Contributions to Born’s New Theory of the Electromagnetic Field

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1—THE INTRODUCTION OF COMPLEX VARIABLES

Born’s theory† starts from describing the field by two vectors (or a “six-vector”), B, E, the magnetic induction and electric field-strength respectively. A second pair of vectors (or a second six-vector) H, D, is introduced, merely an abbreviation, if you please, for the partial derivatives of the Lagrange function with respect to the components of B and E respectively (though with the negative sign for E). H is called magnetic field and D dielectric displacement. It was pointed out by Born‡ that it is possible to choose the independent vectors in different ways. Four different and, to a certain extent, equivalent and symmetrical representations of the theory can be given by combining each of the two “magnetic” vectors with each of the two “electric” vectors to form the set of six independent variables. Every one of these four representations can be derived from a variation principle, using, of course, entirely different Lagrange functions—physically different, that is, though their analytic expressions by the respective variables are either identical or very similar to each other.

In studying Born’s theory I came across a further representation, which is so entirely different from all the aforementioned, and presents such curious analytical aspects, that I desired to have it communicated. The idea is to use two complex combinations of B, E, H, D as independent variables, but in such a way that their “conjugates,” i.e., the partial derivatives of $\mathcal{L}$, equal their complex conjugates. Choosing the following pair of independent variables

$$\mathfrak{f} = B - iD \quad \mathfrak{g} = E + iH \quad (A)$$

(which form a true six-vector) the appropriate Lagrangian works out

$$\mathcal{L} = \frac{\mathfrak{f}^2 - \mathfrak{g}^2}{(\mathfrak{f} \mathfrak{g})} \quad (1)$$


and one has
\[
\begin{align*}
\mathfrak{F}^* &= \frac{\partial \mathcal{L}}{\partial \mathfrak{f}} = -\frac{2\mathfrak{f}}{\mathfrak{f}^2} - \frac{\mathfrak{f}^2 - \mathfrak{f}^2}{\mathfrak{f}^2} \\
\mathfrak{g}^* &= \frac{\partial \mathcal{L}}{\partial \mathfrak{g}} = \frac{2\mathfrak{g}}{\mathfrak{g}^2} - \frac{\mathfrak{g}^2 - \mathfrak{g}^2}{\mathfrak{g}^2} \mathfrak{g}
\end{align*}
\]
(2)

The * indicates the complex conjugate, \(\mathfrak{f}^2\) and \((\mathfrak{g}^2)\) the scalar product of \(\mathfrak{f}\) with \(\mathfrak{f}\) or \(\mathfrak{g}\) respectively. The derivative with respect to a vector is short for: a vector, of which the components are the three derivatives with respect to the components of that vector. The units are “natural” units, Born’s constant \(b\) being equalled to 1 (in other units \(\mathcal{L}\) would take the factor \(b^2\)).

What is so very surprising is that the square root, which is so characteristic for Born’s theory, has disappeared. The Lagrangian is not only rational, but homogeneous of the zeroth degree.

The treatment of the field by the Lagrangian (1) is entirely equivalent to Born’s theory, as I shall prove presently. Therefore it cannot yield any new insight which could not, virtually, be derived from Born’s original treatment as well. Moreover, for actual calculation the use of imaginary vectors will hardly prove useful. Yet for certain theoretical considerations of a general kind I am inclined to consider the present treatment as the standard form on account of its extreme simplicity, the Lagrangian being simply the ratio of the two invariants, whereas in Maxwell’s theory it was equal to one of them.

2—THE FIELD-EQUATIONS AND THE CONDITION OF CONJUGATENESS

The proof of equivalence could be given by a somewhat lengthy analytical transformation, but it will turn out automatically on a closer description of the new treatment, to which we now proceed. In deriving the field-equations from the Lagrangian (1) we must, of course, pay no attention to the connection (A), but actually regard \(\mathfrak{f}, \mathfrak{g}\) as the fundamental variables. Moreover, we must assume (just as in Maxwell’s and in Born’s theories) that the six-vector \(\mathfrak{f}, \mathfrak{g}\) is the four-dimensional curl of a potential four-vector, and that only the four components of the latter are to be varied independently. In other words, we assume that the equations
\[
\begin{align*}
curl \mathfrak{g} + \frac{\partial \mathfrak{f}}{\partial t} &= 0, \quad \text{div } \mathfrak{f} = 0
\end{align*}
\]
are satisfied. We then obtain by variation in the usual way
\[
\begin{align*}
curl \frac{\partial \mathcal{L}}{\partial \mathfrak{f}} + \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \mathfrak{g}} \right) &= 0, \quad \text{div } \left( \frac{\partial \mathcal{L}}{\partial \mathfrak{g}} \right) = 0.
\end{align*}
\]
The partial derivatives of $L$ are given in more detail by the right-hand sides of (2). It should be observed that these expressions, thanks to the first negative sign in the first of them, possess the covariance of $\mathfrak{g}, \mathfrak{\varphi}$ themselves, so that it has an invariant meaning to postulate that they should be equal to the complex conjugates $\mathfrak{g}^*, \mathfrak{\varphi}^*$. But the equality may not be taken for granted, it constitutes an important initial condition as we shall see directly.

Since the time derivatives of $\mathfrak{g}, \mathfrak{\varphi}$ can be calculated uniquely from the "curl"-equations, contained in (3), (4) (apart, maybe, from singular events like vanishing denominators or determinants), the equations determine the future of $\mathfrak{g}, \mathfrak{\varphi}$ uniquely from arbitrary initial values, which only have to comply with the two "div"-equations. Since the latter do not restrict the choice in one point of space, it is clear that we have to postulate the complex conjugateness in question in one moment of time, if we wish it to hold at all. We shall inquire later (§ 4) into what restriction this imposes on the choice of $\mathfrak{g}, \mathfrak{\varphi}$. The first solicitute is to satisfy ourselves that this initial condition will be preserved by the action of the equations (3), (4), in the same way as they take care of conserving the "div"-equations, once they have been imposed in one moment of time.

In order to see that they do, it is not sufficient to state their apparent symmetry at a glance, (4) being the same set of equations as (3), but with respect to $\partial L/\partial \mathfrak{\varphi}$ and $\partial L/\partial \mathfrak{g}$ instead of $\mathfrak{g}, \mathfrak{\varphi}$. This would be sufficient if (3) alone or (4) alone sufficed to determine the future (which they do not), for then one could argue that the quantities and their time-derivatives, being conjugate at the beginning, must always remain so. Now the latter argument actually applies, but not in the incorrect way just referred to. Replace the derivatives in (4) by their expressions in $\mathfrak{g}, \mathfrak{\varphi}$, taken from (2), and you get one set of equations, (3), (4), to determine the future, or one form of the complete set. You get a second form if you inversely replace $\mathfrak{g}, \mathfrak{\varphi}$ in equations (3) by the expressions arrived at by resolving equations (2) algebraically with respect to $\mathfrak{g}, \mathfrak{\varphi}$ (i.e., determining the latter as algebraic functions of $\partial L/\partial \mathfrak{\varphi}, \partial L/\partial \mathfrak{g}$). This would be a difficult task, if the equations (2) were not of the peculiar kind which are their own resolution. That is to say, $\mathfrak{g}, \mathfrak{\varphi}$ are precisely the same functions of $\partial L/\partial \mathfrak{\varphi}, \partial L/\partial \mathfrak{g}$ as the latter are of them. In other words, if you regard (2) as a transformation from a set of six variables to another set of six variables, the "square" of this transformation is the identity. This can easily be proved by straightforward calculation. (Of course, it is not a "chance," but is mathematically connected with the homogeneity of zeroth degree of $L$ and with the peculiar way of "crossing"
the variables.) This has the consequence that the two forms of the complete set (3), (4), of which we spoke before, are identical, the second being obtained from the first by replacing the variables $\xi$, $\psi$ by those functions of them for which the "conservation of conjugateness" is to be proved. Therefore the argument of which we refused an incorrect application at the beginning, now applies correctly and the conservation of conjugateness is proved.

If a solution complies with the condition of conjugateness, its real and imaginary parts, in the co-ordination given by (A), satisfy Born's, or rather Maxwell's equations, as a trivial consequence of our field-equations (3), (4). Whether other solutions have any physical meaning at all, we can leave an open question.

In this paper we shall henceforth deal only with solutions which comply with the condition. Then we are allowed to make use of the extreme left-hand side of equations (2). A further consequence is

\[(\xi^*\psi) + (\psi^*\xi) = 0,\]

owing to \(L\) being homogeneous of degree zero. Moreover, \(L\) becomes purely imaginary. For you can easily calculate from (1), (2) that \(L\) is also equal to

\[L = -\frac{\xi^*\psi^* - \psi^*\xi^*}{(\xi^*\psi^*)},\]

that is to say, it is oppositely equal to its complex conjugate.

We still ignore what restraint it imposes on a set of six numbers, $\xi$, $\psi$, that they should yield their complex conjugates when inserted in the right-hand sides of equations (2). And we equally ignore whether the physical content of the present scheme actually coincides with that of Born's theory. Let us not be preoccupied by these two questions, which, in developing the present scheme, will be settled to the effect that our "condition of conjugateness" is an identical transcription of Born's relations between the real field-vectors.

3—THE TENSOR OF STRESS, MOMENTUM, AND ENERGY

The next thing to do is, of course, to calculate Maxwell's tensor of stress, energy, and momentum. The general method consists in writing down the fact that \(L\) does not depend explicitly on the four co-ordinates and transforming these four statements into "divergence-form" with the help of the field-equations. Thus you arrive at the conservation laws, from which the components of the tensor can be read off—up to
a constant factor common to all of them. We shall not repeat the well-known procedure, but simply give the results, disposing of the multiplier in such a way as to make the agreement with the Born theory complete later on. The result for the 10 components $T_{kl}$ can be expressed in the following way:

$$T_{kl} = \frac{it_{kl}}{(\tilde{\mathcal{E}} \Theta)} + i \frac{L}{2} \delta_{kl}; \ldots \ldots (k, l = 1, 2, 3, 4)$$ (7)

with

$$t_{kl} = \tilde{\mathcal{E}}_{k} \delta_{l} + \theta_{k} \theta_{l} - \frac{1}{2} \delta_{kl} (\tilde{\mathcal{E}}^2 + \Theta^2); \quad (k, l = 1, 2, 3)$$

$$t_{14} = \theta_{2} \delta_{3} - \theta_{3} \delta_{2}, \text{ etc.}$$ (8)

$$t_{44} = \frac{1}{2} (\tilde{\mathcal{E}}^2 + \Theta^2)$$

We observe that the $t_{kl}$ are identical in form with the components of Maxwell's vacuum tensor, $\mathcal{E}$, $\Theta$ being substituted for $H$, $E$. $T_{kl}$ differs in form only by

1. The denominator $(\tilde{\mathcal{E}} \Theta)$, which is a common feature of all our formulas (and a very significant one as will be shown later).

2. The additional term $i/2 L \delta_{kl}$, a multiple of the unity-tensor, making the sum of the diagonal terms of $T_{kl}$ differ from zero and equal to $2iL$.

That all components are real, is proved by the consideration that numerically the same $T_{kl}$ must follow from expression (6) for $L$, but expressed by the variables $\tilde{\mathcal{E}}^*, \Theta^*$. The negative sign in (6) is conserved throughout the process, and we arrive at the same expressions except for the asterisks and a negative sign. Owing to the explicit $i$, which multiplies $t_{kl}$, $L$ in (7), this means that all the $T_{kl}$ are numerically equal to their complex conjugates, therefore real.

4—The Standard Frame. Physical Meaning of Conjugateness

By investigating the transformations of the real tensor $T_{kl}$, it is easy to find a frame of reference, in which the physical meaning of our "condition of conjugateness" is readily disclosed. What distinguishes a Maxwell tensor from the general symmetrical tensor is only that its roots or eigenvalues have the form $\pm \rho$, each double.

The first part of $T_{kl}$, viz.,

$$\frac{it_{kl}}{(\tilde{\mathcal{E}} \Theta)}$$ (9)

is precisely of this type. $\rho$ works out† to

$$\rho = \pm \sqrt{\left(\frac{i}{2} L\right)^2 - 1},$$ (10)

† By considering that in Maxwell's case $\rho$ is known to be

$$\pm \sqrt{\frac{1}{4}(H^2 - E^2)^2 + (HE)^2}.$$
from which, by the way, we conclude that

$$\left(\frac{i}{2} \mathcal{L}\right)^2 \geq 1,$$

since the roots of a real symmetrical tensor must be real. We further infer that a real Lorentz-transformation must exist, which transforms (9) (and thereby also $T_{kl}$) to the diagonal. (There is a well-known exception which we leave aside for the moment, namely, $p = 0$. In this case the transformed tensor would have to vanish, which it actually does, but only in the limit of a “transforming” velocity approaching that of light.) The real Lorentz-transformation in question can, of course, be imposed upon $T_{kl}$ by imposing it on $\mathfrak{F}$, $\mathfrak{G}$, thanks to the covariant form of equations (7), (8).

Let us further expressly exclude the case when $(\mathfrak{F}, \mathfrak{G})$, the scalar product, vanishes (it will turn out to be the same which we excluded a minute ago). And let us pay attention to the fact that, among other things, the cross-product $[\mathfrak{F} \times \mathfrak{G}]$ is made zero by the said Lorentz-transformation:

$$[\mathfrak{F} \times \mathfrak{G}] = 0. \quad (12)$$

Now from equations (2) you can immediately deduce:

$$[\mathfrak{F}^* \times \mathfrak{F}] = [\mathfrak{G}^* \times \mathfrak{G}] = \frac{2}{(\mathfrak{F}, \mathfrak{G})} [\mathfrak{F} \times \mathfrak{G}], \quad (13)$$

which is true quite generally, and in the present case vanishes by (12). The vanishing of the first and second cross-product means that (1) the real and the imaginary three-vector, composing the complex three-vector, $\mathfrak{F}$, have the same direction, and (2) the same holds for $\mathfrak{G}$, though the common direction in the second case might be different from the first.† But this cannot be, else (12) could not hold. So we have (except for the singular cases which we have excluded):

A Lorentz-frame always existing, in which all the four composing three-vectors are parallel in the world point in question. Owing to the two-fold rotational symmetry (two-fold degeneracy; coincidence of roots) of $T_{kl}$ the Lorentz-frame is, of course, not unique. A translation along the common direction and a rotation around it are free.

In order to obtain further simplification, let us make use of the fact that multiplication of all six components $\mathfrak{F}, \mathfrak{G}$ by a factor $e^{i\gamma}$, with $\gamma$ = real constant, has the following consequences:

† In these statements it is to be understood that a zero-vector may be said to have any direction you please.
(1) it does not interfere with the condition of conjugateness, for the right-hand sides in (2) take the factor $e^{-i\gamma}$, as they should;
(2) according to (1), (7), (8) it leaves the numerical values of the Lagrangian $\mathcal{L}$ and of the tensor $T_{kl}$ unaltered;
(3) if you apply the process to a solution of (3) and (4) throughout the world (but with $\gamma$ a constant!) you obtain another solution (though with the same densities of energy, momentum, and stress as before, in every world point).

We shall call that, for shortness' sake, a $\gamma$-transformation—without prejudice, whether it is in the same sense "irrelevant" as a Lorentz-transformation. Maybe it is, for it is strongly reminiscent of the one arbitrary phase-constant of wave-mechanics.

For the moment we make use of it, after having made all other components, except, say, $\mathfrak{F}_1$ and $\mathfrak{B}_1$ vanish by one of the aforesaid Lorentz-transformations; we choose $\gamma$ so as to make $\mathfrak{F}_1$ real ($\mathfrak{F}_1$ cannot vanish, since the case $(\mathfrak{F} \mathfrak{B}) = 0$ was excluded). Equation (5) now reads
\[ \mathfrak{F}_1 (\mathfrak{B}_1 + \mathfrak{B}_1^*) = 0, \]
showing that $\mathfrak{B}_1$ has now become purely imaginary. Put
\[ \mathfrak{B}_1 = i\mathcal{A} \mathfrak{F}_1, \] (14)
where $\mathcal{A}$ is some real constant. By substituting in (2) it can easily be verified that the following set is the only permissible
\[ \mathfrak{F}_1 = \frac{\sqrt{1 - \mathcal{A}^2}}{\mathcal{A}}, \quad \mathfrak{B}_1 = i\sqrt{1 - \mathcal{A}^2}. \] (15)
$\mathcal{A}$ is to be allowed from $-1$ to $+1$; the positive sign of the square root may be taken. We shall call this the "standard field." It is a purely magnetic field with "permeability" equal to $\mathcal{A}^{-1}$, but it might, of course, be transformed into a purely electric field with "dielectric constant" $\mathcal{A}^{-1}$, by a $\gamma$-transformation; or finally, into a "mixture" with both constants equal to $\mathcal{A}^{-1}$. If we called this the standard field, no further use of the $\gamma$-transformation, but only of the Lorentz-transformation, would be necessary to obtain the most general field. For the Lagrange function we calculate in the standard case
\[ \frac{i}{2} \mathcal{L} = \frac{1 + \mathcal{A}^2}{2\mathcal{A}}, \] (16)
in compliance with (11). You can read this equation thus: the Lagrangian $\frac{i}{2} \mathcal{L}$ is equal to the arithmetical mean of that permeability (or dielectric
constant) and its reciprocal, which arise when the world point is transformed to standard-conditions, or (an almost synonymous alternative of expression) when the energy-flux is abolished by Lorentz-transformation.

Though it is arbitrary, whether the electric or the magnetic case be taken as the standard case, i.e., whether \( \gamma \) or \( \beta \) be made real, yet there is by no means symmetry between them. For the first can range (in the standard case) from zero to infinity, the second from zero to 1. The dissymmetry is not one between electricity and magnetism, but between displacement and field; perhaps the dissymmetry in Nature is similar. By the way, the restriction imposed on the magnitude of the field holds good only for the standard case, not in general.

We have now, without expressly looking out for it, arrived at a thorough knowledge of what the "conditions of conjugateness" really mean. They are fulfilled by all sets of values which are derived from (15) by an arbitrary Lorentz- and \( \gamma \)-transformation, and by no other sets of values, for in the excluded case \( (\gamma \beta) = 0 \), one cannot speak of substituting in equations (2); and the second exception, \( \left| \frac{i}{2} L \right| = 1 \), is really the same case, as will be shown in \( \S \ 5 \). The manifoldness of complex six-vectors complying with the condition is six-fold; the two real constants \( \mathcal{A} \) and \( \gamma \) determine, so to speak, the "inner state" of the world point in question; in addition, there are four constants of the arbitrary Lorentz-transformation, four only, because a translation along a rotation round the standard direction does not affect the vector. Thus the 12 real constants of the complex six-vector are reduced to 6, as was to be expected.

Moreover, the identity with Born's theory can now easily be tested by transforming the latter to standard conditions as well. I shall leave that to the reader.†

† In Born's paper (' Proc. Roy. Soc.,' A, vol. 144, p. 425 (1934)), take equations (2.11) to (3.3a) of p. 437. First correct two misprints in the last of them by inserting a minus sign before \( b^2 \) and changing \(-GB\) into \(+GB\) in the numerator. Put \( b^2 = 1 \) for simplicity and choose a frame with \( B \parallel E \). Then obviously \( H \parallel D \parallel B \parallel E \) by (3.3a). In the latter insert \( F, G \) from (2.12a), (2.13a). This shows up the numerical coincidence of dielectric constant and permeability \( (H = \mathcal{A}B, E = \mathcal{A}D) \) with

\[
\mathcal{A} = \sqrt{\frac{1 - E^2}{1 + B^2}}
\]

(\( \mathcal{A} \) is the designation used in the present paper.) From the last equation express \( B^2 \) by \( E^2 \) and replace the latter by \( \mathcal{A}^2D^2 \). Then \( B^2 + D^2 = (1 - \mathcal{A}^2)/\mathcal{A}^2 \), and, of course, \( H^2 + E^2 = 1 - \mathcal{A}^2 \); which reduce to our equation (15) when \( D \) and \( E \) are abolished by a "\( \gamma \)-transformation."
5—The Singular Case

We still have to deal with the two singular cases which we have excepted. The first of them consisted in \( \rho \), equation (10), vanishing, or

\[ \frac{1}{2} \mathcal{L} = \pm i, \]

the second in

\[ (\mathcal{F} \circ \delta) = 0. \]

It goes without saying that the second can only be dealt with as a limiting case, because dividing by actual zero is meaningless. As to the first, we observe that equations (2) can be rewritten identically thus:

\[
\begin{align*}
\mathcal{F}^* &= -\frac{2}{(\mathcal{F} \circ \delta)} \left( \mathcal{F} + \frac{1}{2} \mathcal{L} \mathcal{G} \right), \\
\mathcal{G}^* &= \frac{2}{(\mathcal{F} \circ \delta)} \left( \mathcal{F} - \frac{1}{2} \mathcal{L} \mathcal{G} \right)
\end{align*}
\]

from which by scalar multiplication and using (1) we get

\[ (\mathcal{F}^* \mathcal{G}^*) = -4 \frac{1 + \frac{1}{4} \mathcal{L}^2}{(\mathcal{F} \circ \delta)}. \]

This shows that whenever \( \frac{1}{2} \mathcal{L} \) tends to one of the values \( \pm i \), \( (\mathcal{F} \circ \delta) \) cannot be assumed not to tend to zero, for then \( (\mathcal{F}^* \mathcal{G}^*) \) would do so—in contradiction with the assumption. And inversely: whenever \( (\mathcal{F} \circ \delta) \) tends to zero, \( \frac{1}{2} \mathcal{L} \) has to tend to one of the values \( \pm i \), in order to prevent \( (\mathcal{F}^* \mathcal{G}^*) \) from becoming infinite. So the two cases (17) and (18) are identical.

We now infer from (19) that in the limit we must have

\[ \mathcal{G} = \mp i \mathcal{F} \]

(in order to prevent the starred vectors from going to infinity). For the real vectors (see (A)) this means

\[ \mathbf{B} = \mp \mathbf{H} , \quad \mathbf{D} = \mp \mathbf{E}. \]

Moreover, from (18) and (21):

\[ \mathcal{G}^2 = 0. \]

For the real vectors that means:

\[ (\mathbf{B} - i \mathbf{D})^2 = \mathbf{B}^2 - \mathbf{D}^2 - 2i (\mathbf{B} \cdot \mathbf{D}) = 0. \]

Now we have to distinguish between two cases. (22) and (23) can evidently be fulfilled in the limit by letting all 12 components tend to
infinite smallness. This is by no means trivial, it is contained in our
standard-treatment (cf. equations (15)) in the limit $\mathcal{A} = \pm 1$. The
six-vector $\mathbf{B}, \mathbf{E}$ coincides with either $(+)$ or $(-)$ the six-vector $\mathbf{H}, \mathbf{D}$—in
the special frame of reference chosen in (15), and therefore in every
frame. It is the limiting case corresponding to Maxwell’s theory (the
quaint possibility of the negative sign will be dealt with under a general
aspect later on). If not all the 12 components tend to zero, we infer
from (23) and (22)

$$|\mathbf{B}| = |\mathbf{D}| \quad \text{and} \quad \mathbf{B} \perp \mathbf{D}. \quad (24)$$

This is the second case, the, properly speaking, singular one. In addition
to the coincidence of displacement ($\mathbf{B}, \mathbf{D}$) and field ($\mathbf{H}, \mathbf{E}$) (or negative
of the field), the electric and the magnetic three-vectors have to be equal
in size and orthogonal to each other. It is the case well known from the
plane light-wave, and we shall refer to it as the “light-case.” It is not
very astonishing to find it on theoretical treatment side by side with the
infinitely weak field. For it is the only one which, by a suitable Lorentz-
transformation, can be reduced to arbitrary weakness! This is the
reason why the plane light-wave shares the property of infinitely weak
fields, notably the property of being an exact solution common to
Maxwell’s and Born’s equations.

6—NORMAL AND ABNORMAL FIELDS

Both in the case of the standard-field (15) and in the “light-case” we
encountered the quaint possibility that the displacement-vectors $\mathbf{B}, \mathbf{D}$,
were not bound to have the expected direction with respect to the field-
vectors, $\mathbf{H}, \mathbf{D}$, but that they could also have the opposite direction. This
general feature of the present scheme can immediately be inferred from
its complete symmetry with respect to an exchange of $\mathbf{\gamma}, \mathbf{\delta}$ with $\mathbf{\gamma}^*, \mathbf{\delta}^*$.
Remember that the equations (2) are “their own resolutions,” that
the field-equations (3), (4) are ostensibly symmetrical, and that the supple-
mentary “condition of conjugateness” is also symmetrical! Hence if
the functions $\mathbf{\gamma}, \mathbf{\delta}$ describe a possible field, i.e., satisfy (2), (3), (4) and
“conjugateness,” the functions $\mathbf{\gamma}^*, \mathbf{\delta}^*$ also do so. But according
to (A) the latter field is obtained from the former by inverting the directions
of $\mathbf{D}$ and $\mathbf{H}$. This inversion must result in turning the actual direction
of $\mathbf{B}$ with respect to $\mathbf{H}$ and of $\mathbf{D}$ with respect to $\mathbf{E}$ from the expected to
the unexpected (or vice versa)—provided the case is simple enough to
suggest any expectation at all.
Since this might not always be so, it is better to make the distinction between the two kinds of field in an invariant way. It was stated in (11) that the real quantity
\[ \frac{i}{2} \mathcal{L} \]
is always either \( \geq 1 \) or \( \leq -1 \). This gives the appropriate distinction, the former being the "normal" case, the latter the "quaint" one. What is most unexpected about it is that not only \( \mathcal{L} \) but all the components of the energy-momentum-tensor \( T_{kl} \) change their sign, when you replace \( \mathfrak{F}, \mathfrak{B} \) by their complex conjugates. This can immediately be seen on inspection of (7) and (8), considering that the \( T_{kl} \) are real and therefore would remain unaltered, if the \( i \) that appears explicitly in (7) would also change sign, which, of course, it does not (the present process of shifting over from one field to another field is very liable to be confounded with the one, which we used before in order to prove that the \( T_{kl} \) are real, and which consisted in using the complex conjugate vectors to describe the same field).

Now let us suppose that \( \mathfrak{F}, \mathfrak{B} \) represent a small parcel of waves, travelling along the direction of positive \( x \) (say), with weak field-strength (so, nearly Maxwellian) and of the usual type, i.e.—

(1) the density of energy shall be positive throughout the wave-group (or, to put it more distinctly, with respect to what will follow: larger than in the space that is free of waves);

(2) the Poynting vector shall, on the average, have the direction in which the parcel travels, that is the positive \( x \)-direction.

Then \( \mathfrak{F}^*, \mathfrak{B}^* \) will represent a very similar wave-parcel, which also travels in the direction of positive \( x \). But the energy density will be smaller within the parcel than in the space free of waves, and correspondingly Poynting’s vector is, on the average, directed towards the negative \( x \)!

In order to investigate this more closely, let us consider the standard field (15). There we have
\[ \mathfrak{F}^2 = \frac{1 - \mathcal{A}^2}{\mathcal{A}^2}, \quad \mathfrak{B}^2 = - (1 - \mathcal{A}^2), \quad (\mathfrak{F} \mathfrak{B}) = i \frac{1 - \mathcal{A}^2}{\mathcal{A}}, \]
\[ \frac{i}{2} \mathcal{L} = \frac{1 + \mathcal{A}^2}{2\mathcal{A}}, \quad \frac{i}{2} (\mathfrak{F}^2 + \mathfrak{B}^2) = \frac{1 - \mathcal{A}^2}{2\mathcal{A}}, \]
and the density of energy
\[ T_{44} = \frac{1 - \mathcal{A}^2}{2\mathcal{A}} + \frac{1 + \mathcal{A}^2}{2\mathcal{A}}. \]
The other components are: \( T_{11} = T_{44} \) and
\[
T_{22} = T_{33} = -\frac{1 - \mathcal{A}^2}{2\mathcal{A}} + \frac{1 + \mathcal{A}^2}{2\mathcal{A}}.
\]

We can see that all depends on the sign of \( \mathcal{A} \). I have kept the two parts asunder, the first arising from \( t_{kl} \), the second from the "spherical" tensor. The first is a normal Maxwell-tensor, if \( \mathcal{A} > 0 \), with positive density of energy, a pull (numerically equal to the density) in the direction of the lines of force and a numerically equal pressure orthogonal to them. Moreover, it vanishes with vanishing field \( (\mathcal{A} = 1) \). In an arbitrary frame it will also yield a normal Maxwell-tensor with positive energy, one pull and two pressures (the absolute value in common to all of them, being larger than in the standard frame). If \( \mathcal{A} < 0 \), this first part always yields the negative of a normal Maxwell-tensor, \( i.e., \) with negative energy, two pulls, one pressure. The second part is numerically invariant under Lorentz- and \( \gamma \)-transformations and represents an isotropic pull or pressure with a positive or negative energy density equal to it, according to whether \( \mathcal{A} \gtrless 0 \). With vanishing field \( (\mathcal{A} = \pm 1) \) it does not vanish but approaches \( \pm 1 \). In weak fields (say \( \mathcal{A} = 1 - \varepsilon \), normal case) the second part is ineffective, because, in the first approximation, it does not depend on space and time. The first part gives, in the standard frame, the varying energy-density \( \varepsilon \) (equal to the "susceptibility"). This agrees with Maxwell's expression since, by (15), the field-strength is \( \sqrt{2\varepsilon} \).

Without approximation the second part depends, of course, on the co-ordinates and on time. Yet it is Lorentz-invariant and it contributes nothing to Poynting's vector or to the oblique stresses. But it must not be forgotten that the conservation laws only hold for \( T_{kl} \) as a whole, not for its two parts separately. I venture to believe that we are here approaching the understanding of how Born's equations describe the exchange of energy between matter and radiation.

The fact that the components \( T_{kl} \) do not vanish with vanishing field, and that they approach different values (viz., \( +1 \) or \( -1 \)) according to whether a normal field \( (\mathcal{A} > 0) \) or an abnormal field \( (\mathcal{A} < 0) \) tends to zero—this fact seems very embarrassing. But I think it is essential. You cannot get rid of it by any sort of normalization, unless you wish to drop the one kind of fields altogether. This might be tacitly done in Born's original presentation of his theory. There the abnormal fields correspond to a negative sign of the square root, and one might feel inclined to dictate the positive sign. But a square root, after all,
two signs. I cannot remember a case of a square root occurring in the description of a physical phenomenon with one of its signs totally meaningless! At all events, the presentation offered in this paper would make it seem rather artificial, if to (2), (3), (4) and the condition of conjugateness, we were to add the dictate that only such solutions ought to be admitted, which make the expression (1) negative imaginary.

That is why I am inclined to believe in the analogy—very obvious from the formal point of view—between Born's "abnormal" fields and Dirac's "negative" solutions.

In conclusion, I wish to express my thanks to Imperial Chemical Industries, Limited, whose generosity made it possible for me to carry out this research.

**Summary**

A new representation by complex six-vectors is offered of Born's theory of the electromagnetic field. It is dealt with so far only from the classical point of view, without entering into the question of quantization. The ostensibly simple form of the Lagrangian and of the components of the energy-tensor is remarkable, all these quantities being rational and homogeneous functions (of degree zero) of the components of the field. The following points are disclosed, which I consider to be not merely of formal significance:

1. In addition to the Lorentz transformation a one-parameter-transformation of the field exists (called $\gamma$-transformation in this paper), under which the field equations are invariant.
2. The theory points to a strong dissymmetry between field and displacement, but to none between electricity and magnetism. The only means of accounting for the latter dissymmetry (which is actually met with in Nature) on the lines of the present theory seems to be to avail oneself of the former.
3. The form of the equations emphasizes the extremely singular character of the case in which electric and magnetic field are equal and perpendicular to each other.
4. The complete symmetry between "normal" and "abnormal" fields is thrown into relief, reminding one strongly of the so-called positive and negative solutions in Dirac's theory of the electron.