continued interest in the work, to Dr. H. W. Melville for criticism and advice, and to the Trustees of the Moray Research Fund of Edinburgh University.

Summary.

The low pressure oxidation of sulphur has been studied, particularly the reaction which was found to be initiated on a hot surface and accompanied by a visible glow. The experimental results showed that the observed rate was independent of the observed pressure of oxygen, but this was found to be due to two compensating factors: (1) the rate of oxidation is proportional to the pressure of oxygen, (2) the rate of inhibition by SO_2 is inversely proportional to the pressure of oxygen, hence the net result is independence of oxygen pressure. A residual pressure was observed and a tentative explanation is offered. The effect of foreign gases was not studied in detail, as the one experiment performed indicated that none obtained.

The Internal Conversion Coefficient for Radium C.

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§ 1. *Introduction.*—It is well known that the γ-rays emitted from a radioactive nucleus are often partially absorbed by the atomic system, giving rise to secondary β-rays. From observations of the resultant γ-ray intensity, and that of the β-rays, it is possible to infer the proportion of γ-rays reabsorbed in the atomic system. This factor is called the "internal conversion coefficient." Its theoretical value has been discussed by Miss Swirles† and R. H. Fowler.‡ Miss Swirles treats the nucleus as an oscillating Hertzian doublet, radiating classically, and considers the radiation field as producing photoelectric transitions in the planetary electrons, according to the Schrödinger theory. The rate of emission of γ-rays from the nucleus is taken to be the classical rate of radiation of energy by the dipole, divided by $h\nu$. The values obtained in this way were about 10 times too small, except for the γ-ray of

† 'Proc. Roy. Soc.,' A, vol. 116, p. 491 (1927), and vol. 121, p. 447 (1928).
‡ 'Proc. Roy. Soc.,' A, vol. 129, p. 1.

energy $14 \cdot 26 \times 10^5$ e.v., which has an internal conversion coefficient several hundred times that given by the theory. This special case has been discussed by Fowler (*loc. cit.*), and we shall not consider it here.

An obvious defect in the theory is the use of Schrödinger's equation, which may not be expected to hold so near the nucleus, or for electrons of such high energy. It therefore seemed possible that the more correct, relativistic equation of Dirac might give results in accordance with experiment in the majority of cases, and the calculation has been carried out by Casimir.† The same model is used, and, for purposes of calculation, the interaction of the other electrons is neglected, so that we have a single electron in the field of a charged nucleus. For the β-rays emitted from the K-shell, we may take the actual nuclear charge in carrying out the calculation. In the case of extremely hard γ-rays, whose energies may be considered large compared with mc^2, it is legitimate to use the asymptotic expansion for the wave function representing the β-ray. If we apply this theory to the range covered by experiment, we obtain results (Casimir, *loc. cit.*) which are still much too small, so that we were tempted to attribute the bulk of the conversion to some special type of interaction with the nucleus. It seems fairly certain that this must be the case for the γ-ray with $h\nu = 14 \cdot 26 \times 10^5$ e.v., which has an abnormally high internal conversion coefficient.

At this stage the problem was brought to the notice of the present author by a request from Professor Ehrenfest that Casimir's important conclusions should be checked by an independent calculation. This was done—the formulæ of this paper agree with Casimir's in the limit $h\nu/mc^2 \rightarrow \infty$. But in arranging this check calculation it was found that the calculation could be carried through exactly for all values of $h\nu/mc^2$, subject to a final stage of numerical computation.

In the case of absorption by light elements from a *beam* of γ-rays, calculation shows‡ that we cannot use the asymptotic form of the wave functions for the free electrons in the range covered by experiment, as this gives results which differ from those obtained by using the accurate wave functions. This means that the part of the wave function near the nucleus is important even in the

† 'Phys. Z.,' vol. 32, p. 665 (1931), *cf.* Gamow, "Atomic Nuclei and Radioactivity," p. 77.

‡ Hulme, 'Proc. Roy. Soc.,' A, vol. 133, p. 381 (1931). Referred to as I. The law connecting the absorption with wave-length varies considerably in the experimental range. Asymptotically we find that the absorption is proportional to the wave-length. See Sauter, 'Ann. Physik,' vol. 11, p. 464 (1931).

case of atoms with small nuclear charge. It therefore seemed worth while to go beyond Casimir's calculation, and to take the actual wave functions instead of the asymptotic expansions, particularly as our present knowledge of the nucleus hardly yet justifies us in attempting any further discussion involving its structure.

The result of the calculation is that the asymptotic formula is indeed insufficient and that *the theoretical internal conversion coefficient is in good agreement with Ellis and Aston's*† *observed values for certain of the* γ*-rays of Ra C in the range* 500,000 *e.v.*, ($h\nu/mc^2 = 1$), *to* 1,500,000 *e.v.*, ($h\nu/mc^2 = 3$). There are other rays in the range for which the calculations do not fit at all. This result has been very carefully checked at all stages as explained in the acknowledgments at the end, and is, we believe, entirely reliable. It seems possible that it may lead to important developments in the theory of the nucleus. A few calculations have also been made of the ratios $K : L_I : L_{II} : L_{III}$ absorptions, which are, for K and L_I, in good agreement with the experimental values.

In this extended calculation we still consider a simplified model obtained by neglecting the effect of the outer shells. We shall neglect the exchange degeneracy and magnetic interaction of the two *s*-electrons, and calculate the result for one electron in the presence of a nucleus of charge Ze. This must then be multiplied by two to give the internal conversion coefficient for the two electrons of the K-shell. Following Miss Swirles and Casimir, we shall place an oscillating dipole at the nucleus, to represent the mechanism emitting γ-rays, and the first part of the calculation is very similar to that of these two authors.

§2. *Perturbation Theory.*—We shall use the notation of I, writing the wave equation of the electron as

$$\left[\frac{E}{c} + \frac{e}{c} A_0 + \rho_1 \left(\boldsymbol{\sigma}, \mathbf{p} + \frac{e}{c} \mathbf{A}\right) + \rho_3 mc\right] \Psi = 0, \tag{1}$$

where E is the energy of the system, A_0 and **A** the scalar and vector potentials, **p** the momentum fector (p_x, p_y, p_z) and $\boldsymbol{\sigma}$ the vector $(\sigma_x, \sigma_y, \sigma_z)$, and $e < 0$. The quantities σ_x, σ_y, σ_z, ρ_1, (ρ_2) and ρ_3 do not commute with each other and may be conveniently represented by certain matrices of four rows and columns which obey the same non-commutability relations. In this representation the wave function Ψ has four components.

† 'Proc. Roy. Soc.,' A, vol. 129, p. 180 (1930).

Let us take as the unperturbed system an electron under the influence of a central charge Ze. We have then $\mathbf{A} = 0$ and $A_0 = Ze/r$. The perturbing potentials may be found from the Hertzian vector of the doublet which we shall take to be $\pi_z + \pi^*_z$, where

$$\pi_z = B_0 e \,.\, \exp(-2\pi i\nu t + iqr)/iqr,$$

$q = 2\pi\nu/c$ and the asterisk indicates that the conjugate complex value be taken. This yields for the perturbing potentials the following values :—

$$\left.\begin{aligned} A_0 &= -B_0 e \,.\, \exp.(-2\pi i\nu t + iqr)\cos\theta\left\{\frac{1}{r} - \frac{1}{iqr^2}\right\} + \text{conjugate complex} \\ A_z &= -B_0 e \,.\, \exp.(-2\pi i\nu t + iqr)\frac{1}{r} + \text{conjugate complex} \\ A_x &= A_y = 0, \end{aligned}\right\} \tag{2}$$

where θ is measured from the z-axis.

We may omit the conjugate complex part in the calculation of the transition probabilities, since it is only important for transitions where a γ-ray is emitted. (If $h\nu'$ denote the increase of energy of the system, this part of the perturbation occurs with $\nu + \nu'$ in the denominator, which is therefore always very large in the case of absorption.) Treating the Hertzian oscillator as a classical system we find that the amount of energy radiated per unit time is $4B_0^2e^2cq^2/3$ ergs.† or $8\pi B_0^2e^2q/3h$ quanta.

Suppose we have solved equation (1) for the undisturbed model atom, and let ψ_0 represent the normalised wave function for the ground state, and ψ_k that of a possible final state, where the electron is "free," the time factor being omitted in both cases. If ψ_k be normalised so that its components represent one electron entering or leaving a large sphere, with centre at the origin, per unit time, then we find in the usual way a value for the transition probability, per quantum of γ-radiation, given by

$$\Sigma \frac{3h}{8\pi q B_0^2 e^2}\left(\frac{2\pi}{h}\right)^2 |(\psi_k| - eA_0 - eA_z\rho_1\sigma_z|\psi_0)|^2, \tag{3}$$

where we must sum over all the possible final states, and the time factor has now disappeared from A_0 and A_z.

§3. *Calculation of the Matrix Elements.*—In this calculation we shall take the axis of the dipole along the z-axis, and we shall assume that the atomic system

† See, for example, Abraham, "Theorie der Electrizität," vol. 2, § 8.

is quantised about this axis. In the calculations made by Miss Swirles, the axis of quantisation was fixed and then the results averaged over all directions of the dipole. For an s-state we may easily verify that these two procedures give the same result, as they must do for any spherically symmetrical state. (The averaging introduces a factor of one-third, which is compensated for by the fact that there are more possible final states for an arbitrary direction of the dipole.)

The explicit form of the wave functions depends upon the matrices chosen to represent the σ and ρ's in (1), and in the following we shall use those given by Dirac. The solutions are then found to be of two types†

$$\left.\begin{array}{ll}\psi_1 = -iF_k P_{k+1}^u & \psi_2 = -iF_k P_{k+1}^{u+1} \\ \psi_3 = (k+u+1)\, G_k P_k^u & \psi_4 = (-k+u)\, G_k P_k^{u+1}\end{array}\right\} \quad (4\text{A})$$

and

$$\left.\begin{array}{ll}\psi_1 = -i(k+u)\, F_{-k-1} P_{k-1}^u & \psi_2 = -i(-k+u+1)\, F_{-k-1} P_{k-1}^{u+1} \\ \psi_3 = G_{-k-1} P_k^u & \psi_4 = G_{-k-1} P_k^{u+1}\end{array}\right\}, \quad (4\text{B})$$

where F_k and G_k satisfy

$$\left.\begin{array}{l}\left(A^2 + \dfrac{\gamma}{r}\right) F_k + \dfrac{dG_k}{dr} - \dfrac{k}{r} G_k = 0 \\[2ex] \left(B^2 - \dfrac{\gamma}{r}\right) G_k + \dfrac{dF_k}{dr} + \dfrac{k+2}{r} F_k = 0\end{array}\right\}, \quad (5)$$

with

$$A^2 = \frac{2\pi}{h}\left(mc + \frac{E}{c}\right) \qquad \gamma = \frac{2\pi e^2 Z}{ch} = \frac{Z}{137} \text{ approximately,}$$

$$B^2 = \frac{2\pi}{h}\left(mc - \frac{E}{c}\right)$$

and P_k^u is the associated Legendre function given by

$$P_k^u = (k-u)!\ \sin^u\theta \left(\frac{d}{d\cos\theta}\right)^{k+u} \frac{(\cos^2\theta - 1)^k}{2^k k!}\, e^{iu\phi},$$

u and k being any numbers such that the Legendre functions involved have a meaning. In this representation the axis of quantisation is the z-axis. When $E < mc^2$, suitable solutions of (5) can be obtained in the form of polynomials

† Darwin, 'Proc. Roy. Soc.,' A, vol. 118, p. 654 (1928).

for a set of discrete values of the energy. The two states of lowest energy, corresponding to the two s-orbits, are both of type (4A) and are given by :

$$\left.\begin{aligned} \psi_1 &= -\frac{i\gamma}{1+\sqrt{(1-\gamma^2)}}\, r^\beta e^{-r/a_0} \cos\theta \\ \psi_2 &= -\frac{i\gamma}{1+\sqrt{(1-\gamma^2)}}\, r^\beta e^{-r/a_0} \sin\theta\, e^{i\phi} \\ \psi_3 &= r^\beta e^{-r/a_0} \qquad \psi_4 = 0 \end{aligned}\right\}, \tag{6}$$

for which $k = 0$ and $u = 0$ and

$$\left.\begin{aligned} \psi_1 &= \frac{i\gamma}{1+\sqrt{(1-\gamma^2)}}\, r^\beta e^{-r/a_0} \sin\theta\, e^{-i\phi} \\ \psi_2 &= -\frac{i\gamma}{1+\sqrt{(1-\gamma)^2}}\, r^\beta e^{-r/a_0} \cos\theta \\ \psi_3 &= 0 \qquad \psi_4 = -\, r^\beta e^{-r/a_0} \end{aligned}\right\}, \tag{7}$$

for which $k = 0$ and $u = -1$ and we have put

$$a_0 = \frac{h^2}{4\pi^2 m Z e^2}, \qquad \beta = \sqrt{(1-\gamma^2)} - 1.$$

The energies of both these states are equal to $mc^2\sqrt{(1-\gamma^2)}$ and the normalisation factor is given by

$$[\xi\,(\mathrm{E}_0)]^2\, 4\pi \left(\frac{a_0}{2}\right)^{1+2\sqrt{(1-\gamma^2)}} \Gamma\,\{1 + 2\sqrt{(1-\gamma^2)}\}\, \frac{2}{1+\sqrt{(1-\gamma^2)}} = 1. \tag{8}$$

Any positive value of $(\mathrm{E} - mc^2)$ will yield a permissible solution of (5). We require a normalised solution valid for all r, and this may be obtained in the form of a contour integral. In this case, B is a pure imaginary, and if we put

$$\mathfrak{G}_k = \mathrm{AF}_k + i\,|\mathrm{B}|\,\mathrm{G}_k \tag{9}$$

we find (Hulme, *loc. cit.*),

$$\mathfrak{G}_k = [(k-s) - i\,(b+c)]\, r^s\, a^{2s+2}\, e^{-\pi b} \int_{-1}^{+1} (1-u)^{s-ib+1}\, (1+u)^{s+ib}\, e^{iaru}\, du \tag{10A}$$

and

$$\mathfrak{G}_{-k-1} = [(b-c) + i\,(k-1-s')]\, r^{s'}\, a^{2s'+2}\, e^{-\pi b} \int_{-1}^{+1} (1-u)^{s'-ib+1}\, (1+u)^{s'+ib}\, e^{iaru}\, du, \tag{10B}$$

where

$$\left.\begin{aligned} &AB = ia, & & s = \sqrt{\{(k+1)^2 - \gamma^2\}} - 1, \\ &\frac{\gamma}{2}\left(\frac{A}{B} - \frac{B}{A}\right) = -ib, & & \\ & & & s' = \sqrt{\{k^2 - \gamma^2\}} - 1 \\ &\frac{\gamma}{2}\left(\frac{A}{B} + \frac{B}{A}\right) = -ic, & & \end{aligned}\right\}. \tag{11}$$

We shall express the normalising factor as a product of $\xi(E, k)$, the normalising factor for the radial part of the wave function, and $\xi(k, u)$, that for the angular part. We obtain for the solutions (4A)

$$\left.\begin{aligned} &\xi(E, k) = \left(\frac{2\pi E}{hc^2 a}\right)^{\frac{1}{2}} \frac{A\,|B|}{(A^2 + |B|^2)^{\frac{1}{2}}} K^{-1} \\ &\text{with} \\ &\qquad K = \tfrac{1}{2}\sqrt{\{(k-s)^2 + (b+c)^2\}}\,|\Gamma(s + ib + 1)|\, e^{-3\pi b/2}\,(2a)^{s+1} \\ &\text{and} \\ &\xi(k, u) = \{4\pi\,(k + u + 1)!\,(k-u)!\}^{-\frac{1}{2}}. \end{aligned}\right\} \tag{12}$$

To obtain the corresponding normalising factors for the solutions (4B) we replace k by $k - 1$, s by s' and c by $-c$ in the above.

Let us now consider the values of the matrix elements in (3). For the initial state we should really take a combination of the states given by (6) and (7) with arbitrary phase factors, and then average the results over all possible values of the phase factors. The two states represent the spin pointing in opposite directions along the z-axis, and it is easily seen from the symmetry of the perturbation that they are entirely equivalent physically, and will yield the same result. We shall therefore only consider the first state.

With the representation used we find

$$\rho_1\sigma_z = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Bmatrix}.$$

Since we are dealing with radiation from a dipole, we should expect some accurate selection rule, valid for all values of ν. We find in fact that only two final states are possible, namely those given by

$$\Delta k = 1, \qquad \Delta u = 0.$$

They are therefore $k = 1$, $u = 0$, of types (4A) and (4B). Consider first the

transition to the state of type (4A). Neglecting, for the moment, the normalisation factors, we have

$$\sum_{s=1}^{s=4} \underset{k}{\psi^*_s} \underset{0}{\psi_s} = 2\cos\theta\,[\mathrm{G}^*_k + \gamma\{1+\sqrt{(1-\gamma^2)}\}^{-1}\mathrm{F}^*_k]\, r^\beta e^{-r/a_0},$$

$$\underset{k}{\psi^*_1}\underset{0}{\psi_3} - \underset{k}{\psi^*_2}\underset{0}{\psi_4} + \underset{k}{\psi^*_3}\underset{0}{\psi_1} - \underset{k}{\psi^*_4}\underset{0}{\psi_2}$$
$$= -i\,[\gamma\{1+\sqrt{(1-\gamma^2)}\}^{-1}(1+\cos^2\theta)\,\mathrm{G}^*_k + (1-3\cos^2\theta)\,\mathrm{F}^*_k]\, r^\beta e^{-r/a_0}.$$

For the corresponding matrix element in (3), after dividing by $\mathrm{B}_0{}^2 e^2$, we find a value given by

$$\int_0^\pi\int_0^{2\pi}\int_0^\infty \Big[\mathrm{G}_k\{2(\beta+2)\cos^2\theta - i\gamma(1+\cos^2\theta)\}$$
$$+\mathrm{F}_k\{2\gamma\cos^2\theta - i(\beta+2)(1-3\cos^2\theta)\}$$
$$+\frac{i}{qr}2\cos^2\theta\,\{(\beta+2)\,\mathrm{G}_k+\gamma\mathrm{F}_k\}\Big]$$
$$\times(\beta+2)^{-1} e^{iqr} r^{\beta+1} e^{-r/a_0}\sin\theta\, d\theta\, d\phi\, dr,$$

where we have put $\mathrm{G}^*_k = \mathrm{G}_k$, since it is real.

On integrating with respect to θ and ϕ we obtain the expression

$$\frac{8\pi}{3}\int_0^\infty\Big[\mathrm{G}_k\{(\beta+2)-2i\gamma\}+\gamma\mathrm{F}_k+\frac{i}{qr}\{(\beta+2)\,\mathrm{G}_k+\gamma\mathrm{F}_k\}\Big]$$
$$\times(\beta+2)^{-1} r^{\beta+1} e^{iqr-ra/_0}\, dr. \quad (13\mathrm{A})$$

Similarly, for the transition to the state of type (4B) we obtain for the corresponding expression the value

$$\frac{4\pi}{3}\int_0^\infty\Big[\mathrm{G}_{-k-1}\{(\beta+2)+i\gamma\}+\mathrm{F}_{-k-1}\{3i(\beta+2)+\gamma\}$$
$$+\frac{i}{qr}\{(\beta+2)\,\mathrm{G}_{-k-1}+\gamma\,\mathrm{F}_{-k-1}\}\Big](\beta+2)^{-1} r^{\beta+1} e^{iqr-r/a_0}\, dr. \quad (13\mathrm{B})$$

We now express F_k and G_k in terms of $\mathfrak{G}_k$ and $\mathfrak{G}^*_k$ by means of equation (9), remembering that B is a pure imaginary. We find for the expressions (13A) and (13B) the following values,

$$\frac{4\pi}{3}\int_0^\infty\Big[\mathfrak{G}_k\Big\{\gamma\Big(\frac{1}{\mathrm{A}}-\frac{2}{|\mathrm{B}|}\Big)-i\frac{\beta+2}{|\mathrm{B}|}\Big\}+\mathfrak{G}^*_k\Big\{\gamma\Big(\frac{1}{\mathrm{A}}+\frac{2}{|\mathrm{B}|}\Big)+i\frac{\beta+2}{|\mathrm{B}|}\Big\}$$
$$+\frac{i}{qr}\Big\{\mathfrak{G}_k\Big(\frac{\gamma}{\mathrm{A}}-i\frac{\beta+2}{|\mathrm{B}|}\Big)+\mathfrak{G}^*_k\Big(\frac{\gamma}{\mathrm{A}}+i\frac{\beta+2}{|\mathrm{B}|}\Big)\Big\}\Big](\beta+2)^{-1} e^{iqr-r/a_0} r^{\beta+1}\, dr,$$
$$(14\mathrm{A})$$

and

$$\frac{2\pi}{3}\int_0^\infty \Big[\mathfrak{G}_{-k-1}\left\{\frac{\gamma+3i(\beta+2)}{A}+\frac{\gamma-i(\beta+2)}{|B|}\right\}$$
$$+\mathfrak{G}^*_{-k-1}\left\{\frac{\gamma+3i(\beta+2)}{A}-\frac{\gamma-i(\beta+2)}{|B|}\right\}+\frac{i}{qr}\left\{\mathfrak{G}_{-k-1}\left(\frac{\gamma}{A}-i\frac{\beta+2}{|B|}\right)\right.$$
$$\left.+\mathfrak{G}^*_{-k-1}\left(\frac{\gamma}{A}+i\frac{\beta+2}{|B|}\right)\right\}\Big](\beta+2)^{-1}e^{iqr-r/a_0}r^{\beta+1}\,dr. \tag{14B}$$

§ 4. *Evaluation of the Integrals.*—The integrals occurring above cannot be evaluated in finite terms, but we can express them in terms of hypergeometric series, which can then be evaluated numerically. Consider first the integral

$$\int_0^\infty \mathfrak{G}_k\, e^{iqr-r/a_0}r^{\beta+1}\,dr, \tag{15}$$

where $\mathfrak{G}_k$ is given by (10A). Substituting for $\mathfrak{G}_k$ we obtain

$$\{(k-s)-i(b+c)\}\,a^{2s+2}e^{-\pi b}\int_0^\infty e^{iqr-r/a_0}r^{s+\beta+1}$$
$$\times\int_{-1}^{+1}(1-u)^{s-ib+1}(1+u)^{s+ib}e^{iaru}\,du\,dr. \tag{16}$$

Now if E_0 and E be the energies of the initial and final states of the system we have

$$\begin{aligned} E &= E_0+h\nu \\ &= mc^2\sqrt{(1-\gamma^2)}+hqc/2\pi. \end{aligned}$$

Also from (11) we have

$$a^2=\frac{4\pi^2}{c^2h^2}\{(E+mc^2)(E-mc^2)\},$$

which gives, putting $\theta=mc^2/h\nu$ and $\beta=\sqrt{(1-\gamma^2)}-1$,

$$\begin{aligned} a^2 &= q^2\{1+(2\beta+2)\theta+\beta(\beta+2)\theta^2\} \\ &= q^2/\tau^2,\text{ say.} \end{aligned} \tag{17}$$

Further, if we put $1/a_0=\delta a$ we may change the variable of integration in (16) by writing r instead of ar, obtaining

$$\{(k-s)-i(b+c)\}\,e^{-\pi b}\,a^{s-\beta}$$
$$\times\int_0^\infty\int_{-1}^{+1}e^{-(\delta-i\tau-iu)r}r^{\beta+s+1}(1-u)^{s-ib+1}(1+u)^{s+ib}\,dr\,du.$$

Integrating this with respect to r we find

$$\{(k-s)-i(b+c)\}\,e^{-\pi b}\,a^{s-\beta}\,\Gamma(s+\beta+2)\int_{-1}^{+1}\frac{(1-u)^{s-ib+1}(1+u)^{s+ib}}{(\delta-i\tau-iu)^{s+\beta+2}}\,du.$$

Writing M for the expression outside the integral, and putting $1+u=2u'$, the expression becomes

$$\frac{\mathrm{M}2^{2s+2}}{[\delta-i(\tau-1)]^{s+\beta+2}}\int_0^1(1-u')^{s-ib+1}\,u'^{s+ib}\,(1-zu')^{-(s+\beta+2)}\,du',$$

where $z=2/(1-\tau-i\delta)$. The integral is now expressible as a hypergeometric function,† and we obtain for (15) the value

$$\frac{\mathrm{M}2^{2s+2}}{[\delta-i(\tau-1)]^{s+\beta+2}}\,\frac{\Gamma(s+ib+1)\,\Gamma(s-ib+2)}{\Gamma(2s+3)}\times\mathrm{F}(s+\beta+2,\,s+ib+1\,;\,2s+3\,;\,z).\quad(18)$$

For γ-rays, z is large, and it is convenient to evaluate the hypergeometric function by transforming it into a function of z^{-1}.

We use for formula‡

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\mathrm{F}(a,b\,;\,c\,;\,z)=\frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(c-a)}(-z)^{-a}\,\mathrm{F}(a,1-c+a\,;\,1-b+a\,;\,z^{-1})+\frac{\Gamma(b)\,\Gamma(a-b)}{\Gamma(c-b)}(-z)^{-b}\,\mathrm{F}(b,1-c+b\,;\,1-a+b\,;\,z^{-1}),\quad(19)$$

where $\arg(-z)$ is so chosen that $|\arg(-z)|<\pi$. In this way we obtain a value for the integral (15) given by

$$[(k-s)-i(b+c)]\,.\,\mathrm{L}\,.\Big[\mathrm{F}(s+ib+1,\,ib-s-1\,;\,ib-\beta\,;\,z^{-1})+\frac{\Gamma(s+\beta+2)\,\Gamma(ib-\beta-1)\,\Gamma(s+2-ib)}{\Gamma(s+1-\beta)\,\Gamma(s+ib+1)\,\Gamma(\beta+1-ib)}(-z)^{-(\beta+1-ib)}\times\mathrm{F}(s+\beta+2,\,\beta-s\,;\,\beta+2-ib\,;\,z^{-1})\Big]\quad(20)$$

where

$$\mathrm{L}=\frac{e^{-3\pi b/2}\,a^{s-\beta}\,2^{s+1-ib}\,\Gamma(s+ib+1)\,\Gamma(\beta+1-ib)\,i^{s+1}}{[\delta-i(\tau-1)]^{\beta+1-ib}}.$$

† Whittaker and Watson, "Modern Analysis," ch. 14.

‡ Barnes, 'Proc. Lond. Math. Soc.' vol. 6, p. 141 (1908). The formula is given wrongly in Whittaker and Watson.

Consider now the integral involving $\mathfrak{G}^*_k$—

$$\int_0^\infty \mathfrak{G}^*_k \, e^{iqr - r/a_0} \, r^{\beta+1} \, dr. \tag{21}$$

The *conjugate* of this expression is obtained by replacing q by $-q$ in (15). This leads to an expression equal to (18) with τ replaced by $-\tau$ and z by $z' = 2/(1 + \tau - i\delta)$. We transform the hypergeometric function by the formula

$$\mathrm{F}(a, b;\ c;\ z') = (1 - z')^{-a}\, \mathrm{F}(a, c - b;\ c;\ z'/(z' - 1)).$$

We find $z'/(z' - 1) = z^*$, so that on taking the conjugate complex again, we obtain for the integral involving $\mathfrak{G}^*_k$, an expression in z, similar to (16). On transforming as before, this gives for the integral (21) the value

$$\frac{\mathrm{M}^*\, 2^{2s+2}}{[\delta - i(\tau - 1)]^{s+\beta+2}} \frac{\Gamma(s + ib + 2)\,\Gamma(s - ib + 1)}{\Gamma(2s + 3)} \times \mathrm{F}(s + \beta + 2, s + ib + 2;\ 2s + 3;\ z), \tag{22}$$

and finally,

$$[(k - s) + i(b + c)]\,.\,\mathrm{L}\,.\left[\frac{(s + ib + 1)}{(\beta - ib)}(-z)^{-1} \times \mathrm{F}(s + ib + 2, ib - s, ib + 1 - \beta;\ z^{-1}) + \frac{\Gamma(s + \beta + 2)\,\Gamma(ib - \beta)\,\Gamma(s - ib + 1)}{\Gamma(s + 1 - \beta)\,\Gamma(s + ib + 1)\,\Gamma(\beta + 1 - ib)}(-z)^{-(\beta+1-ib)} \times \mathrm{F}(s + \beta + 2, \beta - s;\ \beta + 1 - ib;\ z^{-1})\right]. \tag{23}$$

In the expression (14A) there occur also the integrals

$$\frac{1}{q}\int_0^\infty \mathfrak{G}_k \, e^{iqr - r/a_0} \, r^{\beta} \, dr \tag{24}$$

and

$$\frac{1}{q}\int_0^\infty \mathfrak{G}^*_k \, e^{iqr - r/a_0} \, r^{\beta} \, dr. \tag{25}$$

The values of these may be obtained from the expressions (20) and (23) by replacing β by $\beta - 1$. For the purposes of numerical calculation, however, it is preferable to express them directly in terms of the two integrals already calculated. This may be done by means of the relations existing between contiguous hypergeometric functions as follows. Let us call the integrals

(15), (21), (24) and (25), P, Q, R and S respectively. Replacing β by $\beta - 1$ in the expressions (18) and (22) for P and Q, we obtain values for R and S.

$$R = \frac{a}{q} \frac{M\, 2^{2s+2}}{(s+\beta+1)\,[\delta - i(\tau-1)]^{\beta+s+1}} \frac{\Gamma(s+ib+1)\,\Gamma(s-ib+2)}{\Gamma(2s+3)} \times F(s+\beta+1, s+ib+1;\; 2s+3;\; z)$$

$$S = \frac{a}{q} \frac{M^*\, 2^{2s+2}}{(s+\beta+1)\,[\delta - i(\tau-1)]^{\beta+s+1}} \frac{\Gamma(s+ib+2)\,\Gamma(s-ib+1)}{\Gamma(2s+3)} \times F(s+\beta+1, s+ib+2;\; 2s+3;\; z).$$

We now use the two recurrence formulæ

$$\left.\begin{aligned} &(c-a)\,F(a-1, b;\; c;\; z) = b(1-z)\,F(a, b+1;\; c;\; z) + (c-b-a)\,F(a, b;\; c;\; z) \\ &\text{and} \\ &(c-a)\,F(a-1, b+1;\; c;\; z) = (b-a+1)(1-z)\,F(a, b+1;\; c;\; z) + (c-b-1)\,F(a, b;\; c;\; z) \end{aligned}\right\} \quad (26)$$

which enable us to express the hypergeometric functions in R and S in terms of those in P and Q. Substituting in these relations we obtain

$$\left.\begin{aligned} &\frac{q}{a}[\delta - i(\tau-1)]^{-1}(s+\beta+1)(s-\beta+1)\,M^*.\,R = (-ib-\beta)\,M^*.\,P + (s-ib+1)(1-z)\,M.\,Q \\ &\text{and} \\ &\frac{q}{a}[\delta - i(\tau-1)]^{-1}(s+\beta+1)(s-\beta+1)\,M.\,S = (s+ib+1)\,M^*.\,P + (ib-\beta)(1-z)\,M.\,Q. \end{aligned}\right\} \quad (27)$$

These two equations enable us to calculate the numerical values of R and S when we know those of P and Q.

When the final state is of type (4B) all the formulæ of this section hold, provided we replace s by s' and $[(k-s) - i(b+c)]$ by $[(b-c) + i(k-1-s')]$. We shall call the new values of L and M, L$'$ and M$'$. In this case a further simplification is possible if we observe that $s' = \beta$ when $k = 1$. We can, in fact, find a relation between the integrals P and Q, which reduces the numerical work. The relation

$$(c-b)\,F(a, b;\; c+1;\; z) + b\,F(a, b+1;\; c+1;\; z) = c\,F(a, b;\; c;\; z)$$

gives

$$\begin{aligned} &(s'-ib+1)\,F(\beta+s'+2, s'+ib+1;\; 2s'+3;\; z) \\ &\quad + (s'+ib+1)\,F(\beta+s'+2, s'+ib+2;\; 2s'+3;\; z) \\ &\qquad = (2s'+2)\,F(\beta+s'+2, s'+ib+1;\; 2s'+2;\; z). \end{aligned}$$

If $s' = \beta$, the last hypergeometric function reduces to

$$(1 - z)^{-(\beta+ib+1)},$$

and from (18) and (22) we have

$$\mathrm{M}^* \mathrm{P}' + \mathrm{M}' \mathrm{Q}' = \frac{\mathrm{M}' . \mathrm{M}'^* . 2^{2\beta+2}}{[\delta - i(\tau - 1)]^{2\beta+2}} \frac{\Gamma(\beta + ib + 1)\Gamma(\beta - ib + 1)}{\Gamma(2\beta + 2)}(1 - z)^{-(\beta+ib+1)}, \tag{28}$$

which is the required equation.

The expressions obtained for the integrals P, Q, etc., have still to be multiplied by the normalising factors for the initial and final states. These are given by (8) and (12). The internal conversion coefficient is then found from the formula (2) and the expressions (14A) and (14B).

The final result is rather cumbersome and we shall express it as follows. The integral (15) is given by the expression (20) which we shall write as

$$|(k - s) - i(b + c)| . \mathrm{L} . \mathrm{I}_1.$$

Similarly for the expression (23) we write

$$|(k - s) - i(b + c)| . \mathrm{L} . \mathrm{I}_2.$$

The integrals (24) and (25) are found in terms of these two, and we shall put them equal to

$$|(k - s) - i(b + c)| . \mathrm{L} . \mathrm{I}_3$$

and

$$|(k - s) - i(b + c)| . \mathrm{L} . \mathrm{I}_4.$$

When the final state is of type (4A) the integrals are

$$|(b - c) + i(k - 1 - s')| . \mathrm{L}' . \mathrm{I}'_1, \text{ etc.}$$

These are now substituted in (14A) and (14B) which are to be multiplied by the normalisation factors for the initial and final states, and substituted into the expression (3) for the internal conversion coefficient. After some algebra we obtain the final result

$$\mathrm{I}_{\mathrm{K}} = \frac{1}{24} \frac{\gamma b}{\mathrm{Z}} \theta \frac{(2\gamma)^{2\beta+2}}{\Gamma(2\beta + 3) . (\beta + 2)} \times \frac{|\Gamma(\beta + 1 - ib)|^2}{\left|\left\{\gamma + \frac{i}{\theta}\left(\frac{a}{q} - 1\right)\right\}^{2(\beta+1-ib)}\right|} \frac{2\mathrm{A}^2|\mathrm{B}|^2}{\mathrm{A}^2 + |\mathrm{B}|^2} (2|\mathfrak{A}|^2 + |\mathfrak{B}|^2), \tag{29}$$

where

$$\mathfrak{A} = \mathrm{I}_1 \left\{\gamma\left(\frac{1}{\mathrm{A}} - \frac{2}{|\mathrm{B}|}\right) - i\frac{\beta + 2}{|\mathrm{B}|}\right\} + \mathrm{I}_2 \left\{\gamma\left(\frac{1}{\mathrm{A}} + \frac{2}{|\mathrm{B}|}\right) + i\frac{\beta + 2}{|\mathrm{B}|}\right\} + \mathrm{I}_3 \left\{\frac{\beta + 2}{|\mathrm{B}|} + \frac{i\gamma}{\mathrm{A}}\right\} + \mathrm{I}_4 \left\{-\frac{\beta + 2}{|\mathrm{B}|} + \frac{i\gamma}{\mathrm{A}}\right\}$$

and

$$\mathfrak{B} = \mathrm{I}'_1 \left\{\gamma\left(\frac{1}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\right) + i(\beta+2)\left(\frac{3}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\right)\right\}$$
$$+ \mathrm{I}'_2 \left\{\gamma\left(\frac{1}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\right) + i(\beta+2)\left(\frac{3}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\right)\right\}$$
$$+ \mathrm{I}'_3 \left\{\frac{\beta+2}{|\mathrm{B}|} + \frac{i\gamma}{\mathrm{A}}\right\} + \mathrm{I}'_4 \left\{-\frac{\beta+2}{|\mathrm{B}|} + \frac{i\gamma}{\mathrm{A}}\right\}.$$

This result is obtained for one electron, and, before comparing it with experiment, we must multiply it by two to allow for the two electrons of the K-shell.

From the above expression we can easily obtain a formula giving the value of I_k as $\lambda \to 0$. This limit implies the following ones :—

$$\theta \to 0, \qquad z \to \infty$$
$$\tau \to 1, \qquad q \to a \to \infty$$

From expressions (20) and (23) we have

$$|\mathrm{I}_1| \to 1, \quad \mathrm{I}_2 \to 0.$$

We then see from (27) that R and $\mathrm{S} \to 0$, so that I_3 and $\mathrm{I}_4 \to 0$.

Similar results hold for I'_1, etc. Also as $\theta \to 0$, $\mathrm{A}/|\mathrm{B}| \to 1$, and $b \to \gamma$, so that

$$\left\{\gamma + \frac{i}{\theta}\left(\frac{a}{q} - 1\right)\right\} \to \gamma + i(\beta+1)$$

and

$$|\mathfrak{A}|^2 \to 2(\beta+2)/\mathrm{A}^2$$
$$|\mathfrak{B}|^2 \to 8(\beta+2)/\mathrm{A}^2.$$

Putting these values in we obtain a value for the internal conversion coefficient per electron given by

$$\mathrm{I}_{\mathrm{K,\,Asymptotic}} = \frac{1}{2}\frac{\gamma^2\theta}{\mathrm{Z}}(2\gamma)^{2\sqrt{(1-\gamma^2)}}\frac{|\Gamma\{\sqrt{(1-\gamma^2)} - i\gamma\}|^2}{\Gamma\{1+2\sqrt{(1-\gamma^2)}\}}e^{-2\gamma \operatorname{arc\,cos}\gamma}, \tag{30}$$

which is the formula given by Casimir.†

§ 5. *Internal Conversion by the Two L_{I} Electrons.*—Experiment shows that the two K-electrons produce the strongest lines in the β-ray spectrum, the next strongest being due to absorption of γ-rays by the two L_{I} electrons. The

† Casimir, *loc. cit.* Dr. Casimir has pointed out that by using the property

$$\psi = \psi_{Asymptotic}\{1 + \mathrm{O}(1/ar)\},$$

we may easily prove that the use of the asymptotic wave functions will give a correct result in this problem if $h\nu \gg mc^2$.

amount of this absorption may be found exactly as for the K-electrons, the algebra, however, is a little more complicated.

For discrete states the values of F_k and G_k have been given by Darwin (*loc. cit.*)

$$\left.\begin{aligned} F_k &= \frac{\gamma}{N+k'+n'}\, e^{-r/a_0 N} \\ &\times \left\{ r^{k'+n'-1}(N+k+1) - r^{k'+n'-2}\, a_0 N (N+k+2)\frac{n'(n'+2k')}{2} \right. \\ &\quad \left. + r^{k'+n'-3}(a_0 N)^2 (N+k+3)\frac{n'(n'-1)(n'+2k')(n'+2k'-1)}{2\,.\,4} - \ldots \right\} \\ G_k &= e^{-r/a_0 N}\left\{ r^{k'+n'-1}(N+k+1) - r^{k'+n'-2}\, a_0 N (N+k)\frac{n'(n'+2k')}{2} \right. \\ &\quad \left. + r^{k'+n'-3}(a_0 N)^2 (N+k-1)\frac{n'(n'-1)(n'+2k')(n'+2k'-1)}{2\,.\,4} - \ldots \right\} \end{aligned}\right\} \quad (31)$$

with

and

$$\left.\begin{aligned} k' &= \sqrt{\{(k+1)^2 - \gamma^2\}} \\ N &= \sqrt{\{(k'+n')^2 + \gamma^2\}}, \end{aligned}\right\} \quad (32)$$

where n' is an integer. The series terminate and the corresponding energy is given by

$$E = mc^2 (k'+n')/N. \quad (33)$$

For F_{-k-1} and G_{-k-1} we must replace k by $-k-1$ in the above. These wave functions are only valid for one electron in the field of a fixed charge Ze. We shall therefore neglect the effect of all the other electrons and assume these wave functions represent the L_I electrons when $n'=1$, $k=0$ and Z is equal to the nuclear charge. Since $k=0$ the wave functions for the two L_I-electrons differ from those of the K-electrons only in the radial part. The possible initial states are therefore given by $u=0$, $u=-1$, and are both of type (4A). As before they are both spherically symmetrical and yield the same result. We shall consider the first state only, keeping the direction of the nuclear dipole fixed along the z-axis. Putting in $n'=1$, $k=0$ and $N^2 = 2(k'+1)$, the equations (31) become

$$\left.\begin{aligned} F_k &= \gamma(N+1)\, a_0\, e^{-r/Na_0}\{N\delta_1 ar - N^2(N-1)/2\} = \gamma(N+1)\, a_0\, e^{-r/Na_0} f, \\ G_k &= (N+1)\, a_0\, e^{-r/Na_0}\{2\delta_1 ar/(N+2) - (N-1)\} = \gamma(N+1)\, a_0\, e^{-r/Na_0} g, \end{aligned}\right\} \quad (34)$$

where we have put $Naa_0 = 1/\delta_1$.

The wave functions for the initial state are normalised in the usual way and we find

$$[\xi(\mathrm{E}_0)]^2\, 4\pi \left(\frac{a_0 \mathrm{N}}{2}\right)^{2\beta+5} \Gamma(\mathrm{N}^2)\, \frac{8\mathrm{N}(\mathrm{N}+1)}{\mathrm{N}+2} = 1.$$

The calculations follow those for the K-shell very closely, and, corresponding to (14A) and (14B), we obtain the following expressions

$$\frac{4\pi}{3}\int_0^\infty \Big[\mathfrak{G}_k \Big\{\gamma \cdot f \cdot \Big(\frac{1}{\mathrm{A}} - \frac{2}{|\mathrm{B}|}\Big) - \frac{ig}{|\mathrm{B}|}\Big\} + \mathfrak{G}^*_k \Big\{\gamma \cdot f \cdot \Big(\frac{1}{\mathrm{A}} + \frac{2}{|\mathrm{B}|}\Big) + \frac{ig}{|\mathrm{B}|}\Big\}$$
$$+ \frac{i}{qr}\Big\{\mathfrak{G}_k\Big(\frac{\gamma \cdot f}{\mathrm{A}} - \frac{ig}{|\mathrm{B}|}\Big) + \mathfrak{G}^*_k\Big(\frac{\gamma \cdot f}{\mathrm{A}} + \frac{ig}{|\mathrm{B}|}\Big)\Big\}\Big] (\mathrm{N}+1)\, a_0 r^{\beta+1} e^{iqr - r/\mathrm{N}a_0}\, dr, \tag{35A}$$

and

$$\frac{2\pi}{3}\int_0^\infty \Big[\mathfrak{G}_{-k-1}\Big\{\gamma \cdot f \cdot \Big(\frac{1}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\Big) + ig\Big(\frac{3}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\Big)\Big\}$$
$$+ \mathfrak{G}^*_{-k-1}\Big\{\gamma \cdot f \cdot \Big(\frac{1}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\Big) + ig\Big(\frac{3}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\Big)\Big\} + \frac{i}{qr}\Big\{\mathfrak{G}_{-k-1}\Big(\frac{\gamma \cdot f}{\mathrm{A}} - \frac{ig}{|\mathrm{B}|}\Big)$$
$$+ \mathfrak{G}^*_{-k-1}\Big(\frac{\gamma \cdot f}{\mathrm{A}} + \frac{ig}{|\mathrm{B}|}\Big)\Big\}\Big] (\mathrm{N}+1)\, a_0 r^{\beta+1} e^{iqr - r/\mathrm{N}a_0}\, dr. \tag{35B}$$

On putting in the values of f and g these become

$$\frac{4\pi}{3}\int_0^\infty \Big[\mathfrak{G}_k\Big\{\gamma(\mathrm{N}-1)\Big(\frac{1}{\mathrm{A}} - \frac{2}{|\mathrm{B}|}\Big) - \frac{a\delta_1}{q}\frac{\mathrm{N}}{|\mathrm{B}|} - \frac{i\mathrm{N}^2(\mathrm{N}-1)}{2|\mathrm{B}|} - \frac{ia\delta_1}{q}\frac{2}{\mathrm{N}+2}\frac{\gamma}{\mathrm{A}}$$
$$+ \frac{1}{qr}\Big(\frac{\mathrm{N}^2(\mathrm{N}-1)}{2|\mathrm{B}|} + \frac{i\gamma(\mathrm{N}-1)}{\mathrm{A}}\Big) + \delta_1 ar\Big(-\frac{2\gamma}{\mathrm{N}+2}\Big(\frac{1}{\mathrm{A}} - \frac{2}{|\mathrm{B}|}\Big) + \frac{i\mathrm{N}}{|\mathrm{B}|}\Big)\Big\}$$
$$+ \mathfrak{G}^*_k \left\{\begin{matrix}\text{same expression with}\\ \text{the sign of } |\mathrm{B}| \text{ changed}\end{matrix}\right\}\Big] \times (\mathrm{N}+1)\, a_0 r^{\beta+1} e^{iqr - r/\mathrm{N}a_0}\, dr, \tag{36A}$$

and

$$\frac{2\pi}{3}\int_0^\infty \Big[\mathfrak{G}_{-k-1}\Big\{\gamma(\mathrm{N}-1)\Big(\frac{1}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\Big) - \frac{a\delta_1}{q}\frac{\mathrm{N}}{|\mathrm{B}|} + \frac{i\mathrm{N}^2(\mathrm{N}-1)}{2}\Big(\frac{3}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\Big)$$
$$- \frac{ia\delta_1}{q}\frac{2}{\mathrm{N}+2}\frac{\gamma}{\mathrm{A}} + \frac{1}{qr}\Big(\frac{\mathrm{N}^2(\mathrm{N}-1)}{2|\mathrm{B}|} + \frac{i\gamma(\mathrm{N}-1)}{\mathrm{A}}\Big)$$
$$+ \delta_1 ar\Big(-\frac{2\gamma}{\mathrm{N}+2}\Big(\frac{1}{\mathrm{A}} + \frac{1}{|\mathrm{B}|}\Big) - i\mathrm{N}\Big(\frac{3}{\mathrm{A}} - \frac{1}{|\mathrm{B}|}\Big)\Big)\Big\}$$
$$+ \mathfrak{G}^*_{-k-1}\left\{\begin{matrix}\text{same expression with}\\ \text{the sign of } |\mathrm{B}| \text{ changed}\end{matrix}\right\}\Big] \times (\mathrm{N}+1)\, a_0 r^{\beta+1} e^{iqr - r/\mathrm{N}a_0}\, dr. \tag{36B}$$

These expressions may be evaluated as before. Expression (36A) contains six integrals

$$X_1 = \delta_1 \int_0^\infty \mathfrak{G}_k \, e^{iqr - r/Na_0} \, r^{\beta+1} \, . \, ar \, dr,$$

$$Y_1 = \delta_1 \int_0^\infty \mathfrak{G}^*_k \, e^{iqr - r/Na_0} \, r^{\beta+1} \, . \, ar \, dr,$$

and four integrals P, Q, R and S, which are the same as P, Q, R and S of § 4 with δ_1 replacing δ (which implies Na_0 replacing a_0). The last four are calculated exactly as before. For a given value of $h\nu$, however, the values of the constants involved are slightly different, since the energy of the emitted electron is different. We have

$$E = h\nu \, (1 + \tfrac{1}{2}\theta N)$$

so that

$$a^2 = q^2 \{1 + \theta N + \theta^2 (N^2/4 - 1)\}$$
$$= q^2/\tau_1^2, \text{ say,}$$

and the constants b and c have different values. Remembering this, P and Q, may be calculated from (20) and (21) and R, S obtained from the former by the formulæ (27), provided δ and τ are replaced by δ_1 and τ_1 in both calculations.

The integrals X_1 and Y_1 are found from (18) and (22) respectively, by replacing β by $\beta + 1$ and multiplying by δ_1. They may be expressed in terms of P_1 and Q_1 by using the same recurrence formulæ (26). We find

$$\left.\begin{aligned} M^* X_1 [\delta_1 - i(\tau_1 - 1)] &= \delta_1 (\beta + 1 - ib) M^* P_1 \\ &\quad + \delta_1 (s + 1 - ib) M Q_1 \\ \text{and} \qquad & \\ M Y_1 [\delta_1 - i(\tau_1 - 1)](1 - z) &= \delta_1 (s + 1 + ib) M^* P_1 \\ &\quad + \delta_1 (\beta + 1 + ib) M Q_1. \end{aligned}\right\} \quad (37)$$

When the final state is of type (4B) the work is exactly the same providing we replace s by s' in the above formulæ. These integrals we call $X'_1, \ldots S'_1$.

For the final result we introduce as before the quantities $J_1, J'_1, \ldots J_6, J'_6$, defined by

$$P_1 = |(k - s) - i(b + c)| \, L_1 \, . \, J_1, \quad P'_1 = |(b - c) + i(k - 1 - s')| \, L'_1 \, J'_1,$$

etc., where L_1 and L'_1 are obtained from L and L′ by replacing δ by δ_1 and τ by τ_1, and the constants b and c are calculated for the L_I-electrons.

On including the normalisation factors we finally obtain for the internal conversion coefficient the value

$$\mathrm{I}_{\mathrm{L_I}} = \frac{1}{24}\frac{\gamma\theta b}{\mathrm{Z}}(2\gamma)^{2\beta+2}\frac{1}{(\beta+2)\,\Gamma(2\beta+3)}\,\frac{\mathrm{N}+2}{2(\mathrm{N}-1)\,\mathrm{N}^{2\beta+4}}$$

$$\times\frac{|\Gamma(\beta+1-ib)|^2}{\left|\left\{\frac{\gamma}{\mathrm{N}}+\frac{i}{\theta}\left(\frac{a}{q}-1\right)\right\}^{2(\beta+1-ib)}\right|}[2|\mathfrak{A}_1|^2+|\mathfrak{B}_1|^2]. \tag{38}$$

The expression for $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are cumbersome and will not be written down. They are obtained from the expressions inside the square brackets of (36A) and (36B) by replacing $\mathfrak{G}_k$, $\mathfrak{G}_k/qr$ and $\delta_1 ar\mathfrak{G}_k$ by J_1, J_3 and J_5, and $\mathfrak{G}^*_k$, $\mathfrak{G}^*_k/qr$ and $\delta_1 ar\mathfrak{G}^*_k$ by J_2, J_4 and J_6. In (36B), $\mathfrak{G}_{-k-1}$ is replaced by J'_1, etc.

The asymptotic form of (38) as $\lambda\to 0$ is easily obtained. As before we find

$$|\mathrm{J}_1|\to 1$$

$$\mathrm{J}_2,\ \mathrm{J}_3,\ \mathrm{J}_4\to 0,$$

and from equations (37)

$$\mathrm{X}_1\to\mathrm{P}_1\frac{(\beta+1-ib)\,\delta_1}{[\delta_1-i(\tau_1-1)]}$$

$$\to\mathrm{P}_1\frac{(\beta+1-i\gamma)\,\gamma}{\gamma+i(\beta+2)}$$

so that

$$\mathrm{J}_5\to\mathrm{J}_1\frac{(\beta+1-i\gamma)\gamma}{\gamma+i(\beta+2)}.$$

Also

$$\frac{\gamma}{\mathrm{N}}+\frac{i}{\theta}\left(\frac{a}{q}-1\right)\to\left(\frac{\gamma}{\mathrm{N}}+\frac{i\mathrm{N}}{2}\right),$$

and on substituting these in (38) we obtain, after some algebra,

$$\mathrm{I}_{\mathrm{L_I}\,\mathrm{Asymptotic}}=\frac{(\mathrm{N}+2)}{2(\mathrm{N}-1)\,\mathrm{N}^{2\beta+4}}\,e^{-2\gamma\tan^{-1}[\gamma/(\beta+2)]}\times\mathrm{I}_{\mathrm{K\ Asymptotic}},$$

both referring to one electron. For radium C this ratio is about 0·149 or 1/6·7.

§ 6. *Internal Conversion by the L_{II} and L_{III} Shells.*—The two $\mathrm{L_{II}}$ electrons are given by $n'=1$, $k=1$, $u=0$ and -1, the wave functions being of type (4B). The selection rule is $\Delta u=0$ and $\Delta k=\pm 1$, but actually there are only two possible final states for each initial one. They are $k=0$, type (4A)

and $k = 2$, type (4B). The calculation is similar to that for the L_I shell, but we shall only give the asymptotic formula. We find

$$\frac{\text{I asymptotic for } L_{II} \text{ shell}}{\text{I asymptotic for } L_I \text{ shell}} = \frac{(2-N)(N-1)}{(2+N)(N+1)},$$

where N has the same value as in § 5. The numerical value of this ratio is about 0·0086.

The L_{III} shell consists of four electrons, whose wave functions are of type (4A) with $n' = 0$, $k = 1$ and $u = 0$, -1, 1 and -2. We have the same selection rule but only three final states are possible for each initial one, since there are no states of type (4B) with $k = 0$. For the four electrons of the L_{III} shell we find asymptotically

$$I_{\text{for } L_{III} \text{ shell}} = \frac{\gamma^2 \theta}{Z} (2\gamma)^{2\sqrt{(4-\beta^2)}} \frac{|\Gamma\{\sqrt{(4-\beta^2)} - i\gamma\}|^2}{\Gamma\{1 + 2\sqrt{(4-\gamma^2)}\}} \times \frac{1}{|\{\gamma + i\sqrt{(4-\gamma^2)}\}^{2[\sqrt{(4-\gamma^2)} - i\gamma]}|},$$

which is the same as the formula for the K shell if we replace $\sqrt{(4-\gamma^2)}$ by $\sqrt{(1-\gamma^2)}$. The ratio of the internal conversion in the L_{III} shell to that in the L_1 shell is about 0·044.

These ratios are only valid in the limit $\lambda \to 0$ but we should expect them to give the general characteristics of the absorption in the experimental range of wave-lengths. For the ratio I_K to I_{L_I} we find the asymptotic value is very close to that calculated for a definite value of λ, given by $mc^2/h\nu = 0{\cdot}709$.

Owing to the roughness of the model, we should not expect the values obtained for the L shells to be very accurate. The neglect of the screening effect results in the coincidence of the L_I and L_{II} lines, since they form a "screening doublet."

In the above calculations we have again kept the direction of the nuclear dipole fixed. If we fix the direction of quantisation and then average over all directions of the dipole, the result must be the same. This follows from the fact that the L_{II} and L_{III} shells are spherically symmetrical even though the states of the separate electrons are not.

§ 7. *Results and Discussion.*—The internal conversion coefficient for the K shell has been calculated for six values of $h\nu$, and the results are given in the Table I.

The experimental values are taken from a table given by Ellis and Aston (*loc. cit.*), and the last two refer to radium B. The first four values calculated

Table I.

$\theta = \dfrac{mc^2}{h\nu}$.	$h\nu$ in electron volts $\times 10^{-5}$.	Internal conversion coefficient, calculated.	Internal conversion coefficient, experimental value.
0·287	17·78	0·0008	0·0016
0·452	11·3	0·0018	0·0062
0·639	8·0	0·0035	—
0·835	6·12	0·0057	0·0061
1·443	3·54	0·0172	(0·10)
2·102	2·43	0·0424	(0·25)

are in the region of the radium C lines, and they are shown on a curve, together with the experimental values, in the figure.

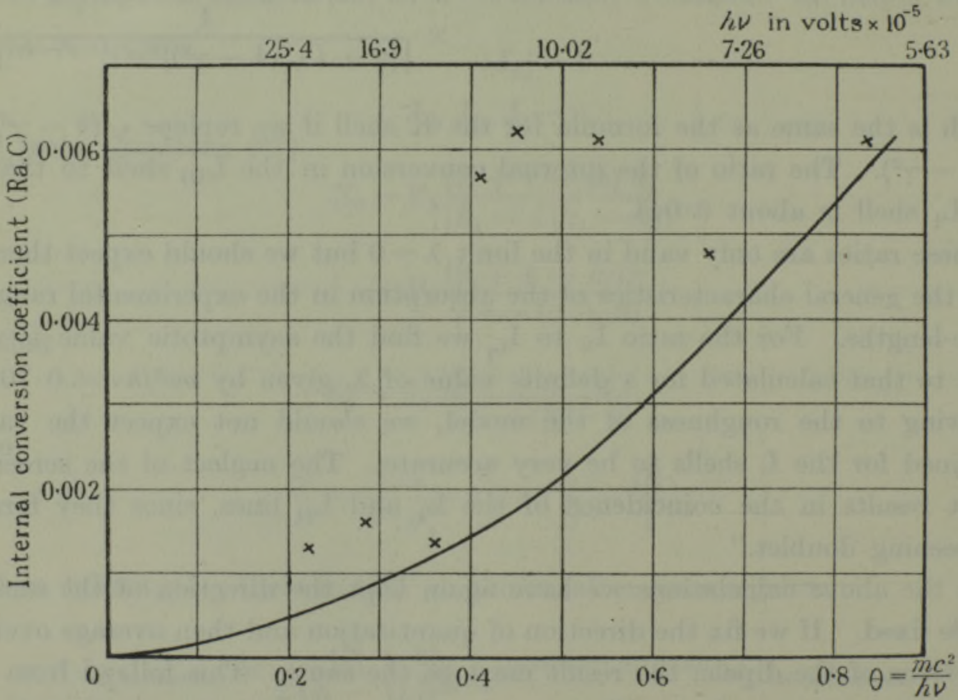

Fig. 1.—The points marked with a cross are the experimental values determined by Ellis and G. H. Aston for radium C for the lines (in volts $\times 10^{-5}$), 6·12, 7·73, 9·41, 11·30, 12·48, 13·89, 17·78 and 22·19.

It will be seen that the curve fits the general order of magnitude if we exclude the three lines between $\theta = 0·4$ and $\theta = 0·6$. Of the remaining lines the three with the lowest energy are certainly within the range of the experimental error.

In the paper following, Taylor and Mott discuss the internal conversion of radiation due to a quadripole placed at the nucleus. They obtain a curve which passes above the experimental points for the highest energies, and in between the two points for $h\nu = 11 \cdot 3 \times 10^5$ e.v. and $h\nu = 9 \cdot 41 \times 10^5$ e.v. It therefore seems probable that the three rays in this region are due to nuclear transitions where the angular momentum of the nucleus changes by two units, and the three rays of lowest energy to transitions where the momentum changes by one unit. For a fuller discussion the reader is referred to the paper following.

It will be seen from the table that the values calculated for radium C are very much smaller than the observed values for radium B. The nuclear charges differ by one unit only, which would reduce the calculated values by about 4 per cent., so that it is quite impossible to explain the internal conversion of radium B on the present dipole theory. Here again Taylor and Mott find that the internal conversion of radiation from a quadripole is considerably greater in this range, and approaches more nearly the experimental values.

The values of the internal conversion coefficients for the L shells have been calculated for the limit $\lambda \to 0$. We find $I_K : I_{L_I} : I_{L_{II}} : I_{L_{III}} = 6 \cdot 7 : 1$ $0 \cdot 0086 : 0 \cdot 044$, the values being approximate, as we have neglected all screening effects. In the experimental range $I_{L_{II}}$ seems to be always greater than $I_{L_{III}}$ from observed values. It is of great interest to know the value of the ratio I_K, I_{L_I} for wave-lengths in the experimental region. This has been calculated for $\theta = 0 \cdot 709$, $h\nu = 7 \cdot 20 \times 10^{+5}$ e.v., and gives a result of about $7 \cdot 0$. As the asymptotic value is $6 \cdot 7$ we are fairly safe in assuming that the ratio does not vary much in the experimental region. Previous theories† gave a value 8 for this ratio, but Dr. Ellis informs me that the present value is in better agreement with experiment.

The results obtained for the absolute magnitude of the internal conversion coefficient differ considerably from those obtained by Miss Swirles and by Casimir. The latter's results are seen to be true in the limit, and some of the terms which vanish as $\lambda \to 0$ are quite large in the experimental region. One would expect, however, that the non-relativistic results of Miss Swirles would not differ considerably from the present ones in the radium B region. Actually the present theory gives results which are about four times as large, which may seem surprising. We must remember, however, that the relativity correction, which arises from the term Z/137, occurs in the wave functions of both initial

† Swirles, *loc. cit.*, and Fowler, *loc. cit.*

and final states. This may be small in the case of light atoms and for the outer electrons of heavier atoms, but for the K shell of radioactive atoms there is no reason why its effect should not be very great.

We may conclude that the internal conversion of γ-rays from the nucleus of radium C is capable of being explained as an ordinary photoelectric effect, provided we exclude the ray with $h\nu = 14 \cdot 26 \times 10^5$ e.v., which seems to be almost completely converted. In some cases the γ-rays are due to transitions where the angular momentum of the nucleus changes by one unit, while Taylor and Mott have shown that the assumption of quadripole radiation will account for the remaining rays within the limits of the experimental accuracy.

In conclusion I wish to thank Professor R. H. Fowler for suggesting that I should check Casimir's calculations, and for his interest and encouragement throughout the work, and Dr. C. D. Ellis for opportunities of discussing the experimental values. The numerical work, which is largely due to Mr. J. McDougall, has been calculated by one of us and checked by the other. The algebra for the K shell has been checked by Mr. H. M. Taylor, and I take this opportunity of thanking him and Mr. McDougall for all their assistance. I should also like to thank Mr. Mott and Mr. Taylor for informing me of their results for quadripole radiation.

Summary.

The internal conversion coefficient for radium C has been calculated on the assumption that the radiation field may be represented by a dipole situated at the nucleus. The accurate relativistic wave functions have been used and the internal conversion coefficient for the K shell has been calculated for several lines. The results are in good agreement with experiment for some lines, which are therefore presumably due to transitions in which the total angular momentum of the nucleus changes by one unit. According to Mott and Taylor the other lines are explainable on the assumption of quadripole radiation. The ratios of the conversion in the K, L_I, L_{II} and L_{III} shells are calculated for the limit $\lambda \rightarrow 0$, and the ratio $I_K : I_{L_I}$, for a value of $mc^2/h\nu = 0 \cdot 709$.