4. It is assumed as a working hypothesis that in the active film, formed at a low temperature, some of the molecules of silver oxide are in relatively unstable positions, but that on raising the temperature, the arrangement of these molecules becomes a more stable one, and the film in consequence becomes less chemically active.

5. The mechanism of the catalysis of the union of hydrogen and oxygen by the film is one of alternate reduction and reoxidation of the film.

Experiments are now being conducted with a view to determining more precisely the character of the change which takes place when an unstable oxide film becomes stable.

A Detailed Study of the "Radioactive Decay" of, and the Penetration of α-Particles into, a Simplified One-Dimensional Nucleus.

By R. H. Fowler, F.R.S., and A. H. Wilson.

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§ 1. *Introduction.*—Gamow's* elegant deduction by general arguments of the law of radioactive decay by α-particle emission and his subsequent investigations on artificial disintegration suggested to us the desirability of investigating as closely as possible any simple model of a decaying nucleus as a verification of his general approximations. For the model chosen the exact investigation of the decay process is almost trivial. Since we obtained this, now some time ago, Dr. Gamow informed us that he had also obtained equivalent detailed results. Still more recently such results have been published by Kudar.† We shall not therefore dwell upon them here. The application of the same ideas, however, to the reverse process of penetration presents points of very definite interest, which we think are well worth discussion. The main point that arises is that the chance of penetration depends (or appears to depend) on whether the energy of the incident α-particle is or is not equal to a characteristic energy of the nucleus itself. This is a point which is not dealt with by

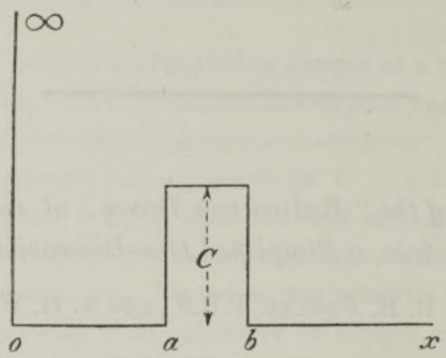
* Gamow, 'Z. Physik,' vol. 51, p. 204 (1928); Gamow and Houtermans, 'Z. Physik,' vol. 52, p. 496 (1928); Gamow, 'Z. Physik,' vol. 52, p. 510 (1928).

† Kudar, 'Z. Physik,' vol. 53, pp. 61, 95, 134 (1929).

Gamow in his paper. We have discussed it with him, and now put forward the results we have obtained.*

Since the solution of the decay problem is required in the main discussion of the penetration of α-particles into the nucleus it is included here in § 2 for reference. We must emphasise that we claim no novelty, except of detail, for the work of § 2; the general lines by now are a matter of fairly common knowledge.

§ 2. *Exact Solution of a Simple Problem of Radioactive Decay.*—We consider a one-dimensional problem with the potential energy U shown in the figure.



The solution we require is to represent at $t = 0$ as nearly as possible a standing wave confined to the " nucleus " between 0 and a, of energy E_0 ($< C$) representing one α-particle of this energy. To the right it is to contain waves travelling to the right only, representing the possibly escaping α-particle. All the wave-functions may be affected by a decay factor $e^{-\lambda' t}$, $\lambda' > 0$. We proceed to verify by construction that such solutions exist.

The wave function satisfies the equation

$$\frac{\partial^2 \Psi}{\partial x^2} - \kappa^2 \left\{ U + \frac{h}{2\pi i} \frac{\partial}{\partial t} \right\} \Psi = 0 \qquad \left(\kappa^2 = \frac{8\pi^2 m}{h^2} \right), \tag{1}$$

where m is the mass of the α-particle. We assume a time factor of the usual form, so that

$$\Psi = \psi e^{-2\pi i E t/h}, \tag{2}$$

but allow that

$$E = E_0 - i\lambda'. \tag{3}$$

* We should like to record here that Mr. R. W. Gurney has also observed in a letter to us that the difference between α-particles with energies equal or unequal to a characteristic of the nucleus had been overlooked by Gamow in his original paper on penetration.

For $0 < x < a$, since $\psi(0) = 0$ on the rigid boundary,

$$\Psi = \mathrm{A} \sin \kappa\sqrt{\mathrm{E}}x\, e^{-2\pi i \mathrm{E}t/h}. \tag{4}$$

For $a < x < b$,

$$\Psi = [\mathrm{B}e^{-\kappa\sqrt{(\mathrm{C}-\mathrm{E})}(x-a)} + \mathrm{B}'e^{\kappa\sqrt{(\mathrm{C}-\mathrm{E})}(x-a)}]\, e^{-2\pi i \mathrm{E}t/h}. \tag{5}$$

For $x > b$,

$$\Psi = \mathrm{D}e^{i\kappa\sqrt{\mathrm{E}}(x-b)-2\pi i\mathrm{E}t/h}. \tag{6}$$

The conditions of continuity at $x = a$ and $x = b$ are

$$\mathrm{A} \sin \kappa\sqrt{\mathrm{E}}a = \mathrm{B} + \mathrm{B}',$$

$$\mathrm{A} \cos \kappa\sqrt{\mathrm{E}}a = (-\mathrm{B} + \mathrm{B}')\sqrt{\frac{\mathrm{C}-\mathrm{E}}{\mathrm{E}}},$$

$$\mathrm{D} = \mathrm{B}/\vartheta + \mathrm{B}'\vartheta,$$

$$i\mathrm{D}\sqrt{\frac{\mathrm{E}}{\mathrm{C}-\mathrm{E}}} = -\mathrm{B}/\vartheta + \mathrm{B}'\vartheta,$$

where

$$\vartheta = e^{\kappa\sqrt{(\mathrm{C}-\mathrm{E})}(b-a)}.$$

These conditions can all be satisfied if and only if

$$0 = \sin\kappa\sqrt{\mathrm{E}}a + \sqrt{\frac{\mathrm{E}}{\mathrm{C}-\mathrm{E}}}\cos\kappa\sqrt{\mathrm{E}}a$$
$$-\frac{1}{\vartheta^2\mathrm{C}}[\mathrm{C}-2\mathrm{E}+2i\sqrt{\{\mathrm{E}(\mathrm{C}-\mathrm{E})\}}]\left[\sin\kappa\sqrt{\mathrm{E}}a - \sqrt{\frac{\mathrm{E}}{\mathrm{C}-\mathrm{E}}}\cos\kappa\sqrt{\mathrm{E}}a\right]. \tag{7}$$

The condition that the solution shall be nearly a standing wave confined to the "nucleus" $0 < x < a$ is that ϑ is very large. The characteristics of the problem—the roots of (7)—are therefore to be obtained on that basis. To a first approximation, $\vartheta = \infty$, the values of E are the roots of

$$\tan\kappa\sqrt{\mathrm{E}}a = -\sqrt{\frac{\mathrm{E}}{\mathrm{C}-\mathrm{E}}}, \tag{8}$$

or

$$\kappa\sqrt{\mathrm{E}}a = n\pi - \arctan\sqrt{\frac{\mathrm{E}}{\mathrm{C}-\mathrm{E}}}, \tag{9}$$

the arc tan being taken between 0 and $\frac{1}{2}\pi$. We denote any one of these by E_0. Then very roughly

$$\mathrm{E}_0 = \frac{n^2\pi^2}{a^2\kappa^2} = \frac{n^2h^2}{8a^2m}. \tag{10}$$

In the next approximation ϑ is so large that the new real term is trivial and one may retain only the new imaginary term. Putting $E = E_0$ in the small part and using (8), the equation (7) becomes

$$\tan \kappa \sqrt{E}a = -\sqrt{\frac{E}{C-E}} - \frac{4iE_0}{\vartheta^2 C}, \tag{11}$$

which to this approximation is equivalent to

$$\kappa \sqrt{E}a = n\pi - \text{arc tan}\left\{\sqrt{\frac{E}{C-E}} + \frac{4iE_0}{\vartheta^2 C}\right\},$$

$$= n\pi - \text{arc tan}\sqrt{\frac{E}{C-E}} - \frac{4i(C-E_0)E_0}{\vartheta^2 C^2}. \tag{12}$$

Hence $E = E_0 - i\lambda'$ where

$$\lambda' = \frac{8(C-E_0)E_0^2}{\vartheta^2 C^2}\frac{1}{\kappa\sqrt{E_0}a + \sqrt{\dfrac{E_0}{C-E_0}}}. \tag{13}$$

In order that ϑ should be large it is not necessary that $C \gg E_0$, but obviously we may assume that it is not true that $C - E_0 \ll E_0$. Thus the last denominator in (13) is nearly $n\pi$.

We now return to the conditions of continuity, which reduce with sufficient accuracy to

$$B = \tfrac{1}{2}A\left(\sin \kappa\sqrt{E}a - \sqrt{\frac{E}{C-E}}\cos\kappa\sqrt{E}a\right) = (-)^{n-1}A\sqrt{\frac{E_0}{C}}, \tag{14}$$

$$B' = \frac{B}{\vartheta^2 C}[C - 2E_0 + 2i\sqrt{\{E_0(C-E_0)\}}], \tag{15}$$

$$D = (-)^{n-1}\frac{2A}{\vartheta C}[C - E_0 + i\sqrt{\{E_0(C-E_0)\}}]\sqrt{\frac{E_0}{C}}. \tag{16}$$

The absolute values must be fixed by a normalising condition at $t = 0$, which is necessarily partly arbitrary. If we take it to be

$$\int_0^b \psi\psi^*\, dx = 1,$$

and neglect terms of order ϑ^2 compared with the leading terms we find

$$A|^2 = \frac{2}{a}\frac{1}{1 + 1/\{\kappa\sqrt{(C-E_0)}a\}}. \tag{17}$$

This is the same result as we should find from the natural normalising condition

$$\int_0^\infty \psi\psi^*\, dx = 1$$

for a potential barrier of height C extending from $x = a$ to $+\infty$. We can only obtain (17) by integrating far enough past $x = a$ for the wave-functions to have become negligible. The result (17) is certainly right. Of course ψ^* is determined by an equation whose characteristics are $E^* (= E_0 + i\lambda')$ but the differences between E, E* and E_0 do not give significant terms in the normalisation at $t = 0$.

We can now check up Gamow's interpretation of λ' and its connection with the outward flux of particles. On account of λ', $\Psi\Psi^*$ contains the time factor

$$e^{-2\pi i(E-E^*)t/h} = e^{-4\pi\lambda' t/h}.$$

The solution normalised to unity in the nucleus at $t = 0$ decays thereafter at a rate $e^{-\lambda t}$, where

$$\lambda = \frac{4\pi\lambda'}{h} = \frac{16\,(C - E_0)\,E_0^{\frac{3}{2}}}{\sqrt{(2m)}\,a\vartheta^2 C^2}\,\frac{1}{1 + 1/\{\kappa\sqrt{(C - E_0)}\,a\}}. \tag{18}$$

On the other hand the outward flux in particles per second is given by the well known formula

$$\frac{h}{4\pi mi}.\{\Psi^* \operatorname{grad} \Psi - \Psi \operatorname{grad} \Psi^*\}.$$

To evaluate this we have

$$\Psi = D\,e^{i\kappa\sqrt{E}(x-b)-2\pi i Et/h}, \qquad \Psi^* = D^*\,e^{-i\kappa\sqrt{E^*}(x-b)+2\pi i E^* t/h},$$

so that the flux is equal to

$$2\sqrt{\frac{E_0}{2m}}\,|D|^2 e^{-\lambda t + \lambda\sqrt{\frac{m}{2E_0}}(x-b)}. \tag{19}$$

The flux past a given plane must, of course, decay at rate $e^{-\lambda t}$ and must depend on the position of the plane as shown by (19), since, as Gamow has pointed out, particles reaching that plane at a given time correspond to a state of affairs at a previous time in the nucleus. Near the nucleus and near $t = 0$ the flux is therefore

$$2\sqrt{\frac{E_0}{2m}}.\frac{4E_0\,(C - E_0)}{\vartheta^2 C^2}.\frac{2}{a}\,\frac{1}{1 + 1/\{\kappa\sqrt{(C - E_0)}\,a\}} = \lambda.$$

This completes the verification in detail of Gamow's theory for this example, correct at every stage to terms of order ϑ^{-2}.

§ 3. *The Converse Problem. The Penetration of α-particles into the Nucleus.*— So far we have merely put down, possibly with new details, results due to Gamow in a form convenient for our purpose. We will now try to study more thoroughly than he has yet done, the converse problem of the penetration of an α-particle into a previously empty "nucleus." We observe at once that when ϑ is large the only possible solutions of the type studied in the last section are those for which the energy is almost exactly a characteristic of the completely isolated nucleus ($\vartheta = \infty$).

We will now investigate a problem in which there is a steady stream of α-particles incident on the nucleus from the right, which are reflected at the nucleus, and at the same time, of course, maintain a standing wave in the nucleus itself. This problem, one finds at once, is really steady, has real E but no characteristics. Solutions exist for all real E. With the potential energy of the figure we may take

$$\Psi = A_1 \sin \kappa \sqrt{E} x \, e^{-2\pi i E t/h} \quad (0 < x < a), \tag{20}$$

$$\Psi = [B_1 e^{-\kappa\sqrt{(C-E)}(x-a)} + B_1' e^{\kappa\sqrt{(C-E)}(x-a)}] \, e^{-2\pi i E t/h} \quad (a < x < b), \tag{21}$$

$$\Psi = [D_1 e^{-i\kappa\sqrt{E}(x-b)} + D_1' e^{i\kappa\sqrt{E}(x-b)}] \, e^{-2\pi i E t/h} \quad (b < x). \tag{22}$$

The conditions of continuity at $x = a$ and $x = b$ are

$$A_1 \sin \kappa\sqrt{E} a = B_1 + B_1',$$

$$A_1 \cos \kappa\sqrt{E} a = (-B_1 + B_1') \sqrt{\frac{C-E}{E}},$$

$$D_1 + D_1' = B_1/\vartheta + B_1'\vartheta,$$

$$i(D_1 - D_1') \sqrt{\frac{E}{C-E}} = B_1/\vartheta - B_1'\vartheta.$$

The equations have solutions for all E, and on solving them we find that D_1 and D_1' are conjugate complex quantities,

$$|D_1| = |D_1'|, \tag{23}$$

$$D_1 = \tfrac{1}{4} A_1 \left[\vartheta \left\{ 1 + i \sqrt{\frac{C-E}{E}} \right\} \left\{ \sin \kappa\sqrt{E} a + \sqrt{\frac{E}{C-E}} \cos \kappa \sqrt{E} a \right\} + \frac{1}{\vartheta} \left\{ 1 - i \sqrt{\frac{C-E}{E}} \right\} \left\{ \sin \kappa\sqrt{E} a - \sqrt{\frac{E}{C-E}} \cos \kappa\sqrt{E} a \right\} \right]. \tag{24}$$

Equation (23) characterises a steady state in which the flux of particles in the incident and reflected beams are equal. Equation (24) shows that in general $|A_1/D_1|^2$ is proportional to ϑ^{-2}, but that if E happens to be equal to E_0, the real part of one of the characteristics of the first problem, then

$$D_1 = (-)^{n-1} \frac{A_1}{2\vartheta} \left\{ 1 - i \sqrt{\frac{C-E_0}{E_0}} \right\} \sqrt{\frac{E_0}{C}}. \tag{25}$$

In this case $|A_1/D_1|^2$ is of the order ϑ^2 instead of ϑ^{-2}. Thus the wave function inside the nucleus is negligible when $E \neq E_0$ compared with its value when $E = E_0$, and so the α-particles practically only penetrate into the nucleus (or at least only penetrate and stay in the nucleus, one cannot here determine which) when their energy is almost exactly that of a characteristic of the nucleus—that is, an energy capable of producing a type of resonance. Of course the agreement need not be absolutely exact—in order that $|A_1/D_1|^2$ should be of order ϑ^2 it is necessary that

$$E = E_0\left\{1 + O\left(\frac{1}{\vartheta^2}\right)\right\}. \tag{26}$$

§ 4. *Chance of Entry of an α-particle into an Empty Nucleus.*—We can now determine the rate at which the α-particle enters the nucleus when (26) is satisfied or more correctly the chance of entry of the α-particle, by fitting together the solutions of §§ 2, 3 so as to give a negligible wave-function in $0 < x < b$ at time $t = 0$. To do this we have only to take $A_1 = -A$ and add the two Ψ's. We then find, to a sufficient approximation,

$$\Psi = A_1 \sin \kappa \sqrt{E_0} x \, e^{-2\pi i E_0 t/h} (1 - e^{-\frac{1}{2}\lambda t}) \quad (0 < x < a), \tag{27}$$

with a corresponding wave-function in $a < x < b$. Inside the nucleus $\Psi\Psi^*$ varies like $(1 - e^{-\frac{1}{2}\lambda t})^2$, which gives the variation with time of the number of particles inside, or rather the chance that the α-particle has entered between the times 0, t. It will be noticed that this number or chance increases initially as t^2, whereas a variation like t rather than t^2 might have been expected. But this is a usual feature of a resonance problem. The proper quantum theory interpretation will appear shortly.

The corresponding incident and reflected beams of α-particles are of intensities controlled by $|D_1|^2$ and $|D + D_1'|^2$ respectively, and it is easily verified after reduction that

$$|D + D_1'|^2 = |D_1|^2\left\{1 - \frac{16(C - E_0)E_0}{C^2}(e^{-\frac{1}{2}\lambda t} - e^{-\lambda t})\right\}.$$

The total number of α-particles incident in the interval 0, t_0 is $\sqrt{\frac{2E_0}{m}}|D_1|^2 t_0$, and the number reflected is therefore

$$\sqrt{\frac{2E_0}{m}}\int_0^{t_0}|D + D_1'|^2\,dt$$
$$= \sqrt{\frac{2E_0}{m}}|D_1|^2 t_0 - \sqrt{\frac{2E_0}{m}}|D_1|^2\frac{16(C - E_0)E_0}{\lambda C^2}(1 - e^{-\frac{1}{2}\lambda t_0})^2. \tag{28}$$

It is easily verified that the last term is equal to

$$\left[\int_0^b \psi\psi^* \, dx\right]_{t_0},$$

so that there is the necessary conservation theorem, the excess of the incident over the reflected particles being equal to the "number" retained in the nucleus.

We have thus obtained a consistent set of formulæ which involve a hitherto unspecified time t_0, and it is now necessary to attempt a proper interpretation. In the first place one observes that the occurrence of t_0 is a necessary consequence of Heisenberg's uncertainty principle and equation (26). Our formulæ only hold on the assumption that we know the energy of the α-particle with an error ΔE of order given by

$$\Delta E = E_0/\vartheta^2.$$

Such α-particles can only be associated with wave trains of length Δt, of order

$$\frac{h\vartheta^2}{2\pi E_0}, \tag{29}$$

and this Δt may therefore be taken to be the t_0 which occurs in our formulæ. The chance that such an incident α-particle stays in the nucleus is then represented after (28) by

$$\frac{8\,(C - E_0)\,E_0}{C^2}\,\frac{(1 - e^{-\frac{1}{2}\lambda t_0})^2}{\frac{1}{2}\lambda t_0}.$$

Using (29) we find that

$$\tfrac{1}{2}\lambda t_0 = \frac{8\,(C - E_0)\,E_0^{\frac{1}{2}}}{C^2}\,\frac{1}{\kappa a + (C - E)^{-\frac{1}{2}}}.$$

This is very rough and a sufficiently accurate value when the hump is high is

$$\tfrac{1}{2}\lambda t_0 = \frac{8\,(C - E_0)\,E_0}{C^2}\,\frac{1}{\kappa a\sqrt{E_0}}.$$

With reasonable numerical values this is moderately large and $\kappa a\sqrt{E_0}$ moderately small. The chance of capture then approximates to

$$\kappa a\sqrt{E_0}.$$

For other values of E than E_0 we should conclude that the chance of capture is negligible.

After having drawn this distinction between $E = E_0$ and $E \neq E_0$ we have to ask if it has any significant application to any conceivable experiment. We must confess at once that it cannot be of practical importance. The length of wave-train required by (29) is so long that our investigation cannot refer to the conditions of any conceivable experiment. We must be content therefore in this investigation in having found and discussed certain interesting theoretical features of the problem of artificial disintegration which were overlooked in earlier investigations.

Summary.

In this note we solve exactly for a simplified nucleus the problem of α-particle disintegration (determination of the *complex* characteristics of the wave-equation with the proper boundary conditions). We then proceed to discuss the converse problem of the penetration of an α-particle into the nucleus from without—a problem which is the basis of Gamow's theory of *artificial* disintegration. We find that the problem *formally* depends very much on whether the energy of the incident α-particle is or is not equal to the real part of a characteristic of the nucleus bombarded. We show that this type of penetration has features characteristic of a resonance effect as one should expect. We show finally, however, that our solutions are not of *practical* importance as they do not correspond to the conditions of any conceivable experiment. Further progress has been made in a forthcoming paper by Gamow and Houtermans.