Mass Defect Curve and Nuclear Constitution.

By G. Gamow, Rockefeller Foundation Fellow, Cambridge.

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In the discussion before the Royal Society on the constitution of the atomic nucleus held on February 7, 1929,* I proposed a simple model of a nucleus built from α-particles in a way very similar to a water-drop held together by surface tension. A certain number of protons (not more than three) and electrons can be bound to such an α-aggregate without forming a new α-particle.† Such additional units of nuclear constitution, usually bound less strongly than those involved in the α-particles, we shall term free nuclear protons and electrons. Their presence will, of course, affect the form of the nuclear energy curve (mass defect curve), not changing, however, its general shape.

In the present paper I shall attempt to treat the problem more closely, analysing from the theoretical point of view the experimental facts concerning the nuclear energy.

To begin with we shall treat a nucleus built from a certain number of α-particles only.‡ To explain the possibility of a staple configuration of positively charged α-particles we must assume some attractive forces, which come into play only for a close approach of two α-particles and overbalance at short distances the forces due to electrostatic repulsion. We have experimental evidence of the presence of these forces in the investigations on the anomalous scattering of α-particles in helium in the case of very close collision.§ Although the law of these forces is not yet exactly determined, we can write the mutual potential energy of two α-particles at a distance r apart in the form

$$U(r) = + 4e^2/r - f(r), \tag{1}$$

the function $f(r)$ decreasing very quickly with distance.

In the classical treatment this additional potential is usually assumed to

* 'Roy. Soc. Proc.,' A, vol. 123, p. 386 (1929); see also 'Phys. Z.,' vol. 30, p. 717 (1929).

† The number of additional protons and electrons can be simply estimated from the nuclear mass (atomic weight) and charge (atomic number).

‡ This case we have for the first 10 elements of atomic weight $4n$ [He_4; B_8 (not known to); C_{12}; O_{16}; Ne_{20}; Mg_{24}; Si_{28}; A_{32}; Ca_{40}]. For the neighbouring elements a certain number of free electrons is always present (see fig. 2).

§ Rutherford and Chadwick, 'Phil. Mag.,' vol. 4, p. 605 (1927).

vary as an inverse high power of distance (polarisation forces, magnetic forces). The modern quantum theory of interaction between two complex particles gives a rather complicated expression for the mutual potential energy.† We have here two kind of forces corresponding to symmetrical antisymmetrical solutions of the wave equation. Both solutions show a strong repulsion at the distances compared with the dimension of particles. At greater distances the symmetrical solution gives an attraction decreasing exponentially with distance, the antisymmetrical one a repulsion of the same type. For our case of α-particles we have to accept the first solution and write the additional potential energy in the form‡ :

$$-f(r) \sim -\mathrm{A}e^{-ar}. \tag{1A}$$

Examining the behaviour of a collection of particles attracting one another with the forces very rapidly decreasing with distance (we neglect at first the coulomb forces which are comparatively small at nuclear distances) we can introduce the well-known ideas made use of in the theory of capillarity. Calculating the potential energy of a particle inside a certain space in which the others are distributed nearly uniformly we must take into account only the action of particles inside a small sphere surrounding the particle in question. In fact the potential energy u of our α-particle is given by the integral :

$$u = -\int_d^\infty f(r) \,.\, 4\pi r^2 \rho \, dr, \tag{2}$$

where d is the smallest distance between the particles and ρ the density of particles. In our case, where the forces diminish rapidly with distance, the integral converges rather quickly and practically need be taken only up to a certain distance r^*. The sphere of radius r^* is well known in the theory of capillarity as "the sphere of molecular action." We can say that the particle inside the liquid has no resultant force acting on it if the distance from the boundary is greater than r^*.

In the surface region very strong forces arise, trying to drag the particle inside the liquid (surface tension). Such a collection of α-particles will be very like a minute drop of water where the inside pressure, due to the kinetic energy of quantised motion, is in equilibrium with the forces of surface-tension trying to diminish the drop-radius.

† See Heitler and London, 'Phys. Z.,' vol. 44, p. 455 (1927), and Sugiura, 'Phys. Z.,' vol. 45, p. 484 (1927), where the simplest case of interaction between two hydrogen atoms is treated.

‡ The sign " ~ " means here an approximate equality.

Of course, in our case, where the number of particles in the drop is rather small, the thickness of the surface layer is of the same order of magnitude as the drop radius; the velocity and the density of the particles decreasing regularly towards the drop-boundary.

The important point for the nuclear drop-model is the question of the quantum number to be ascribed to the different α-particles in the drop. The solution of this question is very simple; all α-particles in the nucleus must be considered to be in the same state with quantum number unity. This is due to the fact that the Pauli principle, which requires the electrons in an atom to be distributed in different shells, is not applicable to α-particles since they carry an even charge.* The number of particles on the same level being not limited, we have to expect that in the ordinary nuclei (not excited ones) all α-particles are in the same state of smallest energy, that is, in the first quantum level.

We shall now, very roughly, and only as regards the order of magnitude, estimate the general behaviour of such a model. If r_0 is the radius of a nucleus consisting of N_α α-particles the average momentum p and kinetic energy κ of a single particle are given by

$$p \sim h/2r_0 \tag{3}$$

and

$$\kappa \sim \frac{1}{2m} p^2 \sim \frac{h^2}{8mr_0^2}. \tag{4}$$

In order to estimate the average potential energy we must remember that the particle coming to the boundary of the drop loses all its kinetic energy. Now, in the case of absolutely sharp walls and constant potential inside, the potential energy of a particle on the drop-boundary is a half of its potential energy inside (as only one half of the "sphere of action" is acting). In this case the kinetic energy of the particle ought to be equal to one-half of the absolute value of potential energy inside. In the real case, with the density of particles in the drop diminishing slightly towards the surface, this relation will hold only approximately and we write:

$$u \sim -2\kappa \sim -h^2/4mr_0^2. \tag{5}$$

* It can be shown from general principles of wave mechanics that in considering a collection of similar particles, each constructed from a certain number of protons and electrons, the Pauli principle must be applied to these particles *only in the case* when the total number of protons and electrons in each is an *odd* number, or in other words when the resultant charge of these particles is odd. [See W. Heisenberg, 'Rapportes et discussions de 5me conseil de physique, Solvay,' p. 271 (1928).]

The whole energy of the particle being

$$u + \kappa \sim - h^2/4mr_0^2. \tag{6}$$

For the preliminary calculation of drop-equilibrium we shall use the Debye formula for surface tension deduced for inverse nth power law of forces*

$$\sigma \sim \left(\frac{N_\alpha}{r_0^3}\right)^2 \frac{A}{d^{n-4}}. \tag{7}$$

Assuming

$$d \sim r_0/N_\alpha^{1/3} \tag{7'}$$

we have

$$\sigma \sim AN_\alpha^{(n+2)/3}/r_0^{n+2}. \tag{7''}$$

Comparing this with the inside pressure which is equal to

$$P \sim \tfrac{2}{3}\kappa\rho \sim \frac{h^2}{m}\frac{N_\alpha}{r_0^5} \tag{8}$$

we have

$$\frac{h^2 N_\alpha}{mr_0^5} \sim \frac{AN_\alpha^{(n+2)/3}}{r_0^{n+2}} \tag{9}$$

from which

$$r_0 \sim \left(\frac{Am}{h^2}\right)^{1/(n-3)} . N_\alpha^{(n-1)/3(n-3)} \sim R_0 . N_\alpha^{(n-1)/3(n-3)}. \tag{9'}$$

The formula (9′) gives the increase of the drop-radius with the number of α-particles, the radius being (for great values of n) nearly proportional to the cube root of N_α. This result fits in with the data about the nuclear radii of different elements obtained from investigations of the anomalous α-scattering† for light elements and the rate of α-disintegration‡ for heavy ones. Since the law of force between two α-particles is not accurately known, we cannot calculate the value of the coefficient R_0 in the equation (9′). For the further calculations we shall assume the value of R_0 deduced from the investigations mentioned above which give $R_0 = 2 . 10^{-13}$ cm.

The total energy of the drop will be now

$$E_\alpha \sim N_\alpha . (u + \kappa) \sim - \frac{h^2 N_\alpha}{4mr_0^2} \tag{10}$$

or, taking r_0 from (9′)

$$E_\alpha \sim - \frac{h^2}{4mR_0^2} . N_\alpha^{1/3}. \tag{10'}$$

* The calculation with exponentially decreasing forces being more complicated gives nearly the same result.

† Bieler, 'Roy. Soc. Proc.,' A, vol. 105, p. 434 (1924); Hardmeier, 'Phys. Z.,' vol. 28, p. 181 (1927); Gamow, 'Z. Physik,' vol. 52, p. 510 (1928).

‡ Gamow and Houtermans, 'Z. Physik,' vol. 52, p. 496 (1928); Atkinson and Houtermans, 'Z. Physik,' vol. 58, p. 478 (1929).

According to (10′) the value of $\Delta E_a/\Delta N_\alpha$ is always negative; the addition of a new α-particle to the drop is an exothermic process and our nuclear model must be stable however heavy it is.

We obtain a different result by taking into account the coulomb repulsive forces which become of importance for heavy nuclei.* The potential energy of an approximately uniformly charged sphere of radius r_0 and charge $+ 2eN_\alpha$ is

$$E_r \sim (2eN_\alpha)^2/r_0 \tag{11}$$

or according to (9′)

$$E_r \sim \frac{(2e)^2}{R_0} N_\alpha^{5/3}. \tag{11'}$$

It is seen that the coulomb energy increases with N_α more rapidly than the energy due to attractive forces and cannot be neglected for heavy nuclei. The whole nuclear energy is given by the formula

$$E \sim E_a + E \sim -\frac{h^2}{4mR_0^2} N_\alpha^{1/3} + \frac{(2e)^2}{R_0} N_\alpha^{5/3} \tag{12}$$

showing the increase for greater values of N_α.

The calculations given above are, of course, only very preliminary. The accurate wave mechanical treatment of this model is to be carried out by means of solution of Hartree's self consistent equation† of the form

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{8\pi^2 m}{h^2}\left[E - (N_\alpha - 1)\int \Psi\bar{\Psi}\, U(r)\, dr\right]\Psi = 0 \tag{13}$$

where the terms in brackets represent the action of the other $(N_\alpha - 1)$ particles. The solution of this equation, with many simplifying assumptions about the law of forces was carried out by D. R. Hartree. The equation yields discrete energy levels giving the energy and nuclear radius of the right order of magnitude. The exact solution of the equation (13) which is necessarily rather complicated cannot be obtained at present in consequence of our meagre knowledge as to the law of the forces between α-particles. For our present purposes, the rough estimate of the general shape of the α-drop energy curve given by previous calculation must be quite sufficient.

The energy as calculated from (12) is shown in fig. 1. The points represent

* The coulomb forces being comparatively feeble inside the nucleus do not have much effect on the structure of the nucleus. They are, however, responsible for the potential drop outside the nucleus which gives the α-particle a chance to escape through the potential barrier surrounding the nucleus.

† Hartree, 'Proc. Camb. Phil. Soc.,' vol. 24, pp. 89, 111 (1927).

the experimental data calculated from the mass defects measured by Aston,* namely, the whole mass of the nucleus less the sum of the masses of α-particles

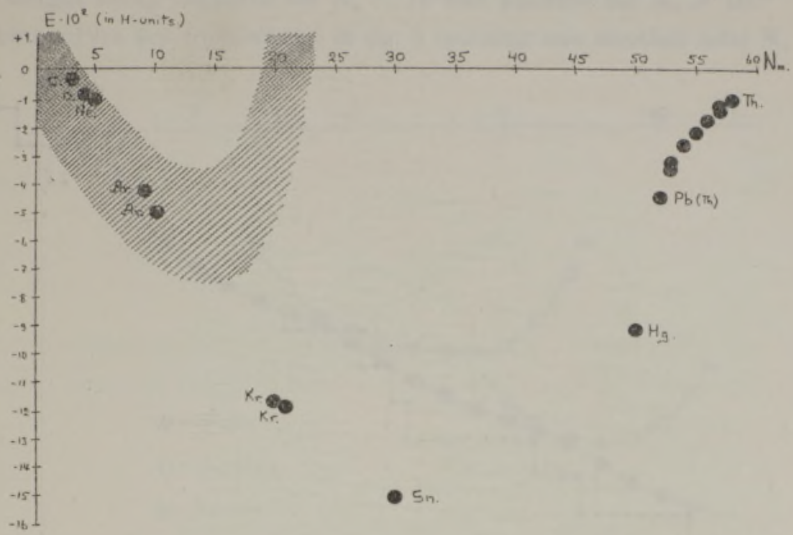

Fig. 1.

and electrons contained in it. We see that the theoretical "curve" fits more or less with the experimental points in the region of light elements, but rises too soon to the zero-axis with increasing mass of the nuclei. This deviation can hardly be due to the roughness of our calculations, but can be simply explained if we remember that for heavier elements the free nuclear electrons come into play. In fact, if the energy curve marked by the experimental points is due entirely to the energy of α-particles, the electrons giving only a comparatively small part of the whole energy, we have to expect a spontaneous α-disintegration to begin for the elements heavier than 120 where the curve has its minimum. The rate of these transformations which can be approximately estimated from the known energy difference would be rather rapid, and the radioactivity of these elements ought to be observed easily if these views were correct. Actually, as we know, none of the elements between atomic weights 120 and 208 are found to be radioactive.

To get over this difficulty we have to turn our attention to the nuclear electrons. The number of free nuclear electrons in nuclei of the type $4n$ is

* The data for mass defects utilised in this paper are taken from Aston's well-known paper, 'Roy. Soc. Proc.,' A, vol. 115, p. 487 (1927).

plotted in fig. 2 against the number of α-particles.* We perceive a striking regularity in it.† The number of free nuclear electrons is always even, a new

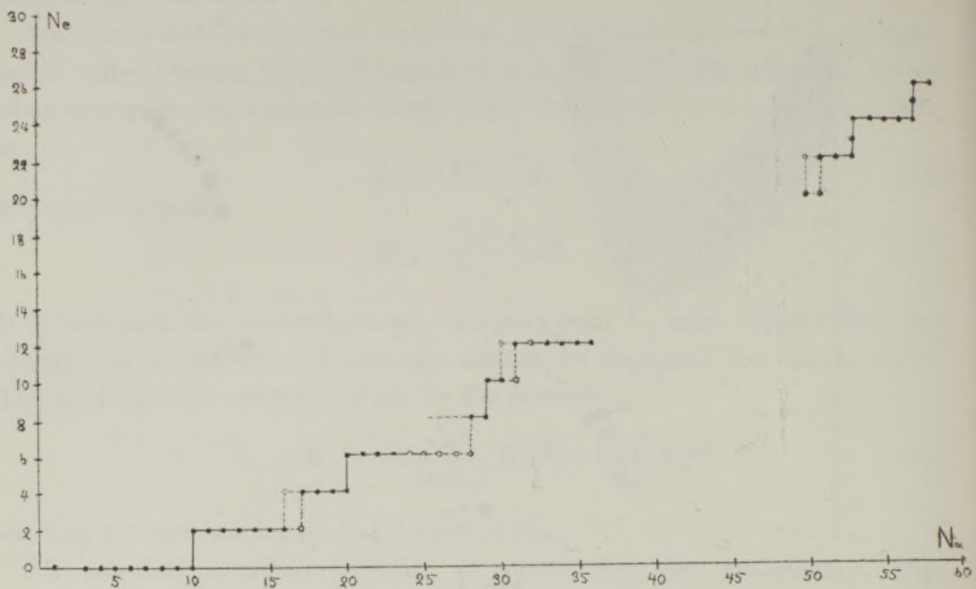

FIG. 2.

pair appearing at certain atomic weights. This fact can be interpreted as follows. For the electrons moving in the α-drop, only the first few levels have negative energy and are stable. According to the Pauli principle, two and only two electrons will stay on each such level.

The final number of negative energy levels in our case is evidently due to the characteristic distribution of potential which is quite different from what we have in the atomic electron system. For the electron inside our drop-model, the potential shows a sharp jump at the boundary and is nearly constant inside. We have here a potential hole with more or less steep walls. In this case the kinetic energy of different quantum levels will increase approximately as the square of the quantum number and for certain level became greater than the depth of the hole, giving to the electron the possibility of escape from the nucleus; thus we can have only a finite number of stable levels. For heavier nuclei with greater radii the energy steps between different levels will be smaller, and we can expect to have a greater number of stable states.

* The known elements are indicated by full circles; the empty circles represent the hypothetical, not known isotopes.

† G. Beck, 'Z. Physik,' vol. 50, p. 548 (1928).

Let us compare now the energy curve for the bare α-drop and for an α-drop with two electrons in it. The energy difference must vary smoothly with atomic weight being negative for $N_\alpha < 10$ and positive for $N_\alpha > 10$.*

The two curves are represented in fig. 3 crossing one another near $N_\alpha = 10$.

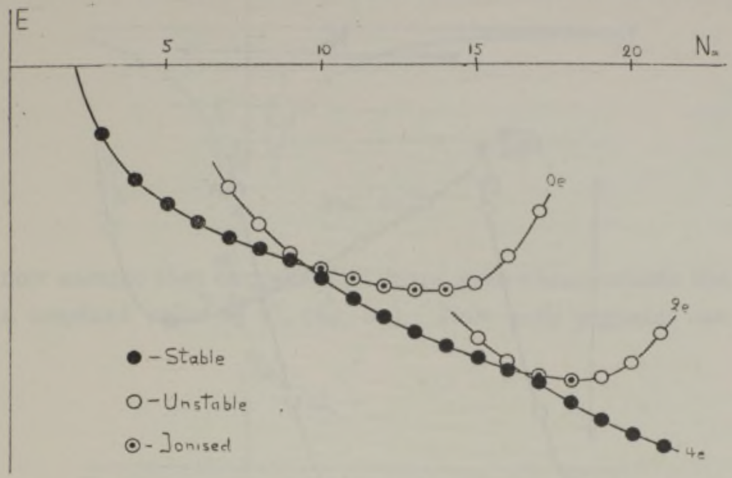

FIG. 3.

The next electron shell appears at $N_\alpha \sim 17$, so that the curve $[N_\alpha \cdot \alpha + 4e]$ must cross the previous curve near this point. The lowest part of our curve (shown by black circles in fig. 3) represent the stable, well-known, nuclei, while the inside branches give either unstable elements with two electrons ready to fly away (left side branches) or the, so to speak, ionised nuclei ready to absorb several pairs of electrons (right side branches). Both of these classes of nuclei are at present unknown though perhaps they could be produced artificially.† Since the curves in fig. 3 are nearly parallel, we cannot hope to perceive the irregularities near the junction of the curves unless very accurate data for mass defects are available. We see that, due to the electronic effect, the stable (descending) part of the energy curve can be continued much farther than for the bare α-drop nucleus.

The most important part of the experimental energy curve is the ascending branch for $N_\alpha > 30$. It seems that the nuclei in this region must inevitably be unstable. The experimental knowledge about this region is, however, rather meagre, most mass defects and even isotopic numbers having not yet been measured. The only accurate values are those for Hg and Pb, showing

* As we know that for $N_\alpha \sim 10$ the first electron level became stable (fig. 2).

† We must notice that the observed β-activity of K and Rb may be connected with existence of such elements.

the general increase of the energy for $N_\alpha > 30$. The only possible way to account for the existence of stable nuclei in this region, in spite of the average increase of the energy, is to assume that the energy curves are of the type shown in fig. 4. We see that the nucleus " A " can eject neither an α-particle (being

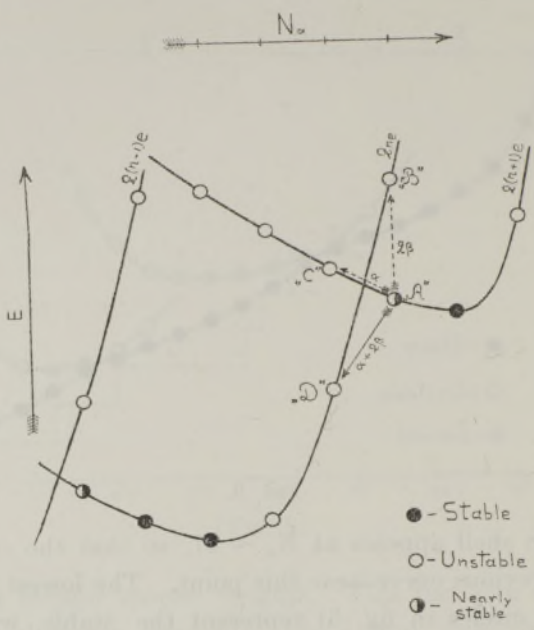

Fig. 4.

transformed to " C ") nor two electrons (being transformed to " B ") since both processes are endothermic. The only possible transformation is to " D " with simultaneous ejection of an α-particle and two electrons, but the probability of such a double disintegration is extremely small. Such, at first sight, curious state of affairs, where, in spite of the total positive energy, one particle cannot get away without leaving the other in the system, will always occur when the particles are not only attracted to the centre of the system, but also attract one another as we see in nuclei.

Let us examine the simple example of two particles moving in an inverse square central field of forces and also attracting each other according to inverse square law (so to speak, helium-model with " attracting electrons "). Both particles are initially in the same level as is shown in fig. 5A ($\underset{1}{\bullet}$ and $\underset{2}{\bullet}$). It is easy to see that if we remove one ($\underset{1}{\circ}$) of the particles (ionise our " atom ") the other ($\underset{2}{\circ}$) will be bound less strongly than before (reverse as in the case for the

real helium atom where the second electron is more difficult to remove than the first one).

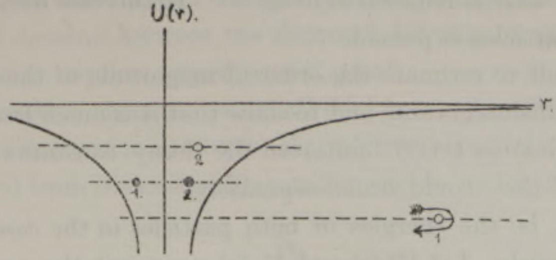

FIG. 5A.

Let us now assume that the potential drops somewhere outside the "atom" reaching a constant value $-U_0$ (fig. 5B). Now both particles can fly away

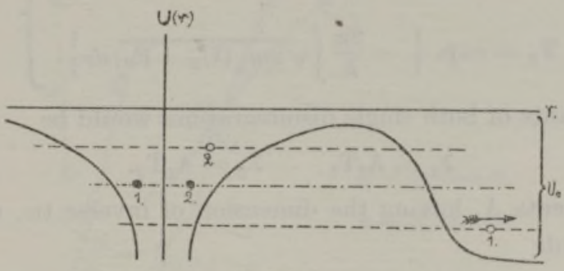

FIG. 5B.

crossing the so arranged potential barrier. We see that the first particle in this case will fly away with an energy smaller than its average energy inside. It will be followed by the second particle carrying a greater amount of energy.

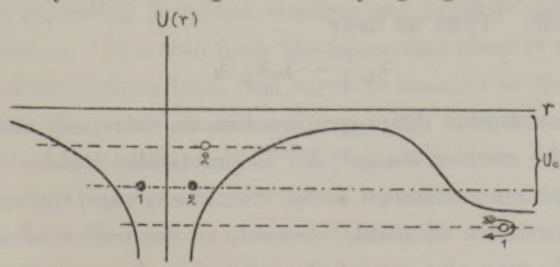

FIG. 5C.

This is the explanation of the known fact that in the radioactive series we always have an increase of disintegration energy (and decay constant) as the α-decay proceeds.

It may be also possible that the energy which either particle would have when the other is left in the system would be insufficient to allow its escape from the " atom " as is demonstrated in fig. 5c. In this case only a simultaneous escape of both particles is possible.

It is not difficult to estimate the order of magnitude of the decay constant of such a double disintegration, and to show that it is much less than even the product (numerical in C.G.S. units) of the decay constants of both single disintegrations if they could occur separately.

Let E_1 and E_2 be the energies of both particles in the case in which they go out simultaneously. Let $U_1(r)$ and $U_2(r)$ represent the potential barriers surrounding the nucleus and preventing the free escape of our particles. The transparency of these barriers is*:

$$\left.\begin{aligned} T_1 &= \exp.\left\{-\frac{2\pi}{h}\int\sqrt{2m_1(U_1-E_1)}\,dr\right\}; \\ T_2 &= \exp.\left\{-\frac{2\pi}{h}\int\sqrt{2m_2(U_2-E_2)}\,dr\right\}. \end{aligned}\right\} \tag{14}$$

The decay constants of both single disintegrations would be

$$\lambda_1 = A_1T_1; \qquad \lambda_2 = A_2T_2. \tag{14'}$$

Here the coefficients A, having the dimension of inverse tie, must be of the order of magnitude

$$A \sim v/r_0 \sim 10^9 \text{ cm. sec.}^{-1}/10^{-12} \text{ cm.} \sim 10^{+22} \text{ sec.}^{-1}. \tag{14''}$$

Now the decay constant of the double disintegration will be

$$\lambda_{1,2} = A_{1,2}\,.\,T_1\,.\,T_2, \tag{15}$$

where $A_{1,2}$ having again the dimension of inverse time, must be of the same order of magnitude. Thus we have

$$\lambda_{1,2} \sim \lambda_1\lambda_2/A \tag{15'}$$

which shows that a double disintegration has an extremely small probability, indicating that the nuclear state " A " is practically stable.

Such an arrangement as shown in fig. 4 is, in any case, the only possible one to explain the existence of stable elements in the region of ascending mass defect curve. We may also expect the existence of some gaps in the series of isotopic numbers corresponding to the unstable regions marked by ○ in fig. 4. Unfortunately the isotopes in this region of atomic weights have not been

* See for example Gamow, 'Z. Physik,' vol. 51, p. 204 (1928); Condon and Gurney, 'Phys. Rev.,' vol. 33, p. 127 (1929).

investigated, and we cannot in consequence give any experimental proof of the accuracy of these deductions. The other consequence is that the energy curve in this region must ascend between the isotopes of some element (differing by $\alpha + 2e$) and descend between the elements differing by one α-particle only, a point which also requires experimental proof.

For the radioactive elements (we shall deal here only with "$4n$"-elements, thorium and its successive products) the relative mass defect of each element can be estimated from the energy liberated in each α- and β-ray transformation.

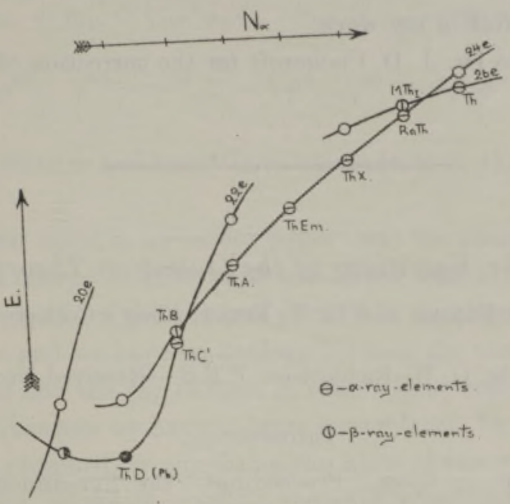

Fig. 6.

The variation of mass defects is shown in fig. 6. If we draw the curves for the elements containing the same number of free electrons, it will be seen that there are three ascending branches crossing one another near the products of β-ray disintegration. It is seen from the figure that there is an easy possibility of one-step transformations from one curve to another at the junction points. This differs from the state of affairs shown in fig. 4 where such a transference requires a highly improbable, double disintegration. We have here the series of successive α-disintegrations interrupted sometimes by two β-ray changes. The transformations obviously end when N_α reaches the first dip of the energy curve.

We have seen that the theory suggests the possibility of formation of a number of radioactive elements (with α-activity) of atomic weight between 120 and 200. The failure to detect these elements is probably due to the fact that all radioactive elements in this region have necessarily a very short life (in

consequence of the very steep rise of the energy curves in this region as shown in fig. 4) and have completely disintegrated.

In order to develop further the point of view proposed in this paper, a more complete knowledge of the distribution of isotopic numbers and more accurate measurement of the masses of the nuclei are required.

Several points treated in this paper are due to the discussion with my friend Dr. L. Landau to whom I should like to express my best thanks.

My thanks are also due to Sir Ernest Rutherford and to Dr. R. H. Fowler for their kind interest in my work.

I am indebted to Dr. J. D. Cockcroft for the correction of the MS.

The Equations of the Quantum Theory.

By J. W. Fisher and H. T. Flint, King's College, London.

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Introduction.

In earlier papers in these 'Proceedings' the five-dimensional system of co-ordinates with a slight modification from that introduced by Klein has been applied to a discussion of certain points and difficulties in the quantum theory.*

In the present paper the work is continued and amplified.

The use of this notation introduces in a natural way the operator

$$\left(\frac{\partial}{\partial x^a} - \frac{2\pi ie}{h}\phi_a\right),$$

which we denote by u_a and which takes the place of $\partial/\partial x^a$ when an electromagnetic field is introduced. The factor mc which is usually introduced arbitrarily into the equations also appears quite naturally.

There is no real objection to the use of such symbolic methods in the attack on problems in physics and the five-dimensional method may also be regarded as symbolic, but it has the advantage that it follows well-known lines and that well-known geometrical terms can be applied to it.

* 'Roy. Soc. Proc.,' A, vol. 123, p. 489 (1929); vol. 124, p. 143 (1929).