The problem of $n$ bodies in general relativity theory

By Sir Arthur Eddington, F.R.S., and G. L. Clark

(Received 22 March 1938)

1. In a recent investigation of the problem of two bodies in general relativity theory, Prof. Levi-Civita (1937b) has reached the conclusion that the centre of gravity has a secular acceleration in the direction of the major axis of the orbit towards the periastron of the larger mass. He gives an example of a binary star in which this acceleration may become detectable in much less than a century, perhaps even in a few years.

Since the periastron slowly revolves, Levi-Civita's result implies that the binary star as a whole spontaneously describes a circle of very large radius. There seems to be no obvious ground for regarding this as impossible; it is conceivable that an unsymmetrical system may radiate momentum by gravitational waves and so experience a recoil. But the behaviour is so peculiar as to inspire doubt; and we determined to re-examine the problem with a view either to detecting an error or to obtaining further light on the nature of the phenomenon.

The conclusion of our investigation is that there is no secular acceleration of the centre of gravity (§8).

Levi-Civita's algebraic calculations are not given in sufficient detail for us to locate the source of the divergence. It may be pointed out, however, that the null result is due to the mutual cancelling of a number of terms, so that a numerical slip in the coefficient of one of the terms may be responsible. Our first calculation gave a positive result, with a coefficient differing from Levi-Civita's but of the same order of magnitude. This was based on de Sitter's general formula for the line-element due to $n$ bodies. We discovered, however, that de Sitter had made an error of theory in the calculation of one of his terms (§2); and when this was corrected, the secular acceleration became zero. De Sitter's investigation is quoted by Levi-Civita in his paper; but we have learnt by correspondence with him that he did not take over the formula containing the error.

For a system of two bodies our result confirms the usual assumption that the system as a whole has no spontaneous acceleration. We have not been able to generalize the proof to apply to $n$ bodies. The obstacle is that, whereas in the two-body problem we use the approximate Newtonian
solution for evaluating small terms, in the \( n \)-body problem the corresponding Newtonian solution does not exist.

Since de Sitter's formula for the line-element has frequently been referred to in the past twenty years, its amendment is of importance apart from the particular application to Levi-Civita's problem. In particular the calculation of the equivalent mass of a system of particles now comes into line with the theory of the equivalent mass of a system of continuous matter developed by Tolman, Whittaker and others (§4). Previously we had been puzzled by a failure to reconcile them.

2. The first discussion of the field of \( n \) bodies was given by Droste (1916). Subsequently de Sitter (1916), making a minor alteration, used the result to investigate problems in gravitational astronomy. The error, mentioned in §1, occurs on pp. 156-7 of de Sitter's paper. He has obtained (equation (65))

\[
\nabla^2 \gamma = \kappa \rho,
\]

and writes the solution as

\[
\gamma = -\frac{\kappa}{4\pi} \int \frac{\rho \, dV}{r}.
\]

The equations (2-1) and (2-2) are most familiar to us as determining a potential in Euclidean space; but the solution is, of course, purely analytical, and does not depend on a geometrical representation of the variables \( x_1, x_2, x_3 \) contained in \( \nabla^2 \). It must, however, be remembered that when, as here, the space is non-Euclidean, \( r \) in (2-2) is a fictitious distance calculated as though \( x_1, x_2, x_3 \) were Euclidean rectangular co-ordinates, and \( dV \) is similarly the fictitious volume \( dx_1 \, dx_2 \, dx_3 \) of the element. De Sitter seems to have overlooked the latter point, and to have taken \( dV \) to be the natural volume; thus he assumes that "in a system of co-ordinates in which a body would be at rest \( \int \rho \, dV \) would be the mass" (de Sitter's \( \rho \) is the invariant density \( T \)). The natural volume is \( \sqrt{(g_{11}g_{22}g_{33})} \, dx_1 \, dx_2 \, dx_3 \), so that the correct formula for the mass is

\[
(g_{11}g_{22}g_{33})^{\frac{1}{2}} \int \rho \, dV,
\]

subject to the usual correction (duly inserted by de Sitter) for change of mass with velocity.

3. It is well known that even the problem of two bodies has not been solved exactly in general relativity theory. The order of approximation aimed at by de Sitter and by Levi-Civita is to determine the accelerations
The problem of n bodies in general relativity theory 467

correctly to the squares of the potentials or equivalently to the fourth powers of the velocities. Treating the Newtonian potential as of the first order, the order of the terms required to be retained in the $g_{\mu\nu}$ is found to be as follows:

$g_{44}$, order 2; $g_{14}$, $g_{24}$, $g_{34}$, order $\frac{3}{2}$; the others, order 1.

The usual method of first approximation (Eddington 1924, §57) gives ($\mu = 1, 2, 3$)

$$g_{\mu\nu} = -1 + \gamma_0, \quad g_{44} = 1 + \gamma_0, \quad g_{12}, g_{23}, g_{31} = 0,$$

$$g = \frac{1}{2} \sum_i m_i \frac{v_i}{r_i},$$

where

$$\gamma_0 = -2 \sum_i m_i \frac{v_i}{r_i}.$$  \hspace{1cm} (3.2)

The units are such that the constant of gravitation and the velocity of light are unity; $(v_\mu)_i$ is the velocity of the $i$th body, and $r_i$ is its distance (calculated as for Euclidean co-ordinates) from the point considered.

This approximation is of the required accuracy except for $g_{44}$. To obtain the higher approximation $g_{44} = 1 + \gamma$, we use the exact equation furnished by

$$G_{44} = -8\pi (T_{44} - \frac{1}{2} g_{44} T),$$

and simplify it by inserting the first approximation for the $g_{\mu\nu}$ other than $g_{44}$, and also for $g_{44}$ when operated on by $\frac{\partial}{\partial t}$. ($\frac{\partial g_{44}}{\partial t}$ consists of terms of the form $\frac{(\partial x_\mu)_i}{\partial t}$, and the second factor is of order $\frac{1}{2}$.) The equation for $\gamma$ is found to be (de Sitter 1916, equation (69))

$$\sum \frac{\partial^2 \gamma}{\partial x^2} = 8\pi (\rho_0 + 2\rho_0 v^2) + \frac{\partial^2 \gamma_0}{\partial t^2} + \sum \left( \frac{\partial \gamma_0}{\partial x_\mu} \right)^2,$$

where $\Sigma'$ denotes summation for $x_1, x_2, x_3$; $\rho_0$ is the invariant density ($T$), and $v$ the velocity of matter at the point considered. The solution (obtained by de Sitter) is

$$\gamma = \frac{1}{2} \gamma_0^2 + \sum \left( \frac{2q_i}{r_i} - \frac{4m_i v_i^2}{r_i} - \frac{4m_i}{r_i} \sum_j \frac{m_j}{\Delta_{ij}} - m_i \frac{\partial^2 r_i}{\partial t^2} \right),$$

where $\Delta_{ij}$ is the distance between the $i$th and $j$th bodies (the value $j = i$ being omitted in the $j$-summation), and

$$q_i = \int \rho_0 dV$$

taken over the $i$th body. De Sitter, taking $dV$ to be the natural volume, gives

$$q_i = m_i (1 - v_i^2)^{\frac{1}{2}}.$$  \hspace{1cm} (30-2)
But taking account of the factor \((g_{11}g_{22}g_{33})^{\frac{1}{2}}\) in (2·3), the correct result is
\[
g_{i} = m_{i}(1 + \frac{3}{2}(\gamma)_{i} - \frac{1}{2}v_{i}^{2})
\] (3·5)
to the required approximation. The value of \((\gamma)_{i}\), i.e. \(\gamma\) evaluated at the \(i\)th body, is
\[
(\gamma)_{i} = -2 \Sigma j m_{j}/A_{ij} \quad (j \neq i).
\] Hence by (3·4) and (3·5)
\[
\gamma = \gamma_{0} + \frac{1}{2}\gamma_{0}^{2} + \Sigma i \left(-\frac{3m_{i}v_{i}^{2}}{r_{i}} + \frac{2m_{i}}{r_{i}} \Sigma j m_{j}/A_{ij} - m_{i}\frac{\partial^{2}r_{i}}{\partial t^{2}}\right).
\] (3·6)
This differs from de Sitter's value (p. 159, equation (70)) by having the coefficient +2 instead of −4 in the middle term in the bracket.

4. If we adopt a co-ordinate system in which the centre of mass of the system of \(n\) particles is at rest at the instant considered, so that \(\Sigma \dot{v}_{i} = 0\), the \(g_{\mu\nu}\) found in § 3 give for the line-element at great distances
\[
d s^{2} = (-1 + \gamma_{0}) (dx_{1}^{2} + dx_{2}^{2} + dx_{3}^{2}) + (1 + \gamma) dx_{4}^{2}.
\] (4·1)
We define the mass of a system (at the instant considered) to be the mass \(M\) of an equivalent particle which gives the same line-element at great distances.* The line-element of the particle in isotropic co-ordinates is (Eddington 1924, §43)
\[
d s^{2} = -(1 + M/2r)^{4} (dx_{1}^{2} + dx_{2}^{2} + dx_{3}^{2}) + (1 - M/2r)^{2} (1 + M/2r)^{2} dx_{4}^{2}.
\]
Writing \(\gamma_{M} = -2M/r\), and expanding in powers of \(\gamma_{M}\), this gives
\[
d s^{2} = (-1 + \gamma_{M} + ...) (dx_{1}^{2} + dx_{2}^{2} + dx_{3}^{2}) + (1 + \gamma_{M} + \frac{1}{2}\gamma_{M}^{2} + ...) dx_{4}^{2}.
\] (4·2)
Comparing (4·1) and (4·2) we see that \(\gamma\) must approach \(\gamma_{M} + \frac{1}{2}\gamma_{M}^{2}\) at great distances.

We have retained the second order term \(\frac{1}{2}\gamma_{M}^{2}\) in \(g_{44}\) because it is the counterpart of the term \(\frac{1}{2}\gamma_{0}^{2}\) in the expression (3·6) for \(\gamma\), and shows that the equivalence of the fields of the system and of the particle extends to the second order. But for the determination of \(M\), the first order terms suffice; \(\gamma_{0}^{2}\) is then negligible, and \(\gamma\) must approach \(-2M/r\) when the \(r_{i}\) become large and equal to \(r\). We then obtain by (3·6)
\[
M = \Sigma i m_{i} + \frac{3}{2} \Sigma i m_{i}v_{i}^{2} - \Sigma i \Sigma j m_{i}m_{j}/A_{ij} + \frac{1}{2}r \Sigma i m_{i}\partial^{2}r_{i}/\partial t^{2}.
\] (4·3)

* The definition will be made more precise later. Meanwhile we contemplate a distance large from the point of view of our approximations, but not so large as to introduce an important time-lag of the potentials.
In the last term, \( \Sigma_i m_i \frac{\partial^2 r_i}{\partial t^2} \) vanishes if the centre of mass of the system has zero acceleration; but we must not assume that this condition is fulfilled. Even if there is no secular acceleration, the centre of mass has usually an instantaneous acceleration. It is understood that the centre of mass is defined in the elementary way, using the rest-masses of the particles; a more complicated definition would be required if we wished to define a point which remains fixed during the motion, bearing in mind the relativistic variations of the masses of the particles.

Accordingly the term \( r \Sigma m_i \frac{\partial^2 r_i}{\partial t^2} \) does not in general vanish. It can be interpreted as the correction for retardation of the potentials contained in \( \gamma_0 \). Denoting retarded values by square brackets, the expansion of a retarded potential in terms of present values is (Eddington 1924, p. 253, equation (4))

\[
\frac{m}{[r(1-v_p)]} = \frac{m}{r} + \frac{1}{2} \frac{d^2 r}{dt^2} - \ldots.
\]

Thus in (3-6), \( \gamma_0 - \Sigma m_i \frac{\partial^2 r_i}{\partial t^2} \) could be written as \([\gamma_0]\).

We can therefore remove the last term of (4-3) by exercising a little more care in defining the equivalent mass of a system. First, we recognize that the comparison of fields at great distances determines the mass of the system at an appropriately antedated instant. Secondly, it is to be understood that the equivalent particle has the same acceleration, as well as the same velocity, as the centre of mass of the system. Then (4-2) must be corrected for the retardation of potential of the accelerated particle, or equivalently the retardation terms must be removed from (4-1). The mass is then

\[
M = \Sigma_i m_i + \frac{3}{2} \Sigma_i m_i v_i^2 - \Sigma_i \Sigma_j \frac{m_i m_j}{\Delta_{ij}},
\]

and the formal difficulty, caused by the divergence of (4-3) as \( r \to \infty \), is avoided.

Let \( T, V \) be the kinetic and potential energies of the system. We have

\[
T = \frac{1}{2} \Sigma_i m_i v_i^2, \quad V = -\frac{1}{2} \Sigma_i \Sigma_j \frac{m_i m_j}{\Delta_{ij}}.
\]

Hence (4-5) becomes

\[
M = \Sigma m_i + 3T + 2V.
\]

It has been shown that in a system of gravitating particles (Eddington 1916, equation (4))

\[
2T + V = \frac{1}{2} \frac{d^2 C}{dt^2},
\]
where \( C \) is the moment of inertia of the system about its centre of mass. The whole energy of the bodies, including rest-energy, is

\[
E = \Sigma m_i + T + V.
\]

Hence by (4.6), (4.7) and (4.8)

\[
M = E + \frac{1}{2} \frac{d^2C}{dt^2}.
\]

Thus, in general the mass of a system is not equal to the energy of the bodies composing it unless its moment of inertia is steady.

The relation \( M = E \) appears in the work of Tolman, Whittaker and others on systems composed of continuous matter; but the supplementary term \( \frac{1}{2} \frac{d^2C}{dt^2} \) in non-static systems appears to be new.

5. The \( g_{\mu\nu} \) of the field having been determined, the track of a body in the field is given by the geodesic

\[
\frac{d^2x_\mu}{ds^2} + \left\{ \chi_\beta, \mu \right\} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0.
\]

Before applying this to one of the bodies of the system we must remove the self-contribution of that body to the field. As Levi-Civita has pointed out, very delicate questions arise as to this "effacing principle".

In the first place, the field is non-linear and is therefore not a simple sum of contributions from the different bodies, so that the rule that the contribution of one body is to be removed is ambiguous. The adopted procedure is determined by the following considerations. When, after performing the differentiations in (5.1), we identify the current co-ordinates \( x_\mu \) with the co-ordinates \( (x_\mu)_i \) of one of the bodies of the system, \( r_i \) vanishes and a number of terms becomes infinite. It is clearly these terms which must be effaced. Therefore in (3.6) we omit the body itself in the \( i \) summation; but it is to be retained in the \( j \) summation in the double summation term.

The justification for this different treatment of the \( i \) and \( j \) summations is that the \( i \)-infinites are spurious, being consequences of treating a finite body as a particle. They should in any case be replaced by volume integrals. In the \( j \) summation, on the other hand, the treatment of the body as a particle remains valid. The effacement is therefore primarily a correction of an invalid approximation; but the assumption that the terms actually vanish requires justification.

This argument removes any possible ambiguity as to how the effacement is to be carried out. On the broader question as to how far the effacement
principle can be logically or physically justified, we cannot enter here. It is
connected with the problem of internal equilibrium of material bodies; and
it is doubtful whether a final answer can be reached without a study of the
combination of quantum and relativity theory. A large part of Levi-
Civita's investigation (1937a) is concerned with it; but we do not think that
the difference of our results for the secular acceleration of the centre of mass
of a double star arises at this point, for he concludes that, when there are
two bodies only, it is certainly justified.

6. The equation resulting from (5·1) is*

\[
\frac{d^2x_\mu}{dt^2} + \sum_i \frac{m_i(x_\mu - x_{\mu i})}{r_i^3} = 4\Sigma_i \Sigma_j m_i m_j \frac{x_\mu - x_{\mu j}}{r_i^3 r_j} + 4\Sigma' m_i (x_\alpha - x_{\alpha i}) v_\mu v_\alpha
\]

\[
- \sum_i m_i (x_\mu - x_{\mu i}) \left( v^2 + 2v_i - 2\eta_i \right)
\]

\[
+ \sum_i \left( \frac{7}{2} \frac{d^2x_{\mu i}}{dt^2} + \frac{3(v_\mu - v_{\mu i})}{r_i^3} \right) e_i + 4\Sigma' \left( (x_\mu - x_{\mu i}) v_{\alpha i} - (x_\alpha - x_{\alpha i}) v_{\mu i} \right) v_\alpha
\]

\[
+ \sum_i \frac{m_i (x_\mu - x_{\mu i})}{r_i^3} \Sigma_{j+\tilde{i}} \frac{m_j}{\Lambda_{ij}} + m \Sigma_i m_i \frac{x_\mu - x_{\mu i}}{r_i^4},
\]

(6·1)

where \[ r_i \eta_i = \Sigma' (x_{\alpha i} - x_\alpha) \frac{d^2x_{\alpha i}}{dt^2}, \quad r_i e_i = \Sigma' (x_{\alpha i} - x_\alpha) v_{\alpha i}. \]

(6·2)

The summations \( \Sigma' \) are over the values \( \alpha = 1, 2, 3 \) of the co-ordinate suffix; the summations \( \Sigma \) are over the \( n - 1 \) bodies of the system excluding the body whose track is being determined. The mass, co-ordinates and velocity of the latter body are denoted by \( m, x_\mu, v_\mu \); its contribution has been separated from the summations and is shown explicitly in the last term. Quantities referring to the other particles carry suffixes \( i \) or \( j \).

The above equation corresponds to (82) in de Sitter's paper. The last two terms are different on account of the emendation made in § 2. Independently of this correction, there is an error (apparently a misprint) in the coefficient of the last term in de Sitter's result.

7. We now consider a system of two bodies of masses \( m_1, m_2 \). The co-
ordinate system is chosen so that the relative motion, on the Newtonian

* In the double summations the value \( j = i \) is not excluded, except in the term
where the exclusion is indicated. A simple rule (applicable throughout this paper)
is to adopt the convention \( \Lambda_{ii} = \infty \), and retain \( j = i \) in all double summations.
level of approximation is in the plane \((x_1, x_2)\). From (6-1) we obtain for the motion of the body of mass \(m_2\)

\[
\frac{d^2 x_{\mu 2}}{dt^2} + \frac{m_1(x_{\mu 2} - x_{\mu 1})}{r^3} = \frac{4m_2^2(x_{\mu 2} - x_{\mu 1})}{r^4} + 4m_1 v_{\mu 2} \frac{\Sigma' v_{x 2}(x_{x 2} - x_{x 1})}{r^3}
\]

\[- \frac{m_1(x_{\mu 2} - x_{\mu 1})}{r^3} (v_2^2 + 2v_1^2 - \frac{3}{2}v_1^2 + \frac{1}{2}r\eta_1)
\]

\[+ m_1 \left( \frac{7}{2} \frac{d^2 x_{\mu 1}}{dt^2} + 3 \frac{(v_{\mu 2} - v_{\mu 1}) c_1}{r^2} + 4 \Sigma' \left\{ (x_{\mu 2} - x_{\mu 1}) v_{x 1} - (x_{x 2} - x_{x 1}) v_{\mu 1} v_{x 2} \right\} \right)
\]

\[+ \frac{m_1 m_2 (x_{\mu 2} - x_{\mu 1})}{r^4},
\]

(7.1)

where \(r\) is the distance between the two bodies.

On the right-hand side of (7.1) we may substitute the approximate Newtonian values

\[
\frac{d^2 x_{\mu 1}}{dt^2} = \frac{m_2 x_{\mu}}{r^3}, \quad \frac{d^2 x_{\mu 2}}{dt^2} = -\frac{m_1 x_{\mu}}{r^3}, \quad v_{\mu 1} = -\frac{m_2}{m} v_{\mu}, \quad v_{\mu 2} = \frac{m_1}{m} v_{\mu},
\]

\[
e_1 = \frac{m_2}{m} \frac{dr}{dt}, \quad e_2 = \frac{m_1}{m} \frac{dr}{dt}, \quad \eta_1 = -\frac{m_2}{r}, \quad \eta_2 = -\frac{m_1}{r},
\]

\[
v^2 = 2E + \frac{2m}{r}, \quad \left( \frac{dr}{dt} \right)^2 = v^2 - \frac{h^2}{r^2}, \quad h^2 = ma(1 - e^2),
\]

where \(m = m_1 + m_2\), \(E\) is the energy of the relative motion, and \(v_{\mu}\) and \(x_{\mu}\) also refer to the relative motion, i.e.

\[
v_{\mu} = v_{\mu 2} - v_{\mu 1}, \quad x_{\mu} = x_{\mu 2} - x_{\mu 1}.
\]

Equation (7.1) then reduces to

\[
\frac{d^2 x_{\mu 2}}{dt^2} + \frac{m_1 x_{\mu}}{r^3}
\]

\[
= \frac{x_{\mu}}{r^3} \left( \left(4m_1^2 - \frac{m_1 m_2^2}{m} + 5m_1 m_2 - \frac{2m_2^3}{m} \right) \frac{1}{r} - 2 \frac{m_1 m_2^2}{m} - \frac{3m_1 m_2^3 h^2}{2m^2 r^2} \right)
\]

\[+ \frac{4m_2^2 \Sigma' v_{x 2}(x_{x 2} - x_{x 1}) v_{x}}{r^3} + \left( \frac{4m_1^3}{m^2} + \frac{3m_1 m_2}{m^2} + \frac{3m_1 m_2^3}{m^2} \right) v_{\mu} \frac{dr}{dt}.
\]

(7.2)
Similarly we obtain for the acceleration of the other particle

\[
\frac{d^2 x_{\mu 1}}{dt^2} - \frac{m_1 x_{\mu}}{r^3} \]

\[
= -\frac{x_{\mu}}{r^3}\left(1 - \frac{m_2}{m} + 5m_1 m_2 - \frac{2m_3^3}{m^2}\right) - 2E\left(\frac{m_2 m_3}{2m^2} + \frac{m_3^2}{m^2}\right) - 3\frac{m_1 m_2 h^2}{2 m^2 r^2} - \frac{4m_1 m_3^2}{m} \sum_r \left(\frac{x_{\mu} v_{\phi} - x_{\phi} v_{\mu}}{r^3} \right) v_{\phi} \left(\frac{4m_3^2}{m} + \frac{3m_1 m_2}{m^2} + \frac{3m_1 m_2}{m^2}\right) v_{\mu} \frac{dr}{r^2 dt}. \tag{7.3}
\]

8. The co-ordinates \( x_{\mu} \) of the centre of mass are

\[
x_{\mu} = (m_2 x_{\mu 2} + m_1 x_{\mu 1})/m,
\]

so that

\[
\frac{d^2 x_{\mu}}{dt^2} = \frac{m_2 d^2 x_{\mu 2}}{dt^2} + \frac{m_1 d^2 x_{\mu 1}}{dt^2}. \tag{8.1}
\]

Substituting from (7-2) and (7-3), we obtain

\[
\frac{d^2 x_{\mu}}{dt^2} = \frac{m_1 m_2 (m_1 - m_2)}{m^2} \left(\frac{x_{\mu}}{r^3} \left(-\frac{2m}{r} - E + \frac{3h^2}{2r^2}\right) + v_{\mu} \frac{dr}{r^2 dt}\right). \tag{8.2}
\]

It is possible at this stage to compare our result with Levi-Civita’s (1937b). Translating his equation (15) into our notation, it agrees with (8.2) except that he has \(-3m/r\) instead of \(-2m/r\) in the inner bracket. The secular acceleration found by him is attributable to this difference.

We now take the \( x_1 \) axis in the direction of periastron of the relative orbit. Then

\[
\frac{1}{r} = \frac{1 + e \cos \theta}{a(1-e^2)}, \quad x_1 = r \cos \theta, \quad x_2 = r \sin \theta.
\]

Introducing the true anomaly \( \theta \) in (8.2) by the relation

\[
\frac{r^2 d\theta}{dt} = h = \sqrt{(ma(1-e^2))},
\]

we have

\[
\frac{d}{d\theta} \left(\frac{dx_{\mu}}{dt}\right) = \frac{m_1 m_2 (m_1 - m_2)}{m^2 h} \left(\frac{x_{\mu}}{r} \left(-\frac{2m}{r} - E + \frac{3h^2}{2r^2}\right) + v_{\mu} \frac{dr}{r^2 dt}\right).
\]

Hence

\[
\frac{d}{d\theta} \left(\frac{dx_1}{dt}\right) = \frac{m_1 m_2 (m_1 - m_2)}{m^2 h} \times \left(\cos \theta \left(-\frac{2m(1 + e \cos \theta)}{a(1-e^2)} - E + \frac{3m(1 + e \cos \theta)^2}{2a(1-e^2)}\right) - \frac{me \sin^2 \theta}{a(1-e^2)}\right). \tag{8.3}
\]
We have to pick out the secular terms on the right-hand side of (8·3). They are

\[
\frac{m_1 m_2 (m_1 - m_2)}{m h a (1 - e^2)} e \left\{ -1 + \frac{3}{2} - \frac{1}{2} \right\},
\]

which is zero. Thus \( \frac{d}{d\theta} \left( \frac{dx_1}{dt} \right) \) is a purely periodic function of \( \theta \), so that the velocity \( \frac{dx_1}{dt} \) is a constant plus a purely periodic function. There is therefore no secular acceleration.

9. The existence of a periodic variation of the velocity of the centre of mass of a binary system illustrates our remark in § 4 on the non-vanishing of \( \Sigma m_t \partial^2 r_i / \partial t^2 \). It only means that the elementary geometrical definition of the centre of mass is too crude to furnish a dynamically fixed point undisturbed by the relative motion.

Since the present calculation, like those of de Sitter and Levi-Civita, is carried only to the second order, it does not exclude a secular acceleration of the third order. Since a secular acceleration implies radiation of momentum by gravitational waves, it is interesting to notice that in a comparable calculation of the radiation of energy by gravitational waves the effect found was of the third order. The radiation of gravitational energy from a double star system has not yet been calculated to the third order; but for a spinning rod a third order radiation (i.e. involving \( v^6 \)) was found by Einstein (Eddington 1924, p. 248). The investigation clearly establishes that no second order radiation of energy occurs; but there is some doubt whether the positive result for the third order radiation can be relied on, owing to uncertainty in the treatment of cohesive forces (Eddington 1924, p. 251).

It is desirable to show that the secular acceleration here computed agrees with that which an observer would measure. The co-ordinates have been chosen so that at great distances from the system they approximate to Galilean co-ordinates. We can therefore surround the system with a sphere beyond which the deviation is negligible. The observer is outside the sphere and employs the Galilean co-ordinates. The acceleration will be determined from the observed change of spectral shift, which includes Doppler shift, Einstein shift, and change of length of the light-track due to the fluctuating metric. The change of Doppler shift, which corresponds to the accelerations here computed, is therefore not directly observable; nor can we, without arbitrariness, apply theoretical corrections to eliminate the other sources of shift, since these depend on the particular co-ordinate system adopted within the sphere. The periodic acceleration given by (8·4) has therefore
The problem of \( n \) bodies in general relativity theory

The problem of \( n \) bodies in general relativity theory has no absolute or observable significance; but it would be possible, with some labour, to compute an observable quantity, namely the periodic variation of the total spectral shift as seen by a distant observer in a specified direction. The computation is, however, of no great interest since the effect is far below the limits of practical detection.

No such difficulty arises in observing the secular acceleration. Since the relative motion is found to be stable (§10), the Einstein shift and light-track correction fluctuate between narrow limits, and any secular change of the observed spectral shift must be shown in our equations as a secular change of velocity.

10. The relative orbit can also be determined from (7·2) and (7·3). The relative acceleration

\[
\frac{d^2x_\mu}{dt^2} = \frac{d^2x_{\mu_2}}{dt^2} - \frac{d^2x_{\mu_1}}{dt^2}
\]

is obtained by subtracting (7·3) from (7·2). It is unnecessary to give the details of a rather long calculation, since we confirm Levi-Civita's calculation of the potential function (1937b, p. 229, equation (I)) and agree with the results he derives.

References

— 1937b Amer. J. Math. 59, 225-34.