a value

\[
\varepsilon = 4v/c,
\]

where \( v \) is the velocity of the photo-electron, the treatment being non-relativistic. The quantities usually measured experimentally are (1) the ratio of the number of photo-electrons emitted in a forward to those emitted in a backward direction; (2) the value of the bipartition angle, which is the semi-angle at the apex of a cone having the incident beam as its axis and dividing the photo-electron beam into two equal parts; and (3) the average value of the forward momentum of an electron. We shall consider the last as being the most convenient for our purposes and also the most significant. To find the

actual dependence on $\theta$ would be more difficult and without any special interest. It would certainly not obey the simple law of equation (34).

Considering first the case of fairly long wave-lengths, we have

$$J \sim (1 + 4v \cos \theta/c) \sin^2 \theta,$$

which gives

$$\cos \theta = 4v/5c.$$ 

Thus the average forward momentum is

$$4mv^2/5c = \frac{2}{5} h\nu_0/c,$$

where we neglect the work required to liberate the electron, which is small compared with $h\nu_0$ when $\nu$ is appreciable. Now $h\nu_0/c$ is the momentum of the light quantum, so that the photo-electrons have an average forward momentum which is $8/5$ times that of the light quantum liberating them. The statement on p. 410 of the paper just quoted, that the average momentum of the photo-electrons is $2h\nu_0/c$, does not seem to be correct. In the calculation on pp. 427-429, the authors assume a distribution $J \sim \sin^2 \theta$, and then add on to the momentum of each photo-electron a quantity $h\nu_0/c$ in the direction of the light ray, obtaining a distribution

$$J \sim (1 + 2v \cos \theta/c) \sin^2 \theta.$$

As the actual distribution is given by equation (36), it is inferred that the average momentum of the electrons is $2h\nu_0/c$. This procedure, however, is incorrect† because the resultant velocities of the electrons do not satisfy the Einstein law, $h\nu_0 = mv^2/c$. Actually if the initial momenta are so chosen that when compounded with $h\nu_0/c$, the resultant momenta all correspond to velocities $v$ given by $h\nu_0 = mv^2/c$, then it can be shown that the average forward momentum is $4h\nu_0/5$.

Let us consider the average forward momentum for very short wave-lengths, using relativistic theory. Neglecting as before the work required to liberate the electron we have

$$h\nu_0 + m_0c^2 = m_0c^2\beta',$$

where

$$\beta' = \sqrt{(1 - v^2/c^2)}.$$

This gives for the ratio of the average forward momentum of an electron to that of the light quantum a value

\[
\frac{v}{c} \frac{\beta'}{\beta' - 1} \cos \theta = \frac{1}{\tau} \cos \theta,
\]

where

\[
\tau^2 = \left(1 + \frac{2mc^2}{h\nu_0}\right)^{-1}
\]

The average forward momentum of the electrons is therefore equal to

\[
\frac{h\nu_0}{c\tau} \cos \theta.
\]

If \( \psi \) be the wave function of the final state, we have

\[
\cos \theta = \frac{\int_0^\pi \int_0^{2\pi} \psi \psi^* \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^\pi \int_0^{2\pi} \psi \psi^* \sin \theta \, d\theta \, d\phi}.
\]

We may use the asymptotic form of \( \psi \) since we are only interested in the distribution at large distances from the nucleus. Here the contributions of the discontinuous spectrum tend to zero very rapidly and we therefore put

\[
\phi = \sum_{k', u'} \int_{\text{mc}^2}^{\infty} a(E', k', u') \psi(E', k', u') e^{\text{i}E'th} dE'/h.
\]

Each \( \psi(E', k', u') \) involves functions of the type given by equations (19), and therefore represents both converging and diverging waves. The quantity \( a(E', k', u') \) involves the matrix element

\[
(E', k', u' | - \sigma_1 e^{\text{i}n\mu z/c} | E'', k'', u''),
\]

but it is only important when \( E' \approx E'' + h\nu_0 \), so that we may put in this value and take it outside the integral. On integrating over this range, the part representing the converging wave vanishes, and we are left with a diverging wave, which is the only physically significant solution.† For sufficiently large \( t \) we find that the denominator in the expression (40) for \( \cos \theta \) is equal to the expression (32) for the total number of electrons emitted, which is what we should expect. We write \( m_a, k', m_b, k'' \) for the matrix elements

\[
(E', k', u' | - \sigma_1 e^{\text{i}n\mu z/c} | E'', k'', u'')
\]

in (32), according as the final state is of type (3A) or (3B). Taking into account
the values of $\delta_a, \delta'_a, \ldots$, at the resonance point $E' = E'' + h\nu_0$, we obtain
finally for $\cos \theta$ the value

$$
2 \sum_{u', k'} \left[ \frac{\sqrt{(k' + u' + 2)(k' - u' + 1)}}{2k' + 3} m_{u', k'}^a m_{u', k' + 1}^a + \frac{\sqrt{(k' + u' + 1)(k' - u')}}{2k' + 1} m_{u', k}^b m_{u', k + 1}^b \right]
+ \frac{2u' + 1}{2(2k' + 1)(2k' + 3)} m_{u', k'}^a m_{u', k'}^b
\sum_{u', k'} [(m_{u', k'}^a)^2 + (m_{u', k'}^b)^2].
$$

(42)

Actually the last term vanishes on summation, owing to equalities between the
matrix elements. In this expression, as well as in the expression (33) for the
absorption coefficient, only $(m_{u', k'}^a)^2$ or $(m_{u', k'}^b)^2$ occur. These decrease
rapidly with $k'$, so that in the region of wave-length $\lambda = h/mc$, about 10 terms
suffice to give an accuracy of one in one hundred.

§ 5. Calculation of the Matrix Elements.—We have

$$
-\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},
$$

so that the matrix element $m_{u', k'}^a$ is really the sum of four elements

$$
(\psi_1' | i e^{2\pi i r_0/\epsilon} | \psi_4''') + (\psi_2' | - i e^{2\pi i r_0/\epsilon} | \psi_3''')
+ (\psi_3' | i e^{2\pi i r_0/\epsilon} | \psi_2''') + (\psi_4' | - i e^{2\pi i r_0/\epsilon} | \psi_1''').
$$

(43)

We shall assume that the atom is initially in the state of lowest energy. The
forms of the wave functions are obtained from equations (3A) and (3B). For
$k = 0$ there are two solutions of type (3A) but none of type (3B). They corre-
spond to $u = 0$ and $u = -1$. Thus the initial state is a combination of two
states, corresponding to two different directions of spin. If we write

$$
a_0 = \frac{\hbar^2}{4\pi^2 mZe^2} \quad \text{(radius of the atom on the Bohr theory)},$$

$$
\beta = \sqrt{(1 - \gamma^2)} - 1,
$$
we have\(^{+}\)

\[
\begin{align*}
\psi_1 &= -\frac{i\gamma}{1 + \sqrt{(1 - \gamma^2)}} r^\theta e^{-r/a_0} \cos \theta \\
\psi_2 &= -\frac{i\gamma}{1 + \sqrt{(1 - \gamma^2)}} r^\theta e^{-r/a_0} \sin \theta e^{i\phi} \\
\psi_3 &= 0 \\
\psi_4 &= 0
\end{align*}
\]

and

\[
\begin{align*}
\psi_1 &= \frac{i\gamma}{1 + \sqrt{(1 - \gamma^2)}} r^\theta e^{-r/a_0} \sin \theta e^{-i\phi} \\
\psi_2 &= -\frac{i\gamma}{1 + \sqrt{(1 - \gamma^2)}} r^\theta e^{-r/a_0} \cos \theta \\
\psi_3 &= 0 \\
\psi_4 &= 0
\end{align*}
\]

We shall label these states \(a_1\) and \(a_2\). The energy in both cases is equal to \(mc^2 \sqrt{1 - \gamma^2}\). These wave functions are easily normalised according to the rule for discrete states, and, if \(\xi(E_0)\) be the normalising factor, we have in both cases

\[
[\xi(E_0)]^2 4\pi \left(\frac{a_0}{2}\right)^{1 + \frac{1}{2}} \sqrt{(1 - \gamma^2)} \Gamma \left[1 + 2 \sqrt{(1 - \gamma^2)}\right] \frac{2}{1 + \sqrt{(1 - \gamma^2)}} = 1. \quad (46)
\]

We take our initial state to be

\[
\frac{1}{\sqrt{2}} \xi(E_0) (\psi (k = 0, u = 0) e^{ip} + \psi (k = 0, u = -1)e^{ip'}), \quad (47)
\]

where \(p\) and \(p'\) are arbitrary phase factors. In our result we must then average over all values of \(p\) and \(p'\).

The matrix elements in (43) now take one of the following forms

\[
\int \int \int \int e^{-r/a_0} r^\theta \left\{ \begin{array}{c}
\cos \theta \\
\sin \theta e^{i\phi}
\end{array} \right\} e^{i2\pi r \cos \theta /\lambda} P_n^m (\cos \theta) f(r) r^2 \sin \theta d\theta d\phi dr, \quad (48)
\]

where \(P_n^m (\cos \theta)\) is one of the wave functions of the final state. We write

\[
e^{i2\pi r \cos \theta /\lambda} = \sum_{j=0}^{\infty} (2j + 1) \frac{i^j}{j!} \sqrt{\left(\frac{\pi}{2\pi \cdot 2\pi /\lambda}\right)} J_{j + \frac{1}{2}} (2\pi r /\lambda) P_j^0 \cos \theta, \quad (49)
\]

where \(J_n(x)\) is Bessel's function of order \(n\). The expression (48) now takes the form

\[
\int \int \int \int \left\{ \begin{array}{c}
\cos \theta \\
\sin \theta e^{i\phi}
\end{array} \right\} P_n^m (\cos \theta) \left( \sum_{j=0}^{\infty} f_j (r) P_j^0 (\cos \theta) \right) r^2 \sin \theta d\theta d\phi dr. \quad (50)
\]

\(^{+}\) Darwin, loc. cit.
The integration with respect to $\theta$ and $\phi$ yields a selection rule for $u$. All the integrals of type (50), involving three Legendre functions, vanish; excepting those below. For these we have

\[
\int_0^{2\pi} \int_0^\pi P_{k+1}^u \cos \theta P_{k+1}^{u+1} \sin \theta d\theta d\phi = \frac{4\pi (k + u + 1)! (k - u + 1)!}{(2k + 1)(2k + 3)},
\]

\[
\int_0^{2\pi} \int_0^\pi P_{k+1}^u \sin \theta e^{-i\phi} P_{k+1}^{u-1} \sin \theta d\theta d\phi = \frac{4\pi (k + u + 2)! (k - u)!}{(2k + 1)(2k + 3)},
\]

\[
\int_0^{2\pi} \int_0^\pi P_{k+1}^{u*} \sin \theta e^{i\phi} P_{k+1}^{u-1} \sin \theta d\theta d\phi = -\frac{4\pi (k + u)! (k - u + 2)!}{(2k + 1)(2k + 3)},
\]

together with the conjugate equations.

We may represent the possible values of $u$ in the scheme below. In the first row we place the $(0, \phi)$ part of the initial wave function, in the second row, that of the two possible types of final wave function—(3a) and (3b). The latter we write in inverted order so that $\psi_4$ (final) is under $\psi_1$ (initial), and then form products in the same column. In the third row we have a term $P_n^0$ from the expansion of $e^{i2\pi \cos \theta/\alpha}$. In the fourth row we place those Legendre functions which give an integral $\neq 0$ when combined with the initial function and $P_n^0$. When $n$ and $u$ have values such that a term in row two is equal to a corresponding term in the last row, we have a possible final state with a given value of $u$, and the values of $n$ show what terms in the expansion (49) contribute towards the integral.

<table>
<thead>
<tr>
<th>Initial state</th>
<th>$-P_0^0 e^{i\phi}$; $-P_0^1 e^{i\phi}$</th>
<th>$-P_1^0 e^{i\phi}$; $-P_1^1 e^{i\phi}$</th>
<th>$P_0^0 e^{i\phi}$</th>
<th>$-P_0^0 e^{i\phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final state</td>
<td>$P_{k+1}^{u*}$</td>
<td>$P_k^{u*}$</td>
<td>$P_{k+1}^{u*}$</td>
<td>$P_{k+1}^{u*}$</td>
</tr>
<tr>
<td>Term in expn. of $e^{i2\pi \cos \theta/\alpha}$</td>
<td>$P_n^0$</td>
<td>$P_n^0$</td>
<td>$P_n^0$</td>
<td>$P_n^0$</td>
</tr>
<tr>
<td>Possible value of Legendre function in final state</td>
<td>$P_n^{u*}$; $P_n^{u*}$</td>
<td>$P_n^{u*}$; $P_n^{u*}$</td>
<td>$P_n^{u*}$</td>
<td>$P_n^{u*}$</td>
</tr>
<tr>
<td>$u = 0$</td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_2$ and $b_2$</td>
<td>$b_2$</td>
</tr>
<tr>
<td></td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_2$ and $b_2$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$u = 1$</td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_1$ and $b_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td></td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_1$ and $b_1$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$u = -1$</td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_2$ and $b_2$</td>
<td>$a_2$</td>
</tr>
<tr>
<td></td>
<td>$n = k - 1$</td>
<td>$n = k + 1$</td>
<td>$a_2$ and $b_2$</td>
<td>$b_2$</td>
</tr>
</tbody>
</table>
We label the combination $a$, $b$, when the final state is of type $(3A)$, $(3B)$ respectively, and with suffix 1, 2, when the initial wave function which contributes to the integral is of type $a_1$, $a_2$, respectively.

We shall consider wave-lengths in the region of $10^{-10}$ cm., corresponding to hard $\gamma$-rays. For these rays the electron emitted has an energy $E$ such that $E - mc^2$ is comparable to $E + mc^2$. Hence $A/|B|$ and $F/G$ are of order unity. It might now appear that the order of magnitude of the matrix elements in (43) depended on the initial state alone, so that we could neglect those involving the $\varphi_1$ and $\varphi_2$ of the initial state, since they are multiplied by a factor $\gamma$. We shall find, however, that the others are also of the same order, so that we must take all four elements of (43) into consideration. When we average the result over all phases of the initial state we find that the square of the matrix element

$$| (\psi \text{ (final)} | P | \psi (k = 0, u = 0) e^{i\varphi} + \psi (k = 0, u = -1) e^{i\varphi'} |^2$$

splits up into

$$| (\psi \text{ (final)} | P | \psi (k = 0, u = 0)) |^2 + | (\psi \text{ (final)} | P | \psi (k = 0, u = -1)) |^2.$$

From the table of combinations it now follows that after applying this, each square in (33) reduces to the sum of three integrals for $u = 0$ or $-1$, and two integrals when $u = 1$ or $-2$. Certain exceptions occur when $k = 0$ or $1$, which are indicated in the following pages. Similar remarks apply to the expression (42) for $\cos \theta$.

We now restrict ourselves to an atom for which the atomic number $Z$ is small, so that $\gamma = Z/137$ approximately, is small compared with unity. In this case

$$\xi(E_0) = (\pi a_0^3)^{-\frac{1}{2}} \text{ approximately.} \quad (52)$$

Let us now consider the integrals represented by $a_2$, $b_2$, etc., in the table, taking the combinations column by column.

I. Combinations in the First Column.

The matrix element is

$$(\psi_4 \text{ (final)} | - i e^{2\pi i r \cos \theta/\lambda} | \psi_1 \text{ (initial)}).$$

We shall calculate this for the various initial and final states, with the general value for the suffix $k$ of the final state.

(A) Initial $a_1$ state, $u = -1$ in final state.

We have for the integrand

$$\left[ \frac{1}{\sqrt{2}} (\pi a_0^3)^{-\frac{1}{2}} \frac{\gamma}{1 + \sqrt{(1 - \gamma^2)}} \right] \left[ -i r e^{-r/a_0} P_0 \right] \left[ -i e^{2\pi i r \cos \theta/\lambda} \psi_4^* \text{ (final)} \right].$$
(1) Final a state.

Expand the exponential and choose one term, say the term in $p_0^0$. This gives, on putting $\sqrt{1 - \gamma^2} = 1$,

$$\left(\frac{\pi a_0}{4} Y \right)^{-\gamma} \frac{r^2 e^{-r/a_0}}{2\pi r/\lambda} \left[ p_0^0 \cdot p_0^0 \cdot p_0^0 \right]$$

$$\times \left[ - \xi(E, k) \xi(k, u) (-k + u) G_k^* \frac{2n + 1}{n!} j^n j_n + \frac{1}{2\pi r/\lambda} \right].$$

Denote the first bracket by $M/\sqrt{4\pi}$. On multiplying the above by $\sin \theta d\theta d\phi$, and integrating over the sphere, we obtain the following results, using (51):

$$n = k + 1$$

$$\frac{k + 1}{2k + 1} \sqrt{(k + 1)} i^{k+1} M \cdot J_{k+1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, k).$$

$$n = k - 1$$

$$\frac{k}{2k + 1} \sqrt{(k + 1)} i^{k-1} M \cdot J_{k-1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, k).$$

For all other values of $n$ the term is zero. The integrand with respect to $r$ is therefore the sum of these two.

The following results are obtained in a similar way:

(2) Final b state. $k \geq 1$

$$n = k + 1$$

$$\frac{k + 1}{2k + 1} \sqrt{k} i^{k+1} M \cdot J_{k+1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, -k - 1).$$

$$n = k - 1$$

$$\frac{k}{2k + 1} \sqrt{k} i^{k-1} M \cdot J_{k-1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, -k - 1).$$

(B) Initial $a_2$ state, $u = -2$ in final state.

(1) Final a state. $k \geq 1$.

$$n = k + 1$$

$$\sqrt{k (k + 1) (k + 2)} \frac{i^{k+1} M \cdot J_{k+1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, k).}{2k + 1}$$

$$n = k - 1$$

$$\sqrt{k (k + 1) (k + 2)} \frac{i^{k-1} M \cdot J_{k-1+\frac{1}{2}} (2\pi r/\lambda) G_k^* \xi(E, k).}{2k + 1}.$$
(2) Final b state. \( k \geq 2 \).

\[ n = k + 1 \]

\[ \frac{\sqrt{[(k - 1) k (k + 1)]}}{2k + 1} \partial^{k+1} M \cdot J_{k+1+i} (2\pi r/\lambda) G^{*}_{-k-1} \xi (E, - k - 1). \]

\[ n = k - 1 \]

\[ - \frac{\sqrt{[(k - 1) k (k + 1)]}}{2k + 1} \partial^{k-1} M \cdot J_{k-1+i} (2\pi r/\lambda) G^{*}_{-k-1} \xi (E, - k - 1). \]

In a similar way we can find the combinations occurring in the second column; they differ only in the first factor. These expressions have then to be multiplied by \( r^2 \), and integrated from \( r = 0 \) to \( r = \infty \).

Let us first consider the integrals for \( G_k \) and \( G_{-k-1} \).

Putting \( t = iau \) in the expression (17a) for \( \Phi_k \), we find

\[ \Phi_k = [(k - s) - i (b + c)] r^s a^{2s+2} e^{-a^2} \int_{-1}^{+1} (1 - u)^{-ib+1} (1 + u)^{s+ib} e^{iaru} du. \]

We write the integral itself, omitting the factors outside

\[ I = \int_{-1}^{+1} (1 - u) (P + iQ) du. \]

On changing \( u \) to \(-u\) this gives

\[ I = \int_{-1}^{+1} (1 + u) (P - iQ) du, \]

so that

\[ \int_{-1}^{+1} P du \neq 0, \quad \int_{-1}^{+1} uP du = 0, \quad \int_{-1}^{+1} uQ du \neq 0, \quad \int_{-1}^{+1} Q du = 0. \]

We find therefore

\[ \Re I = \int_{-1}^{+1} (1 - u)^{-ib} (1 + u)^{s+ib} e^{iaru} du \]

\[ \Im I = -\int_{-1}^{+1} u (1 - u)^{-ib} (1 + u)^{s+ib} e^{iaru} du \]

where

\[ I = \Re I + i \Im I. \]

On integrating by parts the expression

\[ \int_{-1}^{+1} (1 - u)^{-ib+1} (1 + u)^{s+ib+1} e^{iaru} du, \]
we obtain

\[ 3I = -\frac{iar}{2(s+1)} \left\{ \int_{-1}^{+1} (1-u)^{s-ib+1} (1+u)^{s+ib+1} e^{i\alpha_{k+1}} du \right\} \]

\[ -\frac{ib}{(s+1)} \left\{ \int_{-1}^{+1} (1-u)^{s-ib} (1+u)^{s+ib} e^{i\alpha_{k+1}} du \right\}. \]  

(56)

For short wave-lengths we shall see that \( b \) is of the order \( \gamma \), and also that the two integrals in (56), apart from the factor \( b \), contribute quantities of the same order to the radial integral. The presence of \( b \), therefore, enables us to neglect the second integral in (56). Remembering also that \( k - s \approx \gamma^2/k \ll b + c \), we have finally

\[ G_k = -r^{s+2} a^{s+ib} (b + c) \left\{ \int_{-1}^{+1} (1-u)^{s-ib} (1+u)^{s+ib} e^{i\alpha_{k+1}} du \right\}. \]

(57)

Similarly we find

\[ G_{k-1} = -r^{s'+1} a^{s'+ib} (b-c) \left\{ \int_{-1}^{+1} (1-u)^{s'+1-ib} (1+u)^{s'+1+ib} e^{i\alpha_{k-1}} du \right\}. \]

(58)

The radial integrals are of the forms

\[ \int_0^\infty J_{k+1+\frac{1}{2}} (2\pi r/\lambda) G_k \xi^* (E, k) \cdot M \cdot r^2 \, dr \]

and

\[ \int_0^\infty J_{k+1+\frac{1}{2}} (2\pi r/\lambda) G_{k-1} \xi^* (E, -k-1) \cdot M \cdot r^2 \, dr \]

(59)

Further we have

\[ s = k + O(\gamma^2/k), \]

\[ s' = k - 1 + O(\gamma^2/k), \]

and

\[ |\Gamma(k + ib + 1)| = \Gamma(k+1) [1 + O(b^2)] \]

(60)

We shall neglect quantities of order \( b^2 \) or \( \gamma^2 \). On substituting for \( M, \xi (E, k), G_k, \) etc., in (59), we find that both integrals reduce to

\[ L \frac{a^{k+3+b}}{\Gamma(k+1) 2^k} \int_0^\infty r^2 e^{-\frac{r^2}{2}} \frac{r^{k+2}}{\sqrt{(2\pi r/\lambda)}} J_{k+1} (2\pi r/\lambda) \]

\[ \times \left\{ \int_{-1}^{+1} (1-u)^{k-ib} (1+u)^{k+ib} e^{i\alpha_{k+1}} du \right\} \, dr, \]

(61)

where

\[ L = -\frac{(2\pi E)^{\frac{1}{2}}}{\hbar \omega \alpha} \frac{A}{(A^2 + |B|^2)^{\frac{1}{2}}} \frac{e^{-\frac{\nu b}{\pi \gamma a_0^{\frac{1}{2}}}}}{2 \cdot \alpha^{\frac{1}{2}+2}} \]

and

\[ p = k \pm 1 \]

(62)

\[ \text{Vol. CXXXIII.—A.} \]
Let us consider the values of the constants involved. Firstly we have

\[ a = A |B| \frac{2\pi}{\hbar c} (E^2 - m^2c^4)^{\frac{1}{2}}. \]

Since the energy required to liberate an electron is small compared with \( \hbar \nu_0 \) in the region we are considering, \( E = mc^2 + \hbar \nu_0 \), and so

\[ a = \frac{2\pi \nu_0}{c} \left[ 1 + \frac{2mc^2}{\hbar \nu_0} \right]^{\frac{1}{2}}, \]

giving

\[ a = \frac{2\pi}{\lambda \tau} \text{ (equation (38)).} \quad (63) \]

Also \( \beta = \sqrt{(1 - \gamma^2)} - 1 = O (\gamma^2) \) and is therefore small. We put \( r/a_0 = \delta a r \), where \( \delta = 1/aa_0 \) and is also of order \( \gamma \). Lastly

\[ b = \frac{\gamma}{2} \frac{|B|^2 + A^2}{|B|} A = O (\gamma). \]

In (61) we now change the variable of integration by writing \( r \) instead of \( ar \), and making use of the formula

\[ J_{p+\frac{1}{2}}(\tau r) = \frac{(\tau r)^{p+\frac{1}{2}}}{\Gamma (p + 1) 2^{p+\frac{1}{2}} \sqrt{\pi}} \int_{-1}^{+1} (1 - t^2) e^{i\tau t} dt, \quad (64) \]

we find for (61) the expression

\[ Q \int_0^\infty dr \int_{-1}^{+1} du \int_{-1}^{+1} dt e^{-r (\delta - iu - it \tau)} \frac{(1 - t^2)^p (1 - u)^{k+\beta} (1 + u)^{k+ib}}{(1 - (1 - (1 + \gamma)^2))^k(p+ib)} \]

where

\[ Q = \frac{L \tau^p}{\sqrt{\pi} \Gamma (p + 1) \Gamma (k + 1) 2^{k+p+\frac{1}{2}}} \quad (65) \]

Integrating with respect to \( r \) we obtain

\[ Q \Gamma (p + k + \beta + 3) \int_{-1}^{+1} du \int_{-1}^{+1} dt \frac{(1 - t)^p (1 + t)^p (1 - u)^{k+ib} (1 + u)^{k+ib}}{[\delta - i(\tau t + u)]^{p+k+\beta+3}} \cdot (66) \]

This can be expressed as a hypergeometric function of two variables.* If we put \( (1 + u) = 2u' \) and \( (1 + t) = 2t' \), we find for (66) the value

\[ Q \Gamma (p + k + \beta + 3) \frac{2^{k+p+1}}{[\delta + i(\tau + 1)]^{p+k+\beta+3}} \int_0^1 du' \int_0^1 dt' \frac{t'^p (1 - t')^p u'^{k+ib} (1 - u')^{k-ib}}{1 - 2\tau \frac{1 - 1 + i\delta - i\delta}{1 + \tau - i\delta} u'} \frac{2}{1 + \tau - i\delta} \frac{1}{u} \]

* P. Appell et J. Kampé de Feriet, "Fonctions Hypergeometriques et Hyperspheriques," p. 28.
giving,* on neglecting $\beta$

\[
K = \frac{\Gamma(p + 1) \Gamma(k + 1) \Gamma(k + p + 3)}{\Gamma(2p + 2) \Gamma(2k + 2)}
\]

Hence for the integral (61) we have the following—

\[
p = k + 1
\]

\[
L \frac{\Gamma(k + 1) \Gamma(k + 2) 2^{2k+3} \tau^{k+1}}{\sqrt{2\pi} \Gamma(2k + 2) (i (\tau + 1) + \delta)^{2k+1}} F_2(2k + 4, k + 2, k + 1 + ib, 2k + 4, 2k + 2; x, y),
\]

\[
p = k - 1
\]

\[
L \frac{\Gamma(k) \Gamma(k + 1) 2^{2k+1} \tau^{k-1}}{\sqrt{2\pi} \Gamma(2k) (i (\tau + 1) + \delta)^{2k+2}} F_2(2k + 2, k, k + 1 + ib, 2k, 2k + 2; x, y),
\]

where

\[
x = \frac{2\tau}{\tau + 1 - i\delta} \quad \text{and} \quad y = \frac{2}{\tau + 1 - i\delta}
\]

These may be transformed by the following relations*

\[
F_2(\alpha, \beta, \beta', \gamma = \alpha, \gamma'; x, y) = (1 - \alpha)^{-\beta} F_1(\beta', \beta, \alpha - \beta, \gamma'; y/(1 - x), y)
\]

\[
F_2(\alpha, \beta, \beta', \gamma' = \alpha; x, y) = (1 - y)^{-\beta} F_1(\beta, \alpha - \beta', \beta', \gamma; x, x/(1 - y))
\]

Leaving aside the factor outside $F_2$ for the moment, these give respectively

\[
(1 - x)^{-(k+2)} F_1(k + 1 + ib, k + 2, k + 2, 2k + 2; y/(1-x), y)
\]

and

\[
(1 - y)^{-(k+1+ib)} F_1(k, k + 1 - ib, k + 1 + ib, 2k; x, x/(1-y)).
\]

Consider first (70A) and apply the following relation

\[
F_1(\alpha, \beta, \beta', \gamma; u, v) = \frac{(\gamma - \alpha)(\gamma - \alpha + 1)u^2}{\gamma (\gamma + 1) (u - 1)^2} F_1(\alpha, \beta, \beta', \gamma + 2; u, v)
\]

\[
- \frac{2(\gamma - \alpha)u}{\gamma (u - 1)^2} F_1(\alpha, \beta - 1, \beta', \gamma + 1; u, v)
\]

\[
+ \frac{1}{(u - 1)^2} F_1(\alpha, \beta - 2, \beta', \gamma; u, v),
\]

* See Appell, loc. cit., for these and others.
giving
\[
\frac{(1 - x)^{-k}}{(x + y - 1)^2} \left\{ \frac{(k + 1 - ib) (k + 2 - ib)}{(2k + 2) (2k + 3)} \frac{y^2}{(1 - x)^2} F_1 (k + 1 + ib, k + 2, 2k + 4; \frac{y}{1-x}, y) - \frac{2(k + 1 - ib)}{2k + 2} \frac{y}{1-x} F_1 (k + 1 + ib, k + 1, k + 2, 2k + 3; \frac{y}{1-x}, y) + F_1 (k + 1 + ib, k + 2, 2k + 2; \frac{y}{1-x}, y) \right\}.
\]

We now use
\[
F_1 (\alpha, \beta, \gamma + 1; \nu, v) = (1 - u)^{-\alpha} F (\alpha, \beta', \gamma + 1; (u - v)/(u - 1))
\]
and then transform the middle term by the formula
\[
(\gamma - \alpha)z F (\alpha, \beta, \gamma + 1; z) = \gamma F (\alpha, \beta - 1, \gamma; z) - \gamma (1 - z) F (\alpha, \beta, \gamma; z).
\]
We obtain
\[
\frac{(-)^k (x-1)^{1+ib}}{(x + y - 1)^{k+3+ib}} \left\{ \frac{(k + 1 - ib) (k + 2 - ib)}{(2k + 2) (2k + 3)} \frac{y^2}{(1 - x)^2} F (k + 1 + ib, k + 2, 2k + 4; z) - \frac{2y}{(1-x)z} F (k + 1 + ib, k + 1, 2k+2; z) + \frac{2y-z}{z} F (k + 2, k + 1 + ib, 2k + 2; z) \right\},
\]
where
\[
z = \frac{xy}{x + y - 1}.
\]

We transform the hypergeometric series by a formula of Kummer*
\[
F (\alpha, \beta, 2\beta; z) = \left(\frac{2}{1 + \sqrt{1 - z}}\right)^{2\alpha} \times F \left[ \alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}; \left(\frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}\right)^2 \right].
\]

We have
\[
z = \frac{4\tau}{(\tau + 1)^2 + 8\delta^2}
\]
and
\[
\frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} = \tau.
\]

* Kummer, 'Crelle,' vol. 15. The formula is given wrongly owing to a slip in the derivation.
The transformed series converge much more rapidly, about six terms being sufficient for an accuracy of one in one hundred. We may therefore neglect the $ib$ in the hypergeometric functions as well as outside. (For the third function we must transform to $F(x, k+1, 2k+2; z)$ first, using a relation similar to (71), and the result then follows). We also neglect $\delta$, though not $k\delta$, compared with unity, and on applying (73) and substituting for $x, y$ and $z$ we finally obtain for (68a)

$$\frac{L \Gamma (k + 1) \Gamma (k + 2) 2^{2k+3} \tau^{k+1}}{\sqrt{2\pi} \Gamma (2k + 2) (\tau + 1)^3} \left\{ \begin{array}{c} \frac{k + 2}{2 (2k + 3)} \frac{4}{1 - \tau} \\ \times F(k + 1, -\frac{1}{2}, k + \frac{3}{2}; \tau^2) \\ \frac{(1 + \tau)^2}{\tau} \\ \times F(k + 1, \frac{1}{2}, k + \frac{5}{2}; \tau^2) \\ + \frac{1}{\tau (1 - \tau)} \\ \times F(k, -\frac{1}{2}, k + \frac{1}{2}; \tau^2) \end{array} \right\}. (75)$$

In a similar way we may show that $ib$ may be neglected in the function (68b), and we may therefore obtain a result for this by replacing $k$ by $k - 1$ and interchanging $x$ and $y$ in (72). This follows from equations (68) since $F_2$ is symmetrical in $\beta, x$; and $\beta', y$. The result for (68b) is the expression (75) with $k$ replaced by $k - 1$, and the first and last hypergeometric functions multiplied by $-\tau$.

In finding the expressions (57) and (58) for $G_k$ and $G_{k-1}$ we neglected certain integrals which are multiplied by $b$. These lead to expressions similar to (66), which, however, cannot be reduced in the same way. We may show that they do not involve an extra factor $b$ or $\delta$ in the denominator, as follows. They are finite as $b \to 0$, and become integrals involving the product of two Bessel functions, which are again finite as $\delta \to 0$. Thus they may be neglected since they are multiplied by $b$.

In the combinations occurring in the last two columns, the integral no longer contains a factor $\gamma$, but we shall find that it contains factors $b$ or $\delta$, so that we must retain the first powers of these in evaluating it.

For the third column of the table given we find as before

**Initial $a_1$ state, $u = -1$ in final state.**

1. **Final a state.**

$$n = k + 1$$

$$\sqrt{k + 1} i^{k+1} l_u.$$
(2) Final b state. \( k \geq 1 \)

\[ n = k - 1 \]

\[ - \sqrt{k} e^{k-1} I_b, \]

where

\[
I_a = \int_0^\infty a_0^{-3/2} F_k^* r^a e^{-r/\alpha} \xi(E, k) \sqrt{\left( \frac{\pi}{2\tau r/\lambda} \right)} J_{k+1+\frac{1}{2}}(2\pi r/\lambda) r^2 \, dr,
\]

\[
I_b = \int_0^\infty a_0^{-3/2} F_{k-1}^* r^a e^{-r/\alpha} \xi(E, k - 1) \sqrt{\left( \frac{\pi}{2\tau r/\lambda} \right)} J_{k-1+\frac{1}{2}}(2\pi r/\lambda) r^2 \, dr.
\]

(76)

Similar results, involving \( I_a \) and \( I_b \) are obtained for column four.

We now take \( F_k^* = F_k \) as the real part of \( G_k/A \), and similarly for \( F_{k-1}^* \).

In evaluating the integrals (76) we shall keep the first powers of \( b \) and \( \delta \), so that it is more convenient to keep \( G_k \) in the form given by (53). We find by a procedure similar to the one used to evaluate (59)

\[
I_a = \Re \left( \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \right) \left( - \kappa + 1 \right) [(k - s) - i(b + c)] \tau^{k+1} \left( \Gamma(k + 1) \right)^2 (k + 1 - ib) 2^{2k+4} F_{2a},
\]

\[
I_b = \Re \left( \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \right) \left( - \kappa + 1 \right) [(k - 1 - s') - i(b - c)] \tau^k \left( \Gamma(k) \right)^2 (k - ib) 2^{2k+2} F_{2b},
\]

(77)

where

\[
F_{2a} = F_2(2k + 4, k + 2, k + 1 + ib, 2k + 4, 2k + 3; x, y),
\]

\[
F_{2b} = F_2(2k + 1, k, k + ib, 2k + 2k + 1; x, y).
\]

Since \( (k - s) \ll (b + c) \), we must retain the imaginary parts of \( F_{2a} \) and \( F_{2b} \).

We transform these functions as before, and retaining the relevant powers of \( b, \delta \) and \( \gamma \) we find

\[
I_a = \frac{W(k + 1)}{(b + c)} \Re \left( \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \right) \left( \frac{2}{(2k + 3)(\tau + 1)^2} \right) \left( k + 2 - ib \right) (\tau + 1 + i\delta) \times \left( \frac{1}{2\tau} F(k + 1 + ib, ib - \frac{1}{2}, k + \frac{5}{2}; \tau^2) \right)
\]

\[
+ \left( \frac{1 - \tau}{2\tau} \right)^2 F(k + 1 + ib, ib + \frac{1}{2}, k + \frac{3}{2}; \tau^2)
\]

(78a)
Photoelectric Effect for $\gamma$-Rays.

\[ I_\delta = \frac{W(k)}{(b - c)} R. \]

where

\[
\begin{align*}
\left[ (b - c) b - k (k - 1 - s') + i (b - c) k \right] \\
\times (1 - \tau^2)^{-ib}
\end{align*}
\]

\[ \left\{ \frac{(1 - \tau^2)^2}{2\tau} F(k - ib, \frac{1}{2} - ib, k + \frac{1}{2}; \tau^2) \right\} \\
\times \left\{ \frac{(1 - \tau^2)^2ib (1 - \tau^2 + 2i\delta)}{2\tau (1 + \tau^2)} \\
\times F(k - 1 + ib, ib - \frac{1}{2}, k + \frac{1}{2}; \tau^2) \right\} \right), (78B)
\]

These results, together with the analogous ones, (75), for the first two columns, enable us to calculate all the matrix elements.

In carrying out the numerical calculations we note that only two types of hypergeometric functions occur—

\[ F(k + ib, ib - \frac{1}{2}, k + \frac{3}{2}; \tau^2) \] \hspace{1cm} (a)

and

\[ F(k + ib, ib + \frac{1}{2}, k + \frac{1}{2}; \tau^2). \] \hspace{1cm} (b)

Half-a-dozen terms are quite sufficient to calculate the real and imaginary parts of (a) and the real part of (b). For the imaginary part of (b) we use the formula

\[ F(x, \beta, \gamma; z) = (1 - z)^{\gamma - a - \beta} F(\gamma - \beta, \gamma - a, \gamma; z), \]

which gives us the ratio of the real to the imaginary part of (b) in terms of $\tau^2$ only.

§ 6. Numerical Results.—For the coefficient of absorption, $I$, we obtain the following results:

\[
\begin{array}{cccccc}
\lambda & 6.08 \times 10^{-11} \text{ cm.} & 1.22 \times 10^{-10} \text{ cm.} & 2.43 \times 10^{-10} \text{ cm.} & 6.08 \times 10^{-10} \text{ cm.} & 9.73 \times 10^{-10} \text{ cm.} \\
1 \times Z^{-2} & 4.89 \times 10^{-34} & 1.87 \times 10^{-33} & 9.92 \times 10^{-33} & 6.87 \times 10^{-31} & 8.01 \times 10^{-31} \\
\eta & 1.9 & 2.4 & 3.1 & 3.2 & \\
\end{array}
\]

Between any two wave-lengths, we may express the mean variation of $I$ by the formula

\[ I \sim \lambda^n \cdot Z^5. \]

Under each wave-length we have put that value of $n$ which expresses the mean variation of $I$ between that wave-length and the one following. The value of
\( n \) increases with \( \lambda \), which is experimentally observed in the case of heavy atoms. It should be emphasised, however, that these results are only applicable to very light atoms—where \( Z/137 \) may be considered small compared with unity. A comparison with experiment is therefore not yet possible, as it is extremely difficult to measure the photoelectric effect in light atoms.

If \( R \) denote the ratio of the average forward momentum of an electron to that of the light quantum liberating it, we have the following results:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\lambda & \lambda \to 0 & 1.22 \times 10^{-10} \text{ cm} & 2.43 \times 10^{-10} \text{ cm} & 6.08 \times 10^{-10} \text{ cm} & 9.73 \times 10^{-10} \text{ cm} & \lambda \to \infty \\
R & 1 & 1.15 & 1.29 & 1.48 & 1.57 & 1.6 \\
\hline
\end{array}
\]

The first result may be inferred from a graph of the values, or by noting that as \( \lambda \to 0, \tau \to 1 \) and the matrix elements in (42) vary very slowly with \( k \), so that \( \cos \theta \to 1 \). The result for \( \lambda \to \infty \) is obtained from the non-relativistic calculations.

The theoretical value of \( n \) for long waves is 3.5, so that it is interesting to note that the deviations of \( n \) and \( R \), from their values for large wave-lengths, first become serious in the region of the Compton wave-length, given by

\[
\lambda = \frac{h}{mc} = 2.43 \times 10^{-10} \text{ cm}.
\]

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