Summary.

The infra-red absorption spectrum of nitrosyl chloride, NOCl, has been examined from 1 to 18 μ. Five bands were isolated between 4 and 16 μ, but as a consequence of the expected asymmetry of the molecule, no resolution was obtained. The molecule falls into the sulphur dioxide class, having a vertical angle of some 140°, and obeying a valence force system. The force constants have been determined and associated with the heats of linking.

The Quantum Theory of the Neutron.

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1. *Introduction.*

The object of this paper is to show that a plausible theory of the neutron can be developed from Dirac's wave equation without the use of any *ad hoc* assumptions. It is shown that the second order wave equation of the hydrogen atom, which exhibits the relativistic and spin corrections, possesses two sets of solutions "H" and "N" distinguished by their behaviour as $r \rightarrow 0$ (r being the distance of the electron from the proton).

The H-solutions are the accepted wave functions of the hydrogen atom. As $r \rightarrow 0$ these solutions tend to zero if the serial quantum number l differs from zero, and they become infinite of order $r^{[(1-\alpha^2)^{1/2}-1]}$ if $l = 0$ (α is the fine structure constant).

The energy levels in the discrete spectrum are given approximately by the formula

$$\mathrm{E} = m_0c^2(1 - \alpha^2/2n^2), \qquad (n = 1, 2, 3, \ldots),$$

where m_0c^2 is the proper energy of the electron. The energy of the ground state is

$$\mathrm{E_H} = m_0c^2(1 - \tfrac{1}{2}\alpha^2).$$

The N-solutions exist only if $l = 0$. As $r \rightarrow 0$ they become infinite of order $r^{-(1-\alpha^2)^{1/2}}$, but, like the H-solutions, they are quadratically integrable over the range, $0 \leqslant r \rightarrow \infty$. The energy levels of the N-solutions form a continuous spectrum, the values of the energy E covering the half open interval,

$$-m_0c^2 \leqslant \mathrm{E} < +m_0c^2,$$

and the lowest value of the energy being

$$E_N = -m_0c^2.$$

It is suggested here that the N-solutions are the wave functions of the neutron. On making this identification the theory attributes the following properties to the neutron :—

(*a*) The binding energy of the neutron in its normal state is

$$E_H - E_N = m_0c^2 (2 - \tfrac{1}{2}\alpha^2),$$

i.e., approximately $2m_0c^2$ or 10^6 electron volts. Thus the mass of the neutron is less than the mass of the proton by the amount of twice the mass of an electron. This is in agreement with the estimate made by Chadwick in his Bakerian lecture.*

(*b*) The mean value of r^{-1} for the neutron in its normal state is

$$\overline{1/r} = 6/\alpha_0{}^2 a_0,$$

where a_0 is the radius of the first Bohr orbit of hydrogen, *i.e.*, in the usual notation, $a_0 = h^2/(m_0e^2)$,†

The mean value of r is $\bar{r} = 3a_0/10$.

(*c*) At a large distance r from a neutron in its normal state the electrostatic potential of the field is given approximately by the equation,

$$V(r) = \frac{3\sqrt{2}}{16} \cdot \frac{e}{(a_0 r)^{\frac{1}{2}}} \exp\{-2(8r/a_0)^{\frac{1}{2}}\}.$$

(*d*) The serial quantum number of the neutron is $l = 0$. Hence the charge of the neutron is spherically symmetrical and the electron in it has a spin of $\frac{1}{2}h$. Hence the total spin of the neutron must be h or zero. At present it is premature to say whether this deduction is or is not in agreement with experiment.

(*e*) It remains to consider the probabilities of transitions from a hydrogen atom to a neutron. Since $l = 0$ for a neutron, such transitions can occur only when $l = 1$ for the hydrogen atom. Hence the normal state of the hydrogen atom ($n = 1$, $l = 0$) is a meta-stable state, which cannot collapse into a neutron. This result is satisfactory as far as it goes, but it is clearly far from a complete explanation of the rarity of the transitions from hydrogen atoms to neutrons.

* 'Proc. Roy. Soc.,' A, vol. 142, p. 1 (1933).

† h is Planck's constant divided by 2π.

The preceding deductions from the theory of this paper indicate that although it requires some modification and extension, it does succeed in reproducing the main properties attributed to neutrons. The strength of the theory lies in the fact that it is free from any special assumptions, all the results being deduced from the second order wave equation for the hydrogen atom.

2. *The First Order Wave Equation.*

The first order wave equation for the hydrogen atom may be written as

$$\mathrm{F}\psi \equiv \{(p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3)\,\rho_1 + ic^{-1}(\mathrm{E} + e^2/r)\,\rho_3\}\,\psi = \pm\, im_0c\psi, \quad (2.1)$$

where ρ_1, ρ_2, ρ_3 and σ_1, σ_2, σ_3 are two commuting sets of quaternion operators. It is convenient to abbreviate such a quasi-scalar product as

$$(p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3)$$

to (p, σ).

The solution of this wave equation depends upon the identity,

$$(x, \sigma)\,(p, \sigma) \equiv rp_r + i\rho_3\kappa,$$

where p_r is the radial momentum operator,

$$p_r = \tfrac{1}{2}\{(x_1/r)\,p_1 + p_1\,(x_1/r)\} + \ldots,$$

and κ is the angular momentum operator

$$\kappa = \rho_3\{(m, \sigma) + h\}.$$

On multiplying (2.1) by $(x, \sigma)\,\rho_1$ it becomes

$$\{(rp_r + i\rho_3\kappa) + c^{-1}(\mathrm{E}r + e^2)\,\rho'_2\}\,\psi = \pm\, im_0cr\rho'_1\psi, \quad (2.2)$$

where

$$\rho'_1 = r^{-1}(x, \sigma)\,\rho_1,$$

and

$$\rho'_2 = r^{-1}(x, \sigma)\,\rho_2.$$

The operators, ρ'_1, ρ'_2, ρ_3 form a new set of quaternions, which commute with every other operator in (2.2), and which can, therefore, be represented by Pauli's two-rowed matrices,

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

The operator κ commutes with every other operator in (2.2) and hence can be replaced by its proper value μh, where $\mu = \pm 1, \pm 2, \ldots$, (the value zero

being excluded). Hence the equations for ψ_1, ψ_2, the two components of ψ, in the Pauli representation, are

$$(r\partial/\partial_r + 1 - \mu)\,\psi_1 + (\alpha + \mathrm{E}r/hc)\,\psi_2 = \mp\,(m_0c/h)\,r\psi_2,$$

and

$$(r\partial/\partial r + 1 + \mu)\,\psi_2 - (\alpha + \mathrm{E}r/hc)\,\psi_1 = \mp\,(m_0c/h)\,r\psi_1.$$

If a_1r^λ and a_2r^λ are the leading terms in the expansions of ψ_1 and ψ_2 in ascending powers of r, then

$$(\lambda + 1 - \mu)\,a_1 + \alpha a_2 = 0,$$

and

$$(\lambda + 1 + \mu)\,a_2 - \alpha a_1 = 0.$$

Hence

$$\lambda = -1 \pm (\mu^2 - \alpha^2)^{\frac{1}{2}}.$$

Now in order that the integral,

$$\int_a^b (\psi_1{}^2 + \psi_2{}^2)\,r^2\,dr,$$

should converge as $a \to 0$, it is necessary that

$$2\lambda + 2 > -1, \qquad \textit{i.e.}, \qquad \lambda > -\tfrac{3}{2}.$$

Since μ cannot vanish, this condition allows only those solutions for which

$$\lambda = -1 + (\mu^2 - \alpha^2)^{\frac{1}{2}}.$$

In order that the integral should also converge as $b \to \infty$, it is necessary that E should satisfy the equation

$$\frac{\alpha \mathrm{E}}{(m_0{}^2c^4 - \mathrm{E}^2)^{\frac{1}{2}}} = n_0 + (\mu^2 - \alpha^2)^{\frac{1}{2}}. \tag{2.3}$$

where n_0 is a positive integer. Hence the difference between E and the proper energy m_0c^2 is given approximately by the equation

$$\mathrm{E} - m_0c^2 \doteqdot -\,(\alpha^2/2n^2)\,m_0c^2, \qquad (n = n_0 + |\mu|),$$

which is equivalent to Bohr's classical formula. The solutions with the specified values of λ and E are the wave functions for a hydrogen atom with serial quantum l and inner quantum number j, determined by the equations,

$$l = \mu, \qquad j = \mu - \tfrac{1}{2}, \qquad (\mu > 0),$$
$$l = -\mu - 1, \qquad j = -\mu - \tfrac{1}{2}, \qquad (\mu < 0).$$

The first order equation possesses no other solutions which are quadratically integrable over the range, $0 \leqslant r \to \infty$.

3. *The Second Order Wave Equation.*

Since the first order equation has the form (2.1),

$$(\mathrm{F} \pm im_0c)\,\psi = 0,$$

it is natural to take as the second order equation that which is obtained by removing the ambiguous sign of im_0c, *i.e.*,

$$(\mathrm{F} \mp im_0c)\,(\mathrm{F} \pm im_0c)\,\psi = 0,$$

or

$$\mathrm{F}^2\psi = -\,m_0^{\,2}c^2\psi.$$

Now

$$\mathrm{F}^2 = (p_1^{\,2} + p_2^{\,2} + p_3^{\,2}) - c^{-2}\,(\mathrm{E} + e^2/r)^2 + i\,(he^2/cr^2)\,\rho'_2. \tag{3.1}$$

This equation may be considerably simplified by introducing the operator

$$\gamma = -\,\rho_3\kappa + i\alpha h\rho'_2. \tag{3.2}$$

Since ρ'_2 anti-commutes with κ, it follows that

$$\gamma^2 = \kappa^2 - \alpha^2h^2. \tag{3.3}$$

Now

$$\kappa^2 = (m_1^{\,2} + m_2^{\,2} + m_3^{\,2}) - h\,(m, \sigma) + h^2.$$

Hence

$$\gamma^2 + h\gamma = (m_1^{\,2} + m_2^{\,2} + m_3^{\,2}) + i\alpha h^2\rho'_2 - \alpha^2h^2 \tag{3.4}$$

But

$$p_1^{\,2} + p_2^{\,2} + p_3^{\,2} = p_r^{\,2} + (m_1^{\,2} + m_2^{\,2} + m_3^{\,2})\,r^{-2}.$$

Therefore

$$\mathrm{F}^2 = p_2^{\,2} + \frac{\gamma^2 + h\gamma}{r^2} - \frac{\mathrm{E}^2}{c^2} - \frac{2\mathrm{E}e^2}{c^2r},$$

and the second order wave equation is

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2\partial\psi}{r\partial r} + \left\{\frac{\mathrm{E}^2 - m_0^{\,2}c^4}{h^2c^2} + \frac{2\alpha^2\mathrm{E}}{e^2r} - \frac{\gamma^2 + h\gamma}{h^2r^2}\right\}\psi = 0. \tag{3.5}$$

γ and κ commute, and their simultaneous proper values are of the form gh and μh, where

$$g = \pm\,(\mu^2 - \alpha^2)^{\frac{1}{2}}$$

and

$$\mu = \pm\,1, \pm\,2, \ldots \qquad (\mu \neq 0, !).$$

Hence, on replacing γ by its proper value, (3.5) reduces to

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2\partial\psi}{r\partial r} + \left\{\frac{\mathrm{E}^2 - m_0^{\,2}c^4}{h^2c^2} + \frac{2\alpha^2\mathrm{E}}{e^2r} - \frac{g\,(g+1)}{r^2}\right\}\psi = 0. \tag{3.6}$$

This equation is of the same form as Schrödinger's equation for the non-relativistic hydrogen atom, the theory of which is well known.

In view of the fact that the main object of Dirac's investigation was to construct a first order wave equation, linear alike in the energy E and the momentum operators p_1, p_2, p_3, it may appear a retrograde step to replace the first order equation by a second order equation. But the first order equation possesses no solutions other than those describing the orthodox stationary states of the hydrogen atom, as was stated at the end of the last section. Hence the first order equation is clearly incompetent to furnish a theory of the neutron, and it must be modified in some way if the neutron is to be fitted into the quantum theory as a kind of collapsed hydrogen atom. Once the need for some modification is admitted, it will be recognized that the construction of the second order equation is probably the simplest and most natural modification possible.

This second order equation

$$(\mathrm{F}^2 + m_0{}^2c^2)\,\psi = 0,$$

possesses, of course, all the solutions of the first order equation

$$(\mathrm{F} \pm i m_0 c)\,\psi = 0.$$

These are the "H"-solutions, referred to in the Introduction. The second order equation also possesses another set of solutions, quadratically integrable over the range $0 \leqslant r \rightarrow \infty$. These are the "N"-solutions, which are here regarded as the wave functions of a neutron.

4. *The Stationary States of the Neutron.*

It will now be shown that the energy levels of the "N"-solutions form a continuous spectrum, extending over the half-open interval,

$$-\,m_0c^2 \leqslant \mathrm{E} < m_0c^2.$$

The solution for the ground level, $\mathrm{E} = -\,m_0c^2$, requires a separate discussion: the other solutions can be easily constructed as follows:—

If $-\,m_0c^2 < \mathrm{E} < m_0c^2$, the substitutions,

$$\psi = r^{-1}\,\mathrm{W}, \qquad z = 2\,\{(m_0{}^2c^4 - \mathrm{E}^2)/(h^2c^2)\}^{\frac{1}{2}}\,r,$$

reduce the wave equation (3.6) to its canomical form,

$$\frac{d^2\mathrm{W}}{dz^2} + \left\{-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right\}\mathrm{W} = 0, \tag{4.1}$$

where

$$k = \alpha\mathrm{E}/(m_0{}^2c^4 - \mathrm{E}^2)^{\frac{1}{2}}, \qquad m = |g + \tfrac{1}{2}|.$$

It is required to determine the solutions of this equation for which the integral

$$\int_a^b \psi^2 r^2\,dr \qquad \text{or} \qquad \int_a^b \mathrm{W}^2\,dr, \tag{4.2}$$

converges as $a \to 0$ and $b \to \infty$.

Two independent solutions of (4.1) are provided by Whittaker's functions, $\mathrm{W}_{k,m}(z)$ and $\mathrm{W}_{-k,m}(-z)$, whose asymptotic behaviour as $r \to \infty$ is specified by the relations,

$$\mathrm{W}_{k,m}(z) \sim e^{-\frac{1}{2}z} z^k \{1 + \mathrm{O}(z^{-1})\},$$

and

$$\mathrm{W}_{-k,m}(-z) \sim e^{\frac{1}{2}z} (-z)^k \{1 + \mathrm{O}(z^{-1})\}.$$

Hence, for the integrals (4.2) to converge as $b \to \infty$, the solution W must be a numerical multiple of $\mathrm{W}_{k,m}(z)$.

As $z \to 0$, $\mathrm{W}_{k,m}(z)$ is of order $z^{\frac{1}{2}-m}$, unless $(k - m + \frac{1}{2})$ is equal to a positive integer n', in which case $\mathrm{W}_{k,m}(z)$ is of order $z^{\frac{1}{2}+m}$. Hence the integral (4.2) will converge as $a \to 0$ only in the following cases :—

(H)

$$k - m + \tfrac{1}{2} = n',$$

i.e.,

$$\frac{\alpha\mathrm{E}}{(m_0{}^2c^4 - \mathrm{E}^2)^{\frac{1}{2}}} = n' + |g + \tfrac{1}{2}| - \tfrac{1}{2}; \tag{4.3}$$

and (N)

$$1 - 2m > -1,$$

i.e.,

$$|g + \tfrac{1}{2}| = m < 1.$$

Since $g = \pm(\mu^2 - \alpha^2)^{\frac{1}{2}}$, this inequality can be satisfied only if $\mu^2 = 1$ and $g = -(1 - \alpha^2)^{\frac{1}{2}}$.

In case "H," the solutions yield the accepted wave functions of the hydrogen atom. As $r \to 0$, these functions are of order $z^{m-\frac{1}{2}}$. They therefore tend to zero as $r \to 0$, unless $g = -(1 - \alpha^2)^{\frac{1}{2}}$ in which case they present a mild infinity of order $r^{[(1-\alpha^2)^{1/2}-1]}$, *i.e.*, of a lower order than $r^{-\frac{1}{2}\alpha^2}$. Equation (4.3) for the energy levels is equivalent to equation (2.3) obtained from the first order equation.

In case "N," the solutions are quite distinct in character from the "H"-solutions. As $r \to 0$, the corresponding wave functions, ψ, become infinite of order $r^{-(1-\alpha^2)^{1/2}}$, but ψ^2, and even $r^{-1}\psi^2$, remains integrable over the range $(0, \infty)$. In this case the only limitations on E are the inequalities,

$$-m_0c^2 < \mathrm{E} < m_0c^2,$$

so that the energy spectrum is continuous.

The value of the serial quantum number l for these solutions can be obtained by the following argument. It follows from equation (3.4) that, neglecting terms of order α the operator $(m_1^2 + m_2^2 + m_3^2)$ is approximately equal to $(\gamma^2 + h\gamma)$. Hence, approximately,

$$l(l+1)h^2 \doteqdot g(g+1)h^2$$

and

$$l + \tfrac{1}{2} \doteqdot |g + \tfrac{1}{2}| = (1-\alpha^2)^{\frac{1}{2}} - \tfrac{1}{2}.$$

Now l is a positive integer or zero. Therefore the accurate value of l is zero. Also, since $\mu^2 = 1$ the inner quantum number j is $|\mu| - \frac{1}{2}$ or $\frac{1}{2}$.

If $E^2 > m_0^2c^4$ the asymptotic value of the solutions of the wave equation (3.6) is of the form

$$\exp\{\pm i(pr + q\log r)\}, \qquad (p \text{ and } q \text{ real}).$$

Hence no solution of the equation can render the integrals (4.2) convergent as $b \to \infty$. Therefore, if $E^2 > m_0^2c^4$ there are no " closed " states. (The solutions is this case represent electrons fired at a stationary proton.)

5. *The Normal State of the Neutron.*

It remains to consider the solutions when $E = \pm mc^2$. In this case the wave equation (3.6) can be reduced to a standard form by the substitutions,

$$\psi = r^{-\frac{1}{2}}w, \qquad x = r^{\frac{1}{2}},$$

which yield the equation,

$$\frac{d^2w}{dx^2} + \frac{dw}{xdx} + \left\{\frac{8\alpha^2 E}{e^2} - \frac{(2g+1)^2}{x^2}\right\} w = 0. \qquad (5.1)$$

Two independent solutions of this equation are the Bessel functions

$$J_\nu\{(8\alpha^2 E/e^2)^{\frac{1}{2}} x\},$$

of orders

$$\nu = \pm|2g+1|.$$

If $E = m_0c^2$, the arguments of the Bessel functions are real. Now, as $z \to \infty$, $J_\nu(z) \sim (2/\pi z)^{\frac{1}{2}} \cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$. Hence, the asymptotic value of the wave function ψ is of the form

$$\psi \sim r^{-\frac{3}{4}} \cos(pr^{\frac{1}{2}} + q),$$

where p and q are real. Therefore the integral (4.2) cannot converge as $b \to \infty$. Hence $E = m_0^2c^2$ is *not* one of the energy levels allowed by the second order wave equation.

If, however, $E = -m_0c^2$ the arguments of the Bessel functions are wholly imaginary. It is convenient to take as a new variable

$$z = (8\alpha^2 m_0 c^2 r/e^2)^{\frac{1}{2}} = (8r/a_0)^{\frac{1}{2}},$$

where a_0 is the radius of the first Bohr orbit of hydrogen.

Equation (5.1) then becomes

$$\frac{d^2w}{dz^2} + \frac{dw}{zdz} - \left\{1 + \frac{\nu^2}{z^2}\right\} w = 0,$$

where

$$\nu = |2g + 1|.$$

Two independent solutions of this equation are the Bessel functions $I_\nu(z)$ and $I_{-\nu}(z)$ defined by the series,

$$I_\nu(z) = \sum_{s=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2s}}{s!\,\Gamma(\nu+s+1)}. \tag{5.2}$$

As $z \to \infty$, both of these functions have the asymptotic value $(2\pi z)^{-\frac{1}{2}} e^z$, whence the corresponding solutions of the wave equation (3.6) must also become infinite as $r \to \infty$. Hence the only combination of the Bessel functions which can lead to an admissible solution, ψ, is the difference, $I_\nu(z) - I_{-\nu}(z)$, which is of order $z^{-\frac{1}{2}} e^{-z}$ as $z \to \infty$.

The standard function of this type is Macdonald's function

$$K_\nu(z) = \tfrac{1}{2}\pi \operatorname{cosec}(\pi\nu)\{I_{-\nu}(z) - I_\nu(z)\}, \tag{5.3}$$

which can also be expressed as the definite integral,

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh \nu t \, dt.$$

The corresponding solution of the wave equation (3.6) is

$$\psi(r) = Ar^{-\frac{1}{2}} K_\nu\{(8r/a_0)^{\frac{1}{2}}\}. \tag{5.4}$$

(A being a numerical constant). Hence the integral (4.2) will converge as $b \to \infty$.

As $z \to 0$,

$$K_\nu(z) = O(z^{-\nu}),$$

whence,

$$\psi(r) = O(r^\lambda),$$

where

$$\lambda = -\tfrac{1}{2}\nu - \tfrac{1}{2} = -|g + \tfrac{1}{2}| - \tfrac{1}{2}$$
$$= -(\mu^2 - \alpha^2)^{\frac{1}{2}} - 1, \quad \text{if} \quad g > 0,$$

or

$$-(\mu^2 - \alpha^2)^{\frac{1}{2}}, \quad \text{if} \quad g < 0.$$

Hence the integral (4.2) will converge as $a \to 0$, if and only if

$$g < 0, \qquad \mu^2 = 1, \qquad \nu = 2(1 - \alpha^2)^{\frac{1}{2}} - 1, \qquad \text{and} \qquad \lambda = -(1 - \alpha^2)^{\frac{1}{2}}.$$

As before, these values imply that the serial and inner quantum numbers of the normal state of the neutron are $l = 0$ and $j = \frac{1}{2}$. The solution just obtained corresponds to a value of energy E equal to $-m_0c^2$. Hence the energy of the normal state of the neutron is

$$E_N = -m_0c^2.$$

It follows from (4.3) that the energy, E_H, of the normal state of the hydrogen atom, is given by

$$\alpha E_H = (m_0{}^2c^4 - E_H{}^2)^{\frac{1}{2}}$$

i.e.,

$$E_H = m_0c^2(1 + \alpha^2)^{-\frac{1}{2}}.$$

Hence the binding energy of the neutron is

$$E_N - E_N \doteqdot 2m_0c^2,$$

or about one million electron volts.

6. *The Dimensions of the Neutron in its Normal State.*

The spread of the electric charge in a neutron in its normal state can be estimated from the average values of r and r^{-1}, which can be calculated exactly as follows.

The numerical constant, A, in the expression for the wave function,

$$\psi(r) = Ar^{-} K_{\nu}\{(8r/a_0)^{\frac{1}{2}}\},$$

must be chosen so that

$$4\pi \int_0^\infty \psi^2(r)\, r^2\, dr = 1.$$

The average values of r and r^{-1} are then given by the integrals,

$$\bar{r} = 4\pi \int_0^\infty \psi^2(r)\, r^3\, dr,$$

and

$$\overline{r^{-1}} = 4\pi \int_0^\infty \psi^2(r)\, r\, dr.$$

In terms of the dimensionless variable, $z = (8r/a_0)^{\frac{1}{2}}$, these three equations become

$$\overline{r^{-1}} = 4\pi A^2 (2a_0/8) \int_0^\infty z K_\nu^2(z)\, dz,$$

$$1 = 4\pi A^2 (2a_0^2/8^2) \int_0^\infty z^3 K_\nu^2(z)\, dz,$$

$$\bar{r} = 4\pi A^2 (2a_0^3/8^3) \int_0^\infty z^5 K_\nu^2(z)\, dz.$$

The three integrals of the type, $\int_0^\infty z^r K_\nu^2(z)\, dz$, are evaluated in Appendix I. On inserting their values it is found that

$$\left.\begin{aligned} \overline{r^{-1}} &= (12/a_0)\,.\,(1-\nu^2)^{-1} \doteqdot 6/(\alpha^2 a_0) \\ A^2 &= \frac{1}{4\pi}\cdot\frac{8^2}{2a_0^2}\cdot\frac{3}{(1-\nu^2)}\cdot\frac{\sin\pi\nu}{\pi\nu} \doteqdot \frac{12}{\pi a_0^2} \\ \bar{r} &= \tfrac{1}{10}\, a_0 (4-\nu^2) \doteqdot \tfrac{3}{10}\, a_0 \end{aligned}\right\} \qquad (6.1)$$

The mean value of the electron-proton distance, $\bar{r} \doteqdot 3a_0/10 \doteqdot 1\cdot6 \times 10^{-9}$ cm. is decided less than the corresponding distance, $\frac{3}{2} a_0$, for the hydrogen atom in its normal state; but a much better idea of the effective size of the neutron is afforded by the mean value of the reciprocal of r^{-1}.

The value of $\overline{r^{-1}}$ implies that the average electrostatic potential energy of the neutron in its normal state, $\overline{-e^2/r}$, is the same as if the electron were at a fixed distance $\frac{1}{6}\alpha^2 a_0$ from the proton. Since $\alpha \doteqdot 1/137$, this distance is approximately $10^{-5} a_0$, or $0\cdot5 \times 10^{-13}$ cm. This distance is of the order of magnitude of the estimated radius of the neutron.

7. *The External Field of the Neutron in its Normal State.*

The average value of the electrostatic potential energy of a unit positive charge at a distance R from the nucleus of a neutron is equal to the potential of the nucleus, e/R, plus the potential due to a negative charge distributed with volume density $\rho(r) = -e\psi^2(r)$. This potential is the "external field" of the neutron. Its value for large values of R can be calculated from the asymptotic value of the wave function.

Since

$$K_\nu(z) \sim (\pi/2z)^{\frac{1}{2}}\, e^{-z}$$

it follows that

$$\psi(r) \sim A\,(4\pi/a_0)^{\frac{1}{2}}\, z^{-3/2}\, e^{-z},$$

and

$$\rho(r) \sim -48ea_0^{-3} z^{-3} e^{-2z}$$

using the approximate value of A given in equation (6.1). Hence the total charge outside a sphere of radius r, with centre at the nucleus, is approximately

$$(3\pi/8)\, ez^2 e^{-2z}.$$

The external field is therefore given, to a first approximation, by

$$\mathrm{V}(r) = \frac{3\sqrt{2}}{16} \frac{e}{(a_0 r)^{\frac{1}{2}}} \exp\{-2(8r/a_0)^{\frac{1}{2}}\}.$$

This expression is valid only when r is large compared with a_0.

The potential given by this formula decreases much less rapidly, as $r \to \infty$, than the corresponding potential assumed by Massey,* viz.,

$$\mathrm{V}(r) = e\left(\frac{1}{r} + \frac{\mathrm{Z}}{a_0}\right) \exp(-2\mathrm{Z}r/a_0), \qquad (\mathrm{Z} > 25{,}000),$$

but it is difficult to estimate the significance of the result deduced here, until it has been used to calculate the collision cross-section of a neutron.

8. *Summary.*

This paper develops a theory of the neutron on the basis of the second order wave equation for the hydrogen atom,

$$(\mathrm{F}^2 + m_0{}^2c^2)\, \psi = 0,$$

where F is Dirac's wave operator,

$$\mathrm{F} \equiv (p, \sigma)\, \rho_1 + ic^{-1}(\mathrm{E} + e^2/r)\, \rho_3.$$

It is shown that this equation possesses two types of solutions for which $\int_0^\infty \psi^2 r^2\, dz$ is finite—the type " H " which yields the accepted wave functions of the hydrogen atom, and the type " N " which is here identified with the wave functions of the neutron.

The following properties of the neutron are deduced from the form of the " N "-solutions :—

(1) The energy spectrum of the neutron covers the interval,

$$-m_0c^2 \leqslant \mathrm{E} < +m_0c^2,$$

* 'Proc. Roy. Soc.,' A, vol. 138, p. 460 (1932).

the energy of the normal state being $-m_0c^2$, and its binding energy, relative to the normal H-atom being, approximately, $2m_0c^2$ or 10^6 electron volts.

(2) For every energy level of the neutron, $l = 0$ and $j = \frac{1}{2}$. Hence the normal state of the hydrogen is really a metastable state from which no transitions can occur to the neutron states.

(3) The average value of r, the electron-proton distance in the normal neutron, is $\bar{r} = 3a_0/10$, where a_0 is the radius of the first Bohr orbit of hydrogen. The average value of r^{-1} is $\overline{r^{-1}} = 6/\alpha^2 a_0$, where α is the fine structure constant ($\alpha \doteqdot 1/137$).

(4) The external field of the normal neutron is

$$\mathrm{V}(r) = \frac{3\sqrt{2}}{16} \frac{e}{(a_0 r)^{\frac{1}{2}}} \exp\{-2(8r/a_0)^{\frac{1}{2}}\}.$$

Appendix I.

Evaluation of the Integrals of the Form $\int_0^\infty z^n \mathrm{K}_\nu^2(z)\,dz$.

The integrals required in § 6 are not explicitly given in Watson's "Treatise on Bessel Functions," but they can be easily evaluated by means of Lommel's integrals and Schafheitlin's reduction formula, given in §§ 5.11 and 5.14 of this book.

Since $\mathrm{K} \equiv \mathrm{K}_\nu(z)$ satisfies the equation

$$z^2\mathrm{K}'' + z\mathrm{K}' - (z^2 + \nu^2)\mathrm{K} = 0,$$

it follows that

$$\frac{d}{dz}\{z^2\mathrm{K}'^2 - (z^2 + \nu^2)\mathrm{K}^2\} = -2z\mathrm{K}^2.$$

Hence

$$\int_z^\infty z\mathrm{K}^2\,dz = \tfrac{1}{2}[z^2\mathrm{K}'^2 - (z^2 + \nu^2)\mathrm{K}^2].$$

Now the powers of z which appear in the series for K, (5.2) and (5.3) are

$$-\nu + 2s \quad \text{and} \quad \nu + 2s,$$

where $s = 0, 1, 2, \ldots$. Hence the series for K^2 and $z^2\mathrm{K}'^2$ contain terms which are independent of z, and these terms determine the value of the integral, all terms in $z^{-2\nu}$ cancelling, and all other terms tending to zero with z, since $\nu \doteqdot 1 - \alpha^2$. Therefore

$$\int_0^\infty z\mathrm{K}^2\,dz = \frac{\pi\nu}{2\sin\pi\nu}.$$

The reduction formula for the second integral follows from the identity

$$\frac{d}{dz}\{z^3\mathrm{K}'\,\mathrm{K} - z^2\mathrm{K}^2 - \tfrac{1}{2}z^2[z^2\mathrm{K}'^2 - (z^2+\nu^2)\,\mathrm{K}^2]\}$$
$$\equiv 3z^3\,\mathrm{K}^2 + 2\,(\nu^2-1)\,z\mathrm{K}^2.$$

Now the expression differentiated is of order $z^{2-2\nu}$ as $z \to 0$. Hence

$$\int_0^\infty z^3\mathrm{K}^2\,dz = \frac{2}{3}\,(1-\nu^2)\int_0^\infty z\mathrm{K}^2\,dz.$$

Similarly, it follows from the relation

$$\frac{d}{dz}\,\{z^4\,[z^2\,\mathrm{K}'^2 - (z^2+\nu^2)\,\mathrm{K}^2] - 4z^5\,\mathrm{K}'\mathrm{K} + 8z^4\,\mathrm{K}^2\}$$

that

$$\int_0^\infty z^5\mathrm{K}^2dz = \frac{4}{5}\,(4-\nu^2)\int_0^\infty z^3\mathrm{K}^2\,dz.$$

Appendix II.

The "N"-solutions of Schrödinger's Wave Equation for the Hydrogen Atom.

It is interesting to note that Schrödinger's non-relativistic wave equation for the hydrogen atom,

$$\frac{d^2\psi}{dr^2} + \frac{2d\psi}{rdr} + \left\{\frac{2m_0\mathrm{E}}{h^2} + \frac{2}{a_0r} - \frac{l\,(l+1)}{r^2}\right\}\psi = 0,$$

possesses a second set of solutions, quadratically integrable over all space, in addition to its "H"-solutions, which give the non-relativistic wave functions of H.

It is easily seen that such solutions cannot exist if $\mathrm{E} > 0$.

If $\mathrm{E} < 0$, the transformation,

$$\psi = r^{-1}\,\mathrm{W}, \quad z = (-\,8m_0\mathrm{E})^{\frac{1}{2}}\;r/h,$$

reduces the wave equation to its canonical form,

$$\frac{d^2\mathrm{W}}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z^2}\right\}\mathrm{W} = 0,$$

where

$$k = \alpha\,(-\,m_0c^2/2\mathrm{E})^{\frac{1}{2}}, \quad m = l + \tfrac{1}{2}.$$

Hence, as in § 4, it follows that admissible solutions exist only in the following cases :—

(H) $$k - m + \tfrac{1}{2} = n' = 1, 2, 3, \ldots$$

i.e.,

$$E/m_0c^2 = -\alpha^2/2\,(n' + l)^2\,;$$

and (N)

$$1 - 2m > -1,$$

i.e.,

$$l = 0.$$

In case "H," the solutions yield the accepted wave functions of the non-relativistic hydrogen atom. These solutions all remain finite as $r \to 0$. In case "N," the solutions are of the form,

$$\psi\,(r) = Az^{-1}\,W_{k,\frac{1}{2}}\,(z).$$

As $r \to 0$, they are of order r^{-1}. These solutions appear to represent a non-relativistic neutron, with serial quantum number $l = 0$, and with energy E lying in the continuous spectrum, $E < 0$.

If $E = 0$, the solutions of the wave equation are of the form

$$r^{-\frac{1}{2}}\,J_\nu\,\{(8r/a_0)^{\frac{1}{2}}\} \qquad \text{and} \qquad r^{-\frac{1}{2}}\,Y_\nu\,\{(8r/a_0)^{\frac{1}{2}}\},$$

where $\nu = 2l + 1$, and there are therefore no admissible integrals.