The Internal Conversion of $\gamma$-Rays with the Production of Electrons and Positrons

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§ 1 INTRODUCTION

A $\gamma$-ray emitted from the nucleus of a radioactive atom may be absorbed by one of the outer electrons, with the production of a $\beta$-ray. This process has already been treated at length† and fairly satisfactory results have been obtained. If the energy of the $\gamma$-ray is greater than $2mc^2$ it is possible for the $\gamma$-ray to be absorbed by one of the electrons in a state of negative energy. This electron is then emitted with an energy $h\nu_0 - |E'|$, where $h\nu_0$ is the energy of the $\gamma$-ray and $E'$ the energy of the electron in the negative energy state. We are thus left with an electron of energy $h\nu_0 - |E'|$, and a hole, or positron, of energy $|E'|$. The problem has been treated by Oppenheimer and Plesset,‡ who gave an approximate answer in the form

$$I \sim \alpha^2 Z^2,$$

where $I$ is the Internal Conversion Coefficient, that is, the number of pairs created for each $\gamma$-ray emitted from the nucleus, $\alpha$ is the fine structure constant and $Z$ the atomic number of the nucleus. The approximations used were very rough, and as the problem could be treated rigorously it was decided that an accurate computation would be worth attempting.

While these calculations were in progress, Oppenheimer and Nedelski§ gave another calculation of $I$, in which they found that for high energies it was almost independent of the atomic number of the nucleus emitting the $\gamma$-ray. They therefore neglected completely the electrostatic field

of the nucleus. According to these authors the method should be valid when
\[ Zc/137 \nu \ll 1, \]
(where \( \nu \) is the velocity of the particles emitted) but, owing to the neglect of the electrostatic field, it will not give any asymmetry in the numbers of electrons and positrons emitted with given energies. In the nuclear field we should expect the positrons to be repelled, and their average energy to be greater than that of the electrons, which are attracted.

In the following we shall consider the nucleus as a fixed radiating dipole or quadrupole, corresponding to \( \gamma \)-rays produced by the two different types of transition in the nucleus, as shown by Taylor and Mott (loc. cit., I). These authors (loc. cit., II) have introduced a correction to take account of the fact that the nucleus cannot be considered as a strictly closed system, but this will not be considered here, as the effect is only appreciable for large values of \( I \). We shall consider the case of an atom with atomic number 84, which will give us a good idea of the magnitude of the effect in radioactive elements. The results obtained show that the probability of this process is of the order \( 10^{-3} - 10^{-4} \), and an effect of this magnitude has recently been observed by Alichanow and Kosodaew.†

In § 2 we discuss the wave functions for electrons with energies less than \(-mc^2\), and in § 3 we give the perturbation theory. § 4 contains the actual calculation of the matrix elements involved and the results are given in § 5, together with a discussion of the experimental data.

## § 2 The Wave Functions for the States With Negative Energies

We shall write the wave equation of the electron as
\[
\left[ \frac{\mathbf{E}}{c} + \frac{\mathbf{p}}{c} \mathbf{A}_0 + \hat{\rho}_1 \left( \sigma, \mathbf{p} + \frac{\mathbf{e}}{c} \mathbf{A} \right) + \rho_0 mc \right] \Psi = 0, \tag{1}
\]
where \( E \) is the energy of the system, \( A_0 \) and \( \mathbf{A} \) the scalar and vector potentials, \( \mathbf{p} \) the momentum vector \( (p_x, p_y, p_z) \) and \( \sigma \) the vector \( (\sigma_x, \sigma_y, \sigma_z) \), and \( e > 0 \). The quantities \( \sigma_x, \sigma_y, \sigma_z, \rho_2, (\rho_2) \) and \( \rho_3 \) do not commute with each other and may be conveniently represented by certain matrices of four rows and columns which obey the same non-commutability relations. In this representation the wave function \( \Psi \) has four components.

† 'Z. Physik,' vol. 90, p. 249 (1934).
The explicit form of the wave functions depends upon the matrices chosen to represent $\sigma$ and the $p$'s in (1), and in the following we shall use those given by Dirac. The solutions are then found to be of two types:

\begin{align}
\psi_1 &= -i F_k P_{k+1}^u \\
\psi_2 &= -i F_k P_{k+1}^u \\
\psi_3 &= (k + u + 1) G_k P_k^u \\
\psi_4 &= (-k + u) G_k P_k^{u+1}
\end{align}

(2A)

and

\begin{align}
\psi_1 &= -i (k + u) F_{-k-1} P_{k-1}^u \\
\psi_2 &= -i (-k + u + 1) F_{-k-1} P_{k-1}^{u+1} \\
\psi_3 &= G_{-k-1} P_k^u \\
\psi_4 &= G_{-k-1} P_k^{u+1}
\end{align}

(2B)

where $F_k$ and $G_k$ satisfy

\begin{align}
(A^2 + \frac{\gamma}{r}) F_k + \frac{dG_k}{dr} - \frac{k}{r} G_k &= 0 \\
(B^2 - \frac{\gamma}{r}) G_k + \frac{dF_k}{dr} + \frac{k+2}{r} F_k &= 0
\end{align}

(3)

with

\begin{align}
A^2 &= \frac{2\pi}{\hbar} (mc + E) \\
B^2 &= \frac{2\pi}{\hbar} (mc - E), \\
\gamma &= \frac{2\pi e^2 Z}{\hbar c} \approx \frac{Z}{137}
\end{align}

(4)

and $P_k^u$ is the associated Legendre function given by

\[ P_k^u = (k - u)! \sin^u \theta \left( \frac{d}{d \cos \theta} \right)^{k+u} \frac{(\cos^2 \theta - 1)^k}{2^k k!} e^{i\nu \phi}, \]

where $u$ and $k$ being any numbers such that the Legendre functions involved have a meaning. In this representation the axis of quantization is the $z$-axis. When $E < mc^2$, suitable solutions of (3) can be obtained in the form of polynomials for a set of discrete values of the energy. In addition there exist solutions when $E > mc^2$ or $E < -mc^2$. The former represent spherical waves, and correspond to the hyperbolic orbits of the older theory, and the latter are similar solutions representing electrons whose total energy is negative.

Equations (3) may be solved by a method given by Hulme. We first transform them into a standard form by writing

\begin{align}
\Phi_k &= AF_k - BG_k \\
\Theta_k &= AF_k + BG_k
\end{align}

(5)

We then find

\[ \mathfrak{F}_k = C [(k - s) + i(b + c)] r^a a^{2s+2} e^{-n-b} \int_{-1}^{+1} (1 - u)^{s-ib} (1 + u)^{s+ib+1} e^{i\alpha u} \, du \]

\[ \mathfrak{G}_k = C [(k - s) - i(b + c)] r^a a^{2s+2} e^{-n-b} \int_{-1}^{+1} (1 - u)^{s-ib+1} (1 + u)^{s+ib} e^{i\alpha u} \, du \]

for states of type (2A), and

\[ \mathfrak{F}_{-k-1} = D [(b - c) - i(k - 1 - t)] r^a a^{2t+2} e^{-n-b} \int_{-1}^{+1} (1 - u)^{t-ib} (1 + u)^{t+ib+1} e^{i\alpha u} \, du \]

\[ \mathfrak{G}_{-k-1} = D [(b - c) + i(k - 1 - t)] r^a a^{2t+2} e^{-n-b} \int_{-1}^{+1} (1 - u)^{t-ib+1} (1 + u)^{t+ib} e^{i\alpha u} \, du \]

for states of type (2B), where

\[ AB = i\alpha, \]

\[ \frac{\chi}{2} \left( \frac{A}{B} - \frac{B}{A} \right) = -ib, \]

\[ s = \sqrt{(k + 1)^2 - \gamma^2} - 1, \]

\[ t = \sqrt{k^2 - \gamma^2} - 1, \]

\[ \frac{\chi}{2} \left( \frac{A}{B} + \frac{B}{A} \right) = -ic, \]

and so far the constants C and D may have any values. When \( E > mc^2 \) we see from (4) that A is real and B a pure imaginary, so that \( \mathfrak{F} \) and \( \mathfrak{G} \) are conjugate complex quantities (from (5)) and we must restrict C and D to be purely real. On the other hand, when \( E < -mc^2 \), B is real and A is a pure imaginary, so that (5) gives \( \mathfrak{F}^* = -\mathfrak{G} \), and we must restrict C and D to be pure imaginaries in order to satisfy this condition. It is convenient to take

\[ C = e^{\pi b} a^{-s-2}, \quad D = e^{\pi b} a^{-t-2} \quad \text{for } E > mc^2 \]

\[ C = i e^{\pi b} a^{-s-2}, \quad D = i e^{\pi b} a^{-t-2} \quad \text{for } E < -mc^2 \]

giving

\[ \mathfrak{G}_k = [(k - s) - i(b + c)] (ar)^{s-ib+1} (1 + u)^{s+ib} e^{i\alpha u} \, du \]

\[ \mathfrak{G}_{k'} = i [(k' - s') - i(b' + c')] (a'r)^{s'} \int_{-1}^{+1} (1 - u)^{s'-ib'+1} (1 + u)^{s+ib'} e^{i\alpha u} \, du \quad (E > mc^2) \]

\[ \mathfrak{G}_k = [(k - s) - i(b + c)] (ar)^{s-ib+1} (1 + u)^{s+ib} e^{i\alpha u} \, du \quad (E < -mc^2) \]
and corresponding expressions for $\phi_{-k-1}, \phi_{-k'-1},$ etc. We have distinguished the negative energy states by a dash, $s', a'$ and $c'$ are still given by (7) and have the same values as $s, a, c$ in the state with energy $E$; $b'$, however, changes sign and we have $b'(-|E|) = -b(|E|)$.

With these expressions for $\phi_{k},$ etc., the normalizing factors for states of type (2A) are given by

$$\xi(E, k) = \left(\frac{2\pi a |E|}{hc^2}\right)^{\frac{1}{2}} \frac{2 |AB|}{(|A|^2 + |B|^2)^{\frac{1}{2}}},$$

$$= (2/c)^{\frac{1}{2}} (2\pi |p|/h)^{\frac{1}{2}} \Lambda^{-1},$$

where $|p|$ is the momentum of the electron at great distances from the nucleus,

$$\Lambda = \sqrt{(k - s)^2 + (b + c)^2} \left| \Gamma(s + ib + 1) e^{-z_b/2} 2^{s+1} \right|,$$

and

$$\xi(k, u) = \left\{4\pi (k + u + 1)! (k - u)! \right\}^{-\frac{1}{2}}.$$

We have expressed the normalizing factor as a product of $\xi(E, k),$ the normalizing factor for the radial part of the wave function, and $\xi(k, u),$ that for the angular part.

To obtain the corresponding normalizing factors for the solutions (6B) we replace $k$ by $k - 1,$ $s$ by $t$ and $c$ by $-c$ in the above. For negative energy states we have the same normalization factors with $s', a', b', c'$ replacing $s, a, b$ and $c.$

This normalization represents electrons entering or leaving a large sphere with centre at the origin, at a rate of one per unit time.

§ 3 Perturbation Theory

Let us now consider the probability that a $\gamma$-ray emitted from the nucleus will be absorbed by an electron with energy $E' < -mc^2,$ the electron being raised to an energy $h\nu_0 + E' = E,$ where $E > mc^2.$ We shall replace the mechanism emitting the $\gamma$-ray by a radiating dipole or quadrupole as explained, considering first the case of a dipole.

Let us take as the unperturbed system an electron under the influence of a central charge $Ze.$ We have then $A = 0$ and $A_0 = Ze/r.$ The perturbing potentials may be found from the Hertzian vector of the dipole which we shall take to be $\pi_\alpha + \pi_\alpha^*,$ where

$$\pi_\alpha = eB_0 \exp (-2\pi i\nu_0 t + iqr)/iqr,$$
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\( q = 2\pi\nu_0/c \) and the asterisk indicates that the conjugate complex value be taken. This yields for the perturbing potentials the following values:

\[
A_0 = -eB_0 \exp \left(-2\pi i\nu_0 t + iqr\right) \cos \left[ \frac{1}{r} - \frac{1}{iqr^2} \right] + \text{conjugate complex}
\]

\[
A_z = -eB_0 \exp \left(-2\pi i\nu_0 t + iqr\right)/r + \text{conjugate complex}
\]

\[
A_\gamma = A_y = 0
\]

where \( \theta \) is measured from the \( z \)-axis.

We may omit the conjugate complex part in the calculation of the transition probabilities, since it is only important for transitions where a γ-ray is emitted. (If \( h\nu' \) denote the increase of energy of the system, this part of the perturbation occurs with \((\nu_0 + \nu')\) in the denominator, which is therefore always very large in the case of absorption.) Treating the Hertzian oscillator as a classical system we find that the amount of energy radiated per unit time is \( 4B_0^2 e^2 c^2 q^2/3 \) ergs, or \( 8\pi B_0^2 e^2 q^2/3h \) quanta.

Suppose we have solved equation (1) for the undisturbed model atom, and let \( \psi_i \) represent the normalized wave function for the initial state of the electron \((E' < -mc^2)\), and \( \psi_f \) that of a possible final state \((E > mc^2)\). By using a method of variation of parameters we may find the number of electrons which are transferred from energies \( E' \) to \( E = E' + h\nu_0 \). With the normalization given above we find that the number transferred per unit time from a stream of electrons entering a large sphere round the origin at a rate of one per unit time is given by

\[
\left( \frac{2\pi}{\hbar} \right)^2 | \langle \psi_f | - eA_0 - eA_z \rho_1 \sigma_z \psi_i \rangle |^2.
\]

If we divide by the number of quanta which the Hertzian oscillator would emit, we obtain the probability of the absorption of one quantum, when there is one encounter per unit time. If there are \( N (E')dE' \) encounters per unit time of electrons with energies between \( E' \) and \( E' + dE' \) and quantum numbers \( k, u \) we must multiply by \( N (E')dE' \), and then we shall have the probability of a quantum being absorbed and an electron emitted with energy lying between \( E' + h\nu_0 \) and \( E' + dE' + h\nu_0 \). This we shall call the internal conversion coefficient, \( I (E', \nu) dE' \), and we have

\[
I (E', \nu) = \frac{3hN (E')}{8\pi qB_0^2 e^2 \left( \frac{2\pi}{\hbar} \right)^2} | \langle \psi_f | - eA_0 - eA_z \rho_1 \sigma_z \psi_i \rangle |^2,
\]
where we must sum over all possible final states, and the time factor has now disappeared from $A_0$ and $A_z$.

Assuming that all the states with negative energies are occupied, we may calculate $N(E')$ as follows. In each element of phase space, of volume $dx, dy, dz, dp_x, dp_y, dp_z$, there are

$$(2/\hbar^3) \ dx \ dy \ dz \ dp_x \ dp_y \ dp_z$$

possible states for the electron. On transforming to polar co-ordinates in momentum space this becomes

$$\left(2/\hbar^3\right) p^2 \ dp \ d\omega \ dx \ dy \ dz,$$

(14)

where $\omega$ is the element of solid angle. Let us now consider a nucleus at the origin, and assume that its presence does not affect the number of possible states. We wish to find the intensity of the electron stream, or, more precisely, the number of electrons with energies between $E'$ and $E' + dE'$ and with quantum numbers $k, u$, which collide with the nucleus in unit time. A uniform stream of electrons with momentum between $p$ and $p + dp$, moving in a solid angle $d\omega$ along the z-axis may be represented by

$$\psi = A \exp \left(iar \cos \theta\right),$$

(15)

where $A$ is a matrix of one row and four columns if we are using the Dirac wave functions. If this is to be the same as (14) we must have

$$AA^* = \left(2/\hbar^3\right) p^2 \ dp \ d\omega.$$

(16)

We can expand (15) as

$$\psi = A \left(2\pi ar \right)^{1/2} \sum_{k = 0}^{\infty} i^k \frac{2k + 1}{2} J_{k + 1/2} (ar) P_k (\cos \theta),$$

(17)

where $P_k (\cos \theta)$ is the ordinary Legendre function, and each term corresponds to electrons with a definite angular momentum about the origin. Also as $r \to \infty$ we have

$$A \left(2\pi ar \right)^{1/2} J_{k + 1/2} (ar) \to A \frac{2}{ar} \cos (ar - \frac{1}{2} (k - 1) \pi).$$

(18)

Now when the nucleus is present the asymptotic form of the radial wave function, representing a stream of electrons entering a large sphere at the rate of one per unit time, is

$$\left(2\pi \frac{E}{\hbar c^2} \right)^{1/2} \frac{2}{ar} \cos (ar + b \log r + \delta),$$

(19)
which is of the same form as (18), if we omit the logarithmic term which expresses the influence of the nucleus. The normalizing factor for \( P_\kappa \) is \( 4\pi/(2\kappa + 1) \) so that we may write (17) as

\[
\psi = \Sigma A \{ (2\kappa + 1) \frac{hc^2}{2} | E | a \}^{\frac{1}{2}} \psi_\kappa,
\]

where

\[
\psi_\kappa = i^\kappa \left( \frac{2\pi}{hc^2} \right)^{\frac{1}{2}} \left( \frac{2\pi}{ar} \right)^{\frac{1}{2}} J_{k+\frac{1}{2}}(ar) \left( \frac{2\kappa + 1}{4\pi} \right)^{\frac{1}{2}} P_\kappa(\cos \theta)
\]

and \( \psi_\kappa \) represents a unit stream of electrons, with angular momentum \( kh/2\pi \) about the origin, entering the large sphere. The square of the coefficient in (20) then gives the number entering per second in solid angle \( d\Omega \). This gives

\[
2 (2\kappa + 1) 4\pi \frac{hc^2}{2} p^2 \frac{dp}{2} | E | ah^3
\]

on integrating over the solid angle, and reduces to

\[
2 (2\kappa + 1) dE/h.
\]

As there are exactly \( 2 (2\kappa + 1) \) states with quantum number \( k \) (\( 2k + 2 \) of type \( 2a \)) and \( 2k \) of type \( 2b \)), we see that for a given \( k \) and \( \mu \), the number of electrons entering in unit time is just \( dE/h \). Hence

\[
N(E') = 1/h
\]

and

\[
I(E', \nu_0)_{\text{Dipole}} = \Sigma \frac{3}{8\pi qB_0^2 c^2} \left( \frac{2\pi}{h} \right)^2 | (\psi_f - eA_\theta - eA_z P_1 \sigma_z | \psi_i)|^2,
\]

the summation being taken over all initial and final states.

Where we consider the nucleus to be a radiating quadripole the corresponding field is given by†

\[
A_0 = -eC_0 r^{-1} \exp(iqr - 2\pi i\nu_0 t) \left[ 2P_a(\cos \theta) \left( 1 + \frac{3i}{qr} - \frac{3}{q^2 r^2} \right) + 1 \right] + \text{conjugate complex},
\]

\[
A_z = -3eC_0 r^{-3} \exp(iqr - 2\pi i\nu_0 t) \cos \theta [1 + i/qr] + \text{conjugate complex},
\]

\[
A_x = A_y = 0
\]

and the number of quanta emitted by such a field is

\[
24e^2C_0^2 \pi q/5h.
\]

† Taylor and Mott. loc. cit. 1.
Corresponding to expression (22) we obtain
\[
I(E', \nu_0)_{\text{quadrilope}} = \sum_{\nu', r, k} \frac{5}{24\pi qC_0^2} \left( \frac{2\pi^2}{\hbar} \right)^2 \left| (\psi_\nu - eA_0 - eA_z \sigma_z | \psi_\nu) \right|^2 
\]  
(24) 
the summation being taken over all initial and final states.

§ 4 Calculation of the Matrix Elements

Let us consider the evaluation of the matrix elements in (22) and (24), which are given by integrals of the form—
\[
\left[ \psi_{\nu}^* \left( -eA_0 - eA_z \sigma_z \right) \psi_{\nu} r^2 \sin \theta \, d\theta \, d\phi \, dr. \right. 
\]  
(25) 
We shall take the axis of the dipole or quadrilope along the z-axis, and we shall assume that the atomic system is quantized about this axis. It is to be noted that the final result will be the same if we fix the axis of quantization and then average over all directions of the dipole. This is so because we are summing the matrix elements over all values of the magnetic quantum number \( u \), in both initial and final states, and may therefore be inferred by an appeal to the principle of spectroscopic stability.

With the representation used we find
\[
\sigma_z = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}
\]
Consider first a dipole. We should expect some accurate selection rule, valid for all values of \( \nu_0 \), and we find, in fact, that only two final states are possible, namely those given by
\[
\Delta u = 0, \quad \Delta k = \pm 1, \quad \Delta j = \pm 1, 0, 
\]  
(26) 
where \( j \) is the average of the subscripts of the four harmonics occurring in the wave function (see Darwin, *loc. cit.*).

Let us take first the case when both initial and final states are of type (2a). Neglecting, for the moment, the normalizing factors we have, from the solutions (2a)
\[
\psi_k | \psi_{k'} = F_k | F_{k'} (P_{k+1}^u | P_{k+1}^{u'} + P_{k+1}^{u'} | P_{k+1}^u ) \\
+ G_k G_{k'} (k + u + 1) (k' + u' + 1) P_k^u | P_{k'}^{u'} + (-k + u) (-k' + u') P_k^{u+1} | P_{k'}^{u'+1}
\]
and
\[ \psi_k^* \psi_{k'} = iF_k^* G_{k'} \left\{ (k' + u' + 1) P_{k+1}^u * P_{k'}^u - (k' + u') P_{k+1}^{u+1} * P_{k'}^{u+1} \right\} \\
+ i G_k^* F_{k'} \left\{ - (k + u + 1) P_k^u * P_{k+1}^u + (k + u) P_{k+1}^{u+1} * P_{k'}^{u+1} \right\}. \]

Putting these in the expression (25) for the corresponding matrix element and integrating with respect to \( \theta \) and \( \phi \) we obtain the following values for the matrix element —

when \( k = k' - 1, u = u' \)
\[ \xi (E', k', A') \xi (E, k' - 1, A) \frac{\sqrt{\{k' + u' + 1\} (k' - u')}}{2k' + 1} \]
\[ \int_0^\infty [(F_k F_{k'} + G_k G_{k'}) (r - 1/|q|) + 2iF_k G_{k'} r] \times e^{iqr} dr, \tag{27} \]

when \( k = k' + 1, u = u' \)
\[ \xi (E', k', A') \xi (E, k' + 1, A) \frac{\sqrt{\{k' + u' + 2\} (k' - u' + 1)}}{2k' + 3} \]
\[ \int_0^\infty [(F_k F_{k'} + G_k G_{k'}) (r - 1/|q|) - 2iG_k F_{k'} r] \times e^{iqr} dr. \tag{28} \]

and zero for all other values of \( k \) and \( u \). (A and B are used here to indicate states of type (2A) and (2B) respectively, and should not be confused with the A and B defined in equations (4)). This is in accordance with the selection rule given by (26). We shall write the expression (27) as
\[ \sqrt{\{k' + u' + 1\} (k' - u')} M (A', k'; A, k' - 1) \]
with an obvious notation, and we shall use corresponding expressions when \( k = k' + 1 \), or the B states are involved. In the formula (22) for \( I (E', \nu) \) these expressions occur squared, and have to be summed over all values of \( u \). Performing this summation we find —
\[
I (E', \nu)_{\text{Dipole}} = \frac{3\pi e^2}{2q\hbar^2} \sum_k \left[ \frac{2k' (k' + 1)}{3 (2k' + 1)} |M (A', k'; A, k' - 1)|^2 \\
+ \frac{2 (k' + 1) (k' + 2)}{3 (2k' + 3)} |M (A', k'; A, k' + 1)|^2 \\
+ \frac{2 (k' - 1) k'}{3 (2k' - 1)} |M (B', k'; B, k' - 1)|^2 \\
+ \frac{2k' (k' + 1)}{3 (2k' + 1)} |M (B', k'; B, k' + 1)|^2 \\
+ \frac{2k'}{3 (2k' - 1) (2k' + 1)} |M (B', k'; A, k' - 1)|^2 \\
+ \frac{2 (k' + 1)}{3 (2k' + 1) (2k' + 3)} |M (A', k'; B, k' + 1)|^2 \right]. \tag{29} \]
The quantities $M$ are given by equations analogous to (28). We have

$$M(A', k'; A, k' - 1) = N \int_0^\infty \left\{ (F_k F_{k'} + G_k G_{k'}) X_1 + 2i F_k G_{k'} Y_1 \right\} dr$$

$$M(A', k'; A, k' + 1) = N \int_0^\infty \left\{ (F_k F_{k'} + G_k G_{k'}) X_1 - 2i G_k F_{k'} Y_1 \right\} dr$$

$$M(B', k'; B, k' - 1) = N \int_0^\infty \left\{ (F_{-k-1} F_{-k'-1} + G_{-k-1} G_{-k'-1}) X_1 \right\} dr$$

$$M(B', k'; B, k' + 1) = N \int_0^\infty \left\{ (F_{-k-1} F_{-k'-1} + G_{-k-1} G_{-k'-1}) X_1 \right\} dr$$

$$+ 2i G_{-k-1} F_{-k'-1} Y_1 \right\} dr$$

$$M(B', k'; A, k' - 1) = N \int_0^\infty \left\{ (F_{-k-1} F_{-k'-1} + G_{-k-1} G_{-k'-1}) X_1 \right\} dr$$

$$+ 2i G_{-k-1} F_{-k'-1} Y_1 \right\} dr$$

$$M(B', k'; A, k' + 1) = N \int_0^\infty \left\{ (F_{-k-1} F_{-k'-1} + G_{-k-1} G_{-k'-1}) X_1 \right\} dr$$

$$+ 2i G_{-k-1} F_{-k'-1} Y_1 \right\} dr,$$

where

$$X_1 = (r - 1/jq) e^{jqr}$$

$$Y_1 = r e^{jqr}$$

and $N$ denotes the product of the two appropriate normalizing factors.

For the quadripole we have the selection rules

$$\Delta u = 0; \quad \Delta k = 0, \pm 2; \quad \Delta j = 0, \pm 1, \pm 2,$$

and the corresponding expression for the internal conversion coefficient is

$$I(E', \nu_0) \quad \text{Quadripole} = \frac{\pi e^2}{3q\hbar^2} \sum_{k'} \left[ \frac{4k' (k' + 1) (k' + 2)}{(2k' + 1) (2k' + 3)} \right] |M(A', k'; A, k')|^2$$

$$+ 6 \frac{(k' - 1) k' (k' + 1)}{(2k' - 1) (2k' + 1)} |M(A', k'; A, k' - 2)|^2$$

$$+ 6 \frac{(k' + 1) (k' + 2) (k' + 3)}{(2k' + 3) (2k' + 5)} |M(A', k'; A, k' + 2)|^2$$

$$+ \text{three expressions similar to the above with } A, A' \text{ replaced by } B, B',$$

and $k'$ by $k' - 1$ in the numerical factors,

$$+ \frac{3k' (k' + 1)}{(2k' - 1) (2k' + 1) (2k' + 3)} |M(A', k'; B, k')|^2$$

$$+ \frac{3 (k' + 1) (k' + 2)}{(2k' + 1) (2k' + 3) (2k' + 5)} |M(A', k'; B, k' + 2)|^2$$

$$+ \frac{3k' (k' + 1)}{(2k' - 1) (2k' + 1) (2k' + 3)} |M(B', k'; A, k')|^2$$

$$+ \frac{3 (k' - 1) k'}{(2k' - 3) (2k' - 1) (2k' + 1)} |M(B', k'; A, k' - 2)|^2.$$
where the quantities $M$ are given by

\[
M (A', k'; A, k') = N \int_0^\infty \{ X_2 (F_k F_{k'} + G_k G_{k'}) \\
+ i Y_2 (F_k G_{k'} - G_k F_{k'}) \} \, dr,
\]

\[
M (A', k'; A, k' + 2) = N \int_0^\infty \{ X_2 (F_k F_{k'} + G_k G_{k'}) - 2i Y_2 G_k F_{k'} \} \, dr,
\]

\[
M (A', k'; A, k' - 2) = N \int_0^\infty \{ X_2 (F_k F_{k'} + G_k G_{k'}) + 2i Y_2 F_k G_{k'} \} \, dr,
\]

\[
M (B', k'; B, k') = N \int_0^\infty \{ X_2 (F_{k-1 -1} F_{k'-1} + G_{k-1} G_{k'-1}) \\
+ i Y_2 (F_{k-1} G_{k'-1} - G_{k-1} F_{k'-1}) \} \, dr,
\]

\[
M (B', k'; B, k' + 2) = N \int_0^\infty \{ X_2 (F_{k-1 -1} F_{k'-1} + G_{k-1} G_{k'-1}) \\
+ 2i Y_2 F_{k-1} G_{k'-1} \} \, dt,
\]

\[
M (B', k'; B, k' - 2) = N \int_0^\infty \{ X_2 (F_{k-1 -1} F_{k'-1} + G_{k-1} G_{k'-1}) \\
- 2i Y_2 G_{k-1} F_{k'-1} \} \, dr,
\]

\[
M (A, k' ; B, k') = N \int_0^\infty \{ 2X_2 (F_{k-1} F_{k'} + G_{k-1} G_{k'}) \\
+ i Y_2 (2k' + 3) F_{k-1} G_{k'} \\
+ i Y_2 (2k' - 1) G_{k-1} F_{k'} \} \, dr,
\]

\[
M (A, k' ; B, k' + 2) = N \int_0^\infty \{ 2X_2 (F_{k-1} F_{k'} + G_{k-1} G_{k'}) \\
+ i Y_2 (2k' + 5) F_{k-1} G_{k'} \\
+ i Y_2 (2k' + 1) G_{k-1} F_{k'} \} \, dr,
\]

\[
M (B', k' ; A, k') = N \int_0^\infty \{ 2X_2 (F_{k} F_{k'-1} + G_{k} G_{k'-1}) \\
- i Y_2 (2k' - 1) F_{k} G_{k'-1} \\
- i Y_2 (2k' + 3) G_{k} F_{k'-1} \} \, dr,
\]

\[
M (B, k' ; A, k' - 2) = N \int_0^\infty \{ 2X_2 (F_{k} F_{k'-1} + G_{k} G_{k'-1}) \\
- i Y_2 (2k' - 3) F_{k} G_{k'-1} \\
- i Y_2 (2k' + 1) G_{k} F_{k'-1} \} \, dr,
\]

where

\[
X_2 = \left\{ r + \frac{3i}{q} - \frac{3}{q^2 r} \right\} e^{iqr},
\]

\[
Y_2 = \left\{ r + \frac{i}{q} \right\} e^{iqr},
\]

and $N$ is again the product of the two appropriate normalizing factors.
In calculating the integrals $M(A', k'; A, k)$ and $M(B', k'; B, k)$ it is more convenient to use the symmetrical form for the potentials as given by Taylor and Mott. This eliminates the last term, unity, in the expression for $A_0$. This term makes no contribution to the other matrix elements, and so we use the form given in (23) for these.

It will be seen from (30) and (33) that the integrals required are of the type

$$\int_0^\infty W e^{iqr} \, dr \quad \text{and} \quad \int_0^\infty W r e^{iqr} \, dr$$

for dipole transitions, with the additional $\int_0^\infty W r^{-1} e^{iqr} \, dr$ for the quadrupole, where $W$ is the product of two radial wave functions, one referring to positive and the other to negative energy states, for example

$$W = F_k G_{k'}$$

with

- $k = k' \pm 1$ for dipole transitions,
- $k = k', k' \pm 2$, for quadrupole transitions.

We shall calculate the corresponding integrals involving $\mathbf{G}_k$ and $\mathbf{G}_k^*$, etc., and then derive the integrals we require by using the relations (5) which become

$$\mathbf{G}_k = A F_k + i |B| G_k$$

(35A)

$$\mathbf{G}_{k'} = i |A'| F_{k'} + B' G_{k'}$$

(35B)

If we calculate $A$ and $B$ for the positive energy corresponding to $E'$, i.e., for $-E' = |E'|$, then

$$A' = B$$

$$B' = A$$

and the second equation becomes

$$\mathbf{G}_{k'} = A G_{k'} + i |B| F_{k'}$$

(36B)

which is the same as (35A) with $F$ and $G$ interchanged.

Let us consider a typical integral

$$\int_0^\infty \mathbf{G}_k \mathbf{G}_{k'} e^{iqr} r^\lambda \, dr$$

(37)

† Loc. cit. I, equations (4.16), (4.20) and (4.21).
where \( \lambda = 0, 1 \) for dipole, and \( \lambda = -1, 0, 1 \) for quadripole. We shall put \( a = \sigma s, a' = \sigma' s \) and then we have

\[
\mathcal{G}_k = \Omega (k) (\sigma s q)^s \int_{-1}^{+1} (1 - u)^s + ib + 1 (1 + u)^s - ib e^{i\sigma s q u r} du,
\]

\[
\mathcal{G}_{k'} = \Omega (k') (\sigma' s q')^{s'} \int_{-1}^{+1} (1 - u')^{s'} - ib' + 1 (1 + u')^{s'} + ib' e^{i\sigma' s' q u'} du',
\]

where

\[
\Omega (k) = \left[ (k - s) - i (b + c) \right],
\]

\[
\Omega (k') = i \left[ (k' - s') - i (b' + c') \right].
\]

The integral is

\[
\Omega (k) \Omega (k') \int_0^\infty (\sigma s q)^s (\sigma' s q')^{s'} r^s \int_{-1}^{+1} e^{is q u r} (1 - u)^s + ib + 1 (1 + u)^s - ib du
\]

\[
\times \int_{-1}^{+1} e^{is' q u'} (1 - u')^{s'} - ib' + 1 (1 + u')^{s'} + ib' du' dr.
\]

Let us change the variables of integration by writing \( r \) for \( q u r \) and changing the sign of \( u \) and \( u' \). The integral now becomes

\[
\{ \Omega (k) \Omega (k') \sigma^s \sigma'^{s'} q^{-(\lambda + 1)} \} \int_0^\infty \int_{-1}^{+1} r^{s + s' + \lambda} e^{-r [\delta - i (1 - \sigma u - \sigma' u')]}
\]

\[
\times (1 + u)^s + ib + 1 (1 - u)^s - ib (1 + u')^{s'} - ib' + 1 (1 - u')^{s'} + ib' dr du du',
\]

where we have put in a positive quantity \( \delta \) to facilitate the change of order of integration, \( \delta \) afterwards tending to zero. We may now integrate with respect to \( r \) obtaining

\[
Q \int_{-1}^{+1} \int_{-1}^{+1} (1 + u)^s + ib + 1 (1 - u)^s - ib (1 + u')^{s'} - ib' + 1 (1 - u')^{s'} + ib' du du',
\]

where

\[
Q = \Omega (k) \Omega (k') \sigma^s \sigma'^{s'} q^{-(\lambda + 1)} \Gamma (s + s' + \lambda + 1).
\]

This becomes, on putting \( 1 + u = 2 \tau, 1 + u' = 2 \tau' \)

\[
M \int_{-1}^{+1} \int_{-1}^{+1} \frac{(1 - \tau)^s + ib \tau^{s - ib' + 1} (1 - \tau')^{s'} - ib' + 1 (1 - \tau')^{s'} + ib' d\tau d\tau'}{2 \sigma \tau} \left[ \frac{1 + \sigma + \sigma' + i \delta}{1 + \sigma + \sigma' + i \delta} \right]^{s + s' + \lambda + 1},
\]

where

\[
M = \frac{Q \sb{s + s' + \lambda + 1}}{(1 + \sigma + \sigma')^{s + s' + \lambda + 1}}
\]

on letting \( \delta \to 0 \) in \( M \).
The double integral may be expressed as a double hypergeometric series and we have the integral equal to\(^\dagger\)

\[
M \frac{\Gamma(s - \i b + 2) \Gamma(s + \i b + 1)}{\Gamma(2s + 3)} \cdot \frac{\Gamma(s' - \i b' + 2)}{\Gamma(2s' + 3)} \quad _2F_2 \tag{41}
\]

where

\[
_F_2 = _F_2(s + s' + \lambda + 1; s - \i b + 2, s' - \i b' + 2; 2s + 3, 2s' + 3; x, y)
\]

with

\[
x = \frac{2\sigma}{1 + \sigma + \sigma'}, \quad y = \frac{2\sigma'}{1 + \sigma + \sigma'}
\tag{42}
\]
on letting \(\delta \to 0\), since the series is convergent. The condition for convergence is

\[
| x | + | y | < 1,
\]

which gives

\[
\sigma + \sigma' < 1
\tag{43}
\]
or

\[
\frac{c}{\hbar y} (| p | + | p' |) < 1,
\]

which is always true.

The other integrals can be obtained in a similar way. We have only to note that the change of \(\mathbf{6}_k\) to \(\mathbf{6}_k^*\) involves changing \(\Omega(k)\) to \(\Omega^*(k)\) and the change of \(ib\) to \((ib + 1)\).

When \(G_{-k-1}\) is involved we have

\[
\begin{align*}
\Omega(-k - 1) &= [(b - c) + i(k - t - 1)] \\
\Omega(-k' - 1) &= i[(b' - c') + i(k' - t' - 1)]
\end{align*}
\tag{44}
\]

and in the expression (39) we have to substitute \(\Omega(-k - 1)\) etc., and \(t, t'\) instead of \(\Omega(k)\) and \(s, s'\).

The four types of double hypergeometric function required for each value of \(\lambda\) are of the form

\[
\begin{align*}
A &= _F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y), \\
B &= _F_2(\alpha; \beta + 1, \beta'; \gamma, \gamma'; x, y), \\
C &= _F_2(\alpha; \beta, \beta' + 1; \gamma, \gamma'; x, y), \\
D &= _F_2(\alpha; \beta + 1, \beta' + 1; \gamma, \gamma'; x, y).
\end{align*}
\]

A considerable simplification is introduced if we use the recurrence formulæ connecting these functions with the corresponding ones with \(\alpha\)

\(^\dagger\) Appell, ‘‘Fonctions Hypergéométriques,’’ p. 28.
replaced by \( \alpha + 1 \), which we shall call \( A_1 \), etc. This enables us to obtain the integrals with \( \lambda = 0 \) from those with \( \lambda = -1 \) and those with \( \lambda = 1 \) from \( \lambda = 0 \). With the above notation these formulae are

\[
\begin{align*}
\alpha A_1 &= (\alpha - \beta - \beta') A + \beta B + \beta' C, \\
(1 - x) \alpha B_1 &= (\gamma - 1 - \beta) A + (\beta + 1 - \gamma + \alpha - \beta') B + \beta' D, \\
(1 - y) \alpha C_1 &= (\gamma' - 1 - \beta') A + (\beta' + 1 - \gamma' + \alpha - \beta) C + \beta D \\
(1 - x - y) \alpha D_1 &= (\gamma' - 1 - \beta') B + (\gamma - 1 - \beta) C + (\alpha + \beta + \beta' - \gamma - \gamma' + 2) D
\end{align*}
\]

They may be derived from the recurrence formulae for the hypergeometric series of one variable.

In this way we may calculate all the integrals of type (37), and hence derive the required integrals by using equations (35). These have now to be put in the matrix elements (30) and (33) and then the equations (29) and (32) give the values of the internal conversion coefficient in the two cases of dipole and quadripole transitions in the nucleus. Theoretically we have to sum over all values of \( k' \), the azimuthal quantum number of the electron in its initial state of negative energy. It happens, however, that only the first two or three values of \( k' \) are needed, indicating that only those electrons with small angular momentum about the nucleus are affected by the \( \gamma \)-ray.

§ 5 Results

We have calculated the internal conversion coefficient for an element of atomic number 84, in both dipole and quadripole cases, for the ranges of energy shown in fig. 2. The method used gives the number of positrons emitted with a definite energy \( E' \), the electron having an energy \( h\nu_0 - E' \), and to find the total number of pairs created we must integrate over all values of \( E' \). This has been done in the case \( \varepsilon = h\nu_0/mc^2 = 3 \), for example, by finding the number of electrons emitted with energies \( mc^2 \), \( 1.05 mc^2 \), \( 1.25 mc^2 \), \( 1.5 mc^2 \) and \( 1.75 mc^2 \), and then drawing a curve through these points. The area under the curve then gives the total number of pairs created. The two curves, dipole and quadripole, for this value of \( \varepsilon \) are shown in fig. 1, and are labelled D and Q respectively. We have also calculated, by the same method, the curves for \( Z = 0 \), which is covered by the calculations of Oppenheimer and Nedelski (loc.
and we give our results as dotted curves in the same figure. The curves for other values of $\varepsilon$ are similar.

The ordinate $N$ gives the number of positrons emitted per unit energy range, and the abscissa gives the total energy of the positron, the unit of energy being $mc^2$.

The end points of the curves have been calculated by a limiting process, and it will be seen that the number of positrons emitted per unit energy range increases steadily with the energy of the positrons. This increase is less marked for larger values of $\varepsilon$, and the curves flatten out. As $\varepsilon$ increases we should therefore expect a maximum to develop, and move towards the centre of the range. We see, in fact, from Table I, that the quantity

$$\Delta = \frac{E_+ - E_-}{E_+ + E_-} - 2mc^2$$

decreases as $\varepsilon$ increases. For energies comparable to those of cosmic rays it should be small, when the curves will have a maximum near the centre of the range. Quanta of such high energies are not emitted from nuclei, but in the pairs created by a beam of cosmic rays the quantity $\Delta$ is small, as has been shown by Anderson.†‡

‡ [Note added in proof—We have since extended the method to give the number of pairs created by a beam of $\gamma$-rays. In this case the asymmetry of the curves is not
There is thus a marked difference in the average energies of the electrons and positrons emitted, the asymmetry arising when the influence of the nucleus is included. This is due to the Coulomb forces acting on the two particles, which are created in a region of high potential. We give in Table I the difference of the average values of the energies of the positron and electron in units of \( mc^2 \).

<table>
<thead>
<tr>
<th>( e )</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_+ - E_- )</td>
<td>0.40</td>
<td>0.73</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Table I

Roughly speaking we may say that the pair is created in the region given by

\[
E_+ - E_- \approx 2Ze^2/r
\]

or

\[
r \approx \frac{Z}{137 \pi} \frac{mc^2}{E_+ - E_-} \lambda_c,
\]

where \( \lambda_c \) is the Compton wave-length. We may write this

\[
r \approx \left( \frac{Z}{137} \right)^2 \frac{2mc^2}{E_+ - E_-} a_0,
\]

where \( a_0 \) = radius of K-shell on the Bohr theory. We should, therefore, expect some shielding by the K-electrons, but this will not affect the results appreciably.

The results for the total number of pairs created—the Internal Conversion Coefficient—are given in fig. 2, and the curves for \( Z = 0 \) are put in for comparison. We have labelled the curves as in fig. 1.

It will be seen that the magnitude of the effect does not depend much on the nuclear charge \( Z \). The reason for this is that the density of the electrons in negative energy states is not affected much by the presence of the nucleus. In considering the internal conversion in the K-shell, the electron density depends very much on the atomic number, \( Z \), of the nucleus, and so the effect varies strongly with \( Z \).

It happens that most of the pairs are produced in transitions where \( \Delta j = 0 \) in the dipole case, and \( \Delta j = \pm 1 \) in the quadripole case, particularly for low energies. We have calculated all the transitions for the cases \( \varepsilon = 3 \) and 5, but for the other cases (except for dipole \( Z = 84 \)) so pronounced and even for \( \varepsilon = 3 \) they have a maximum for positron energies of about 1.7 \( mc^2 \).]
we have only calculated the main transitions, and have added an appropriate correction for the remaining ones. A dotted curve has therefore been drawn in the region $\varepsilon > 5$.

Our results for $Z = 0$ should be identical with those of Oppenheimer and Nedelski, and we have verified that this is so for both dipole and quadripole radiation.

Experimental confirmation of internal pair formation has recently been obtained by Alichanow and Kosodaew (loc cit.). These authors have examined the number of positrons emitted by a radioactive source, RaC. After applying certain corrections to account for pairs created outside the source by $\gamma$-rays, they obtain the curve given in fig. 3, where the number of positrons emitted is plotted against the kinetic energy of the positrons, expressed in kilo-volts.

According to Alichanow and Kosodaew the main effect must be sought in the direct emission of positrons from the nucleus, or else in the internal conversion of $\beta$-rays from the nucleus, part of the energy of the $\beta$-ray going to create a pair. Both these effects may be expected to give a continuous curve. The existence of the small maximum at 700 kV, and the (doubtful) one at 1000 kV, are due to the internal conversion of $\gamma$-rays of definite energies, as explained in the present paper.
Alichanow and Kosodaew assume that the excess energy of the \( \gamma \)-rays, viz., \( h\nu_0 - 2mc^2 \), is distributed so that the average energies of the positron and electron are equal, and that the maximum number of positrons (per unit energy range) are emitted with energies equal to \( \frac{1}{2} (h\nu_0 - 2mc^2) \). From the dotted curves in fig. 1 we see that this is true when \( Z = 0 \), and therefore should hold approximately for light elements. For RaC, however, the full curve in fig. 1 shows that the maximum number of positives are emitted with energies equal to \( h\nu_0 - 2mc^2 \).

We therefore conclude that the maximum occurring at 700 kV is due to the internal conversion of the \( \gamma \)-ray of energy 1760 kV in the spectrum of RaC, and not due to the \( \gamma \)-ray of energy 2200 kV as Alichanow and Kosodaew suppose. This latter \( \gamma \)-ray accounts for the small maximum in the region of 1000 kV. Let us consider the value of the internal conversion coefficient obtained from the experimental data of Alichanow and Kosodaew.

The number of positives per disintegration is 0.2\% and the number of \( \gamma \)-rays of energy 1760 kV is 26\%. Further, the area of the hump is about 1/25 times that of the whole curve. We therefore have

\[
I = \frac{1}{25} \cdot \frac{2}{1000} \cdot \frac{100}{26} \approx 3 \times 10^{-4}.
\]
The theoretical results for this value are

Dipole \[ I = 3 \times 10^{-4} \]
Quadripole \[ I = 1.5 \times 10^{-4} \]

The evidence from the values of the Internal Conversion Coefficients in the K shell (Taylor and Mott, loc. cit. I) indicate that this line is probably due to quadripole radiation, so that our results are small. From the nature of the experiment, however, we can only expect a very rough value for the Internal Conversion Coefficient, and the agreement would seem to be well within the limit of experimental error.

In order to carry out the lengthy calculations involved one of us (J. C. J.) has worked as assistant to Professor R. H. Fowler, F.R.S., who obtained a grant from the Royal Society (Trevelyan Fund) for this purpose. Our thanks are due to him and to the Society for providing these facilities.

§ 6 Summary

A radioactive nucleus is considered as a radiating dipole or quadripole, and expressions are obtained for the number of pairs created by the absorption of the \( \gamma \)-rays emitted by the nucleus. The effect is not sensitive to changes in the atomic number of the nucleus, and increases with the energy of the \( \gamma \)-ray. For energies in the region of \( 5mc^2 \) it is of the order \( 10^{-4} \) to \( 10^{-3} \) pairs per \( \gamma \)-ray quantum.

In this range it is found that the number of positrons emitted per unit energy range increases steadily with the energy of the positron, which therefore takes up most of the available energy. For very light elements, typified by the case \( Z = 0 \), we have symmetrical distribution of the energy between positron and electron.

The results have been compared with those obtained experimentally by Alichanow and Kosodaew (loc. cit.) who have examined the number of positrons emitted by a radioactive source, and the agreement is satisfactory when we consider the difficulty of obtaining a good experimental value.