A Contribution to Modern Ideas on the Quantum Theory.

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I.—*The de Broglie Phase Wave in Generalised Space-Time.*

The relativity theory of gravitation indicates that space-time is a four dimensional continuum in which the line element is measured by the equation

$$(ds)^2 = g_{mn}dx_m dx_n, \tag{1}$$

the notation being that generally adopted.

The world-lines or natural tracks of free particles in this space are geodesics. From (1) we have

$$g_{mn}\frac{dx_m}{ds}\cdot\frac{dx_n}{ds} = 1, \tag{2}$$

the quantity on the left being an expression corresponding to the kinetic energy of ordinary dynamics for a particle of unit mass. This correspondence is readily appreciated if it be noted that dx_m/ds is the natural extension of the velocity, dx_m/dt.

dx_m/ds $(m = 1, 2, 3, 4)$ is one of the direction cosines of the track of the particle.

Equation (2) corresponds to

$$g_{mn}\frac{dx_m}{dt}\cdot\frac{dx_n}{dt} = \left(\frac{ds}{dt}\right)^2, \qquad m, n = 1, 2, 3,$$

in three dimensions.

Hence since ds/dt is the speed of the particle we could write

$$2\mathrm{T} = g_{mn}\frac{dx_m}{dt}\cdot\frac{dx_n}{dt}, \qquad m, n = 1, 2, 3,$$

where T is the kinetic energy.

Thus in a conservative system with total energy, E, and potential energy V, we have

$$g_{mn}\frac{dx_m}{dt}\cdot\frac{dx_n}{dt} = 2(\mathrm{E} - \mathrm{V}), \qquad m, n = 1, 2, 3. \tag{3}$$

This point is mentioned to show what is meant by the statement that the left side of (2) corresponds to the kinetic energy in ordinary mechanics and it appears that instead of 2T or 2 (E — V) we have here to do with the invariant expression of (2).

We may define as the momentum of our unit mass the covariant vector whose typical component, $p_m = g_{m\alpha} \frac{dx_\alpha}{ds}$.

Then (2) is replaced by

$$g^{mn} p_m p_n = 1, \qquad (4)$$

where again g^{mn} has the usual significance.

(4) is the expression of the invariant expression in terms of covariant quantities and corresponds to the expression of kinetic energy in terms of generalised momenta instead of velocities. We may make the substitution

$$p_m = \frac{\partial W}{\partial x_m},$$

and so express (4) as

$$g^{mn} \cdot \frac{\partial W}{\partial x_m} \frac{\partial W}{\partial x_n} = 1,$$

which is equivalent to $|\text{grad. W}|^2 = 1$* where the gradient operator is to be understood as the generalised gradient of the analysis of tensors.

Now surfaces, W = constant, have normals whose direction cosines are $\frac{\partial W}{\partial x^m}$ ($m = 1, 2, 3, 4$) and since from (2) we have

$$p_n \frac{dx_n}{ds} = 1,$$

it follows that

$$\frac{\partial W}{\partial x^n} \cdot \frac{dx_n}{ds} = 1,$$

i.e., the geodesic is normal to the surfaces.

It becomes possible to regard the continuum as filled with geodesics and a family of surfaces which bear to each other the same relation as lines of force and equipotential surfaces in an electrostatic field.

This mutual relation appears to be the basis of de Broglie's view that the motion of a particle can be associated with a phase-wave. The particle in the continuum may be located as a point of the geodesic or it may be associated with the particular W-surface through that point.

* *Cf.* "Quantisierung als Eigenwertproblem," Shroedinger, 'Ann. d. Physik,' vol. 79, p. 492 (1926).

To a three-dimensional observer the particle appears as a point travelling along its world line and to him the W-surface will appear as a wave.*

II.—*The Wave Equation as a Fundamental Law of Nature.*

The relativity theory of gravitation teaches that gravitational phenomena are the physical representation of the geometry of space-time described in terms of the components, g_{mn}, which fix the nature of the geometry. It would be a natural extension of these principles to proceed by a similar method to include the phenomena of electromagnetism. It would be very satisfactory if, by choosing appropriate components, g_{mn}, we could include in one system of geometry both gravitation and electromagnetism.

Natural phenomena do not appear to occur in accordance with such a simple scheme.

Weyl and Eddington have made the most natural extension to space-time geometry by the introduction of local scales of measurement, a standard for the measurement of $(ds)^2$ being appropriate to each point of the continuum. They have in this way found it possible to include the second class of phenomena in the scheme.

It is possible to make this inclusion in another way.

Let it be supposed that the energy, momentum and stresses associated with electromagnetic phenomena exert an influence upon the geometry of space, contributing along with gravitation to the components, g_{mn}.† Let the line-element be still measured by the rule,

$$(ds)^2 = g_{mn}\, dx_m\, dx_n.$$

This is not enough for the description of the phenomena. We must add a new element to the theory.

Let a four-vector, $\mathbf{s}'$, the current four-vector, be associated with elements of three-dimensional volume. The affix denotes the contravariant character of the vector and its components are (s^1, s^2, s^3, s^4). The three-dimensional volume element may be considered as a covariant vector with components $(d\sigma_1, d\sigma_2, d\sigma_3, d\sigma_4)$ and the integral

$$\iiint s^n\, d\sigma_n,$$

is of importance in the theory.

* [*Added March* 16, 1927.—We have found it possible to extend the view outlined here in a way which leads to the conception of a world-line possessing structure, from which one passes naturally to the idea of a quantum of action. We hope to communicate our conclusions shortly.]

† *Cf.* Eddington, 'The Mathematical Theory of Relativity,' p. 185.

This integral may be shown to be equivalent to the four-dimensional integral

$$\iiiint \text{div. } \mathbf{s}' \,.\, dv.$$

In the case when $\mathbf{s}'$ is the current four-vector div. $\mathbf{s}'$ vanishes. Associated with the current four-vector is a six-vector, which we may regard as a tensor of the second order, known as the Minkowski six-vector. If we denote its typical components by F_{mn} where

$$F_{mm} = 0, \qquad F_{mn} = F_{nm}$$

we have

$$\iint \tfrac{1}{2} F_{mn}\, dA^{mn} = \iiint (\text{Lor. } \mathbf{F})^n \, d\sigma_n,$$

where dA^{mn} is a typical component of the contravariant element of area.

The Lorentz operator is to be understood in its general sense appropriate to generalised space-time.

There is a connection between $\mathbf{s}'$ and $\mathbf{F}$, for the equations

$$(\text{Lor. } \mathbf{F})^n = s^n \; (n = 1, 2, 3, 4)$$

give us half the Maxwell equations of electrodynamics in their most general form. The other half are given by

$$(\text{Lor. } \mathbf{G})^n = 0.$$

We might combine these two and write

$$\text{Lor. } (\mathbf{F} + \mathbf{G})^n = s^n,$$

if it were possible to dissociate $\mathbf{F}$ and $\mathbf{G}$ so as to reproduce the equations

$$(\text{Lor. } \mathbf{F})^n = s^n$$

$$(\text{Lor. } \mathbf{G})^n = 0.$$

The derivation of the general electrodynamical equations may be carried out by remembering that the components of the two reciprocal tensors are respectively

$$F^{23} = H_x,\ F^{31} = H_y,\ F^{12} = H_z,\ F^{14} = -E_x,\ F^{24} = -E_y,\ F^{34} = -E_z,$$

$$G^{23} = E_x,\ G^{31} = E_y,\ G^{12} = E_z,\ G^{14} = H_x,\ G^{24} = H_y,\ G^{34} = H_z.$$

We may note that the equations

$$\iint \tfrac{1}{2} G_{mn}\, dA^{mn} = 0,$$

and

$$\iiint s^n d\sigma_n = 0,$$

or what is the same thing

$$\text{Lor. } \mathbf{G} = 0$$

and

$$\text{div. } \mathbf{s}' = 0$$

are similar relations, applying to two and three-dimensions respectively. Lor. $\mathbf{G} = 0$ is not sufficient for the description of the electromagnetic phenomena, we require to add to it the equation

$$\text{Lor. } \mathbf{F} = \mathbf{s}'.$$

We shall suppose that in the same way div. $\mathbf{s}' = 0$ is also insufficient to cover all the phenomena possible in nature, and shall suppose that we must consider another four-vector $\mathbf{r}'$ whose divergence does not vanish but is equal to some quantity ψ which is, of course, a scalar.

We thus complete our equations by adding

$$\text{div. } \mathbf{r}' = \psi.$$

The system of equations so obtained is strikingly simple and uniform. It is

$$\left.\begin{aligned} \text{div. } \mathbf{s}' &= 0 \\ \text{div. } \mathbf{r}' &= \psi \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{Lor. } \mathbf{G} &= 0 \\ \text{Lor. } \mathbf{F} &= \mathbf{s}' \end{aligned}\right\}$$

and to these we add the equation for the four-vector potential

$$\text{curl } \mathbf{a}_1 = \mathbf{F}.$$

In these equations, ψ may be regarded as the most fundamental quantity. It is to be associated with four-dimensional volume elements in the same way that $\mathbf{s}'$ is associated with three-dimensional volume elements.

We may combine the first two equations and write

$$\text{div. } (\mathbf{s}' + \mathbf{r}') = \psi,$$

and regard the two vectors as combined into one. Of this single vector the part $\mathbf{s}'$ controls the phenomena of electrodynamics, it is related to $\mathbf{F}$ by the fourth equation and through $\mathbf{F}$ to the vector potential $\mathbf{a}$.

In empty space, at any rate, we may deduce $\mathbf{G}$ from it for $\mathbf{G}$ and $\mathbf{F}$ are simply related in this case, $\mathbf{G}$ being the reciprocal of $\mathbf{F}$.

It would be a very simple scheme of things if we could refer $\mathbf{s}'$ one step farther back to some fundamental scalar ψ, and thus complete our geometrical conception of the universe by associating ψ with the four-dimensional volume.

We cannot do this directly, but we can refer to ψ a fundamental vector $(\mathbf{s}' + \mathbf{r}')$ which is such that its divergence is ψ.

This is what is meant by stating that of all these quantities ψ is the most fundamental. ψ cannot be a perfectly arbitrary scalar, it must satisfy some equation, this equation being the mathematical expression of a law of nature. We cannot hope to deduce such a law by a series of logical deductions applied to a four-dimensional world, for it will be of the same character as the principle of least action, the law of gravitation or the laws of thermodynamics, the truth of which rests upon experimental results.

Let us try the law contained in the equation

$$\mathbf{s}' + \mathbf{r}' = \kappa \text{ grad. } \psi,$$

where κ is a constant to be determined by application to special problems.

We therefore deduce the differential equation

$$\text{div. } (\mathbf{s}' + \mathbf{r}') = \psi = \kappa \text{ div. (grad. } \psi).$$

This is the relation which ψ must satisfy, and it is the fundamental law we are seeking.

The operations of divergence and gradient are to be understood in their general significance.

Grad. ψ is strictly a vector with components $\partial\psi/\partial x_m$, etc., whereas the divergence is an operator concerning a contravariant vector so that (grad. ψ) in this equation must be regarded as having for its typical component the quantity $(g^{mn}\,\partial\psi/\partial x_m)$, and the expression div. (grad. ψ) is equivalent to

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_m}\left(\sqrt{g}.g^{mn}\frac{\partial\psi}{\partial x_n}\right).$$

Thus the equation for ψ is

$$\frac{\kappa}{\sqrt{g}}\frac{\partial}{\partial x_m}\left(\sqrt{g}.g^{mn}\frac{\partial\psi}{\partial x_n}\right) = \psi.$$

It is significant that this is the four-dimensional mode of writing Shroedingers' wave equation.

The equation div. $\mathbf{s}' = 0$, is the equation of continuity of electricity. If we are to combine $\mathbf{s}'$ and $\mathbf{r}'$ into a single vector we must be careful that the relation

$$\text{div. } (\mathbf{s}' + \mathbf{r}') = \psi,$$

does not contradict the macroscopic equation of continuity. This appears to impose upon ψ a microscopic variation so that ψ takes positive and negative values, and so that macroscopically

$$\overline{\text{div.}\,(\mathbf{s}' + \mathbf{r}')} = \overline{\psi} = 0,$$

where the bars denote average values.

Possibly this may be associated with a periodicity in time for the quantity ψ, corresponding with conclusions reached by Schroedinger. ('Ann. d. Physik,' vol. 81, p. 135 (1926).)

It may be that this is connected with the existence of point charges in space, and, if this is so, doubtless the electronic charge will enter into the relation. There will then be the possibility of linking up in this theory the fundamental electronic charge and the element of action, h, which determines the value of κ.

We must confess that we do not yet understand this point very clearly, but the theory would appear to admit of interesting enquiries in this direction.

It has been recognised since the first introduction of the Quantum Theory that a new element had to be introduced into Physics.

The point of view taken is that the four-vector $\mathbf{s}'$ is incomplete, and must be extended by the introduction of a four-vector $\mathbf{w}'$ made up of $\mathbf{s}'$ and the additional $\mathbf{r}'$, so that

$$\mathbf{w}' = \mathbf{s}' + \mathbf{r}' .$$

$\mathbf{r}'$ is the new element introduced.

It seems possible in this way to include in one uniform scheme gravitational, electrical and quantum phenomena.