Escape of molecules from a plane surface into a still atmosphere

References

Fuchs 1934 Phys. Z. Sovjet. 6, 224.
Maxwell 1890 Scientific Papers, 2, 635. Cambridge University Press.
Nernst 1904 Z. phys. Chem. 47, 52.
Noyes & Whitney 1897 Z. phys. Chem. 23, 689.

On the propagation of electromagnetic waves through the atmosphere

BY B. K. BANERJEA, M.Sc., Palit Laboratory of Physics,
University College of Science, Calcutta, India

(Communicated by M. N. Saha, F.R.S.—Received 18 July 1946)

A general method of tackling the problem of the propagation of electromagnetic waves in the ionosphere has been developed and the current methods of Appleton, Hartree, Saha, Rai and Mathur, etc., have been deduced as special cases from the general results. The different assumptions by Appleton, Hartree, Bose, Booker and Rai, as regards the condition of reflection of the waves from the ionosphere, have been shown to be identical. A symbol-correspondence chart for the different symbols used by the different workers has been given to facilitate the understanding of the parallelism between the different methods. Polarization of the radio waves have been discussed fully.

Introduction

The existence of the region of the atmosphere which we now call ‘ionosphere’ was first postulated by Balfour Stewart (1878) to explain the daily variation of the earth’s magnetic field, but it was in 1902 that Kennelly and Heaviside pointed out that the hypothesis of a conducting layer at a height of about 100 km. can explain the paradoxical result that wireless signals can, in spite of the curvature of the earth, be propagated over great distances. It was, however, as late as 1925, when its existence was first directly proved by Appleton & Barnett (1925) in England by the slow variation of the transmitter wave-length and, later, by Breit & Tuve (1926) in the U.S.A. by the use of the pulse technique method. The latter method, which is now in universal use for investigation of the ionosphere, gives apparently spectacular exhibition of its existence, from the appearance of the reflected echo on the cathode ray oscillograph on a properly calibrated time axis.

The detailed interpretation of these echoes, however, is not so simple, as has been emphasized by various workers, and the reason is twofold: (a) experimental, and (b) theoretical.
Experimental methods in common use are: (i) The angle of incidence method developed by Appleton & Barnett (1925) to find out the height of the ionosphere consists mainly in comparing the signal intensity received on a vertical with that received on a loop aerial.

(ii) Wave-length change method also developed by Appleton & Barnett (1925). In this method the interference pattern of the ground wave and the sky wave are studied and the number of resultant signal maxima and minima are recorded under an artificial continuous change of phase of the transmitted wave-train.

(iii) Group retardation method: Breit & Tuve (1926) developed a novel method of measuring the time taken by the signal group to reach the ionosphere and then to come down to the receiver. A pulse of very short duration is transmitted and received at a moderate distance from the transmitter. Both the ground wave and the reflected wave are received on the same calibrated time axis of the cathode ray oscillograph. The sharp and regular echo received on the oscillograph screen points towards the existence of a reflecting layer. Various other workers like White (1931), Bruewitch (1934), Goubau & Zenneck (1930), Colwell, Friend & Hall (1936), Banerjee (1945) have developed new techniques for intensifying and shortening the duration of the transmitted pulse, and pulses of 20 to 50 kW peak power, and duration of $10^{-5}$ sec. are at present available for studying the properties of the ionosphere.

The appearance of an echo pattern adds weight to the assumption underlying the ray method of tackling the propagation problem that the ionosphere may be regarded as a sharply stratified medium, the width of each stratum being less than the wave used. It may be pointed out that the methods of observation are not yet refined enough to prove the correctness or otherwise of the assumption of gradual stratification in the ionosphere. Thus in ‘Breit and Tuve experiment’ on the group retardation method, let

$$\alpha \text{ mm. on the oscillograph screen } \equiv \tau \text{ sec.}$$

and

$$\Delta \text{ mm. be the least measurable width on the screen } \equiv 1 \text{ mm.}$$

Then the minimum width of the reflecting layer, as can be inferred from the sharpest echo pattern, is for the best type of transmitting and receiving arrangement $= \tau/\alpha \Delta n \simeq 1 \text{ km.} \simeq 10\lambda$ for the $E$ layer, where $n = \text{velocity of propagation of the waves}$. Thus even the sharpest echo pattern observed in the oscillograph screen does not justify the assumption underlying the ray method. Improved methods of generating very sharp pulses have already been devised, but the received echo is broadened owing to the fact that the band of reception of the receiver itself has a finite width. Still with these specially designed sharp pulse transmitters, Colwell, Friend & Hall (1936) have observed diffused echoes, showing that the wave is reflected not from a sharp layer, but from a diffused layer.

The theoretical difficulties are of a fundamental nature, for though after the great success of the group retardation method for investigating the properties of the ionosphere by sending wireless waves to and receiving them back from the ionosphere both vertically and obliquely, the actual problem of the propagation of the electro-
magnetic waves through the ionosphere has not been solved rigorously. The reasons are:

(a) The equation of propagation of electromagnetic waves through the ionosphere had not been rigorously developed.

(b) The boundary conditions, by which are meant the arrangement of the antenna, the conditions of the ground on which the antenna is situated and the physical conditions of the ionosphere through which the electromagnetic waves are propagated, are difficult to be properly formulated and utilized in the solutions of the equations.

The theoretical method most in vogue is the magneto-ionic theory developed by Appleton (1925) in England. It is also known as the ray method. The final form to this theory has been given by Appleton (1932) and Hartree (1931) and discussed further by Appleton (1935). A critical review of the methods based on the magneto-ionic theory is given below.

PRESENT THEORETICAL METHODS

In order to compare the merits of the different methods as well as to show their interconnexion, the different works will be transcribed to the same uniform notation and the different reference framework will be shown in the same diagram. In fact, it has been shown that the works of Appleton (1932), Hartree (1931), Saha, Rai & Mathur (1937) and Saha & Banerjea (1945) may be deduced from the same general results. At a later stage, a symbol correspondence chart will be given which should enable the reader to follow the works of different authors in their original form.

The theoretical work done on this subject mainly consists of two parts, viz.:

(1) Solution of the equation of motion of the charged particles in the earth’s magnetic field, under the influence of the electric vector of the incident electromagnetic wave. This displacement \( \rho(\xi, \eta, \zeta) \) has been used by the different workers for different purposes.

(2) Solution of the Maxwell equations.

1. Solution of the equation of motion

Taking a general framework \( XYZ \) the equation of motion of an electron in the earth’s magnetic field \( \mathbf{H} \) under the influence of the incident electric vector \( E e^{ipt} \) is

\[
\ddot{\rho} + \nu \dot{\rho} - \frac{e}{m_0 c} (\dot{\rho} \times \mathbf{H}) = \frac{e}{m_0} E e^{ipt},
\]

where \( \rho(\xi, \eta, \zeta) \) is the displacement of the electron, \( \nu \) the collisional frequency, \( e, m_0 \) the charge and mass of the electron (the sign of \( e \) has been already taken into account in the above equation) and \( c \) the velocity of light. The effect of Lorentz’ polarization term has been neglected, as the discussions of Darwin (1934) have shown it to be negligible.
Assuming that \( \rho \propto e^{ipt} \), equation (1) becomes

\[
\rho(-p^2 + ivp) - \frac{i\rho}{m_0c}[\rho \times \mathbf{H}] = \frac{e}{m_0} E e^{ipt}.
\]

Introducing a new dyad \([l \times \mathbf{H}]\) given by the matrix

\[
[l \times \mathbf{H}] = \begin{vmatrix} 0 & -H_z & H_y \\ H_z & 0 & -H_x \\ -H_y & H_x & 0 \end{vmatrix}, \quad l = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},
\]

we can express the vector product \([\rho \times \mathbf{H}]\) as the scalar product of the dyad \([l \times \mathbf{H}]\) with \(\rho\), i.e.

\[-[\rho \times \mathbf{H}] = [l \times \mathbf{H}] \cdot \rho.
\]

Then the equation of motion becomes

\[
-\frac{m_0p^2}{e} \left( \beta l - \frac{ie}{m_0pc}[l \times \mathbf{H}] \right) \rho = E e^{ipt},
\]

where \(\beta = 1 - iv/p\).

Putting \(\omega = \frac{eH}{m_0cp}\) and breaking the above vector equation into components, we get

\[
\begin{align*}
\begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} e^{ipt} &= -\frac{m_0p^2}{e} \begin{vmatrix} \beta & i\omega_z & -i\omega_y \\ -i\omega_z & \beta & i\omega_x \\ i\omega_y & -i\omega_x & \beta \end{vmatrix} \begin{vmatrix} \xi \\ \eta \\ \zeta \end{vmatrix}.
\end{align*}
\]

We can also express \(\rho\) in terms of \(E\) either by finding out the dyad reciprocal to \(\beta l - \frac{ie}{m_0cp}[l \times \mathbf{H}]\) by the usual method or from (3) by the usual determinant method. Thus

\[
\begin{vmatrix} \xi \\ \eta \\ \zeta \end{vmatrix} = \frac{A}{N e} \begin{vmatrix} \beta^2 - \omega^2 & \omega_x\omega_y + i\beta\omega_z & \omega_x\omega_z - i\beta\omega_y \\ \omega_y\omega_x - i\beta\omega_z & \beta^2 - \omega_y^2 & \omega_y\omega_z + i\beta\omega_x \\ \omega_z\omega_x + i\beta\omega_y & \omega_z\omega_y - i\beta\omega_x & \beta^2 - \omega_z^2 \end{vmatrix} \begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} e^{ipt}
\]

or \(\rho = S \cdot E\), where \(S\) is the dyad reciprocal to \(-\frac{m_0p^2}{e} \left( \beta l - \frac{ie}{m_0cp}[l \times \mathbf{H}] \right)\) occurring in equation (2) and given by the matrix in (4) with

\[
A = \frac{r}{4\pi\beta^2 - \omega^2}, \quad r = \frac{p_0^2}{\beta^2} = \frac{4\pi Ne^2}{m_0p^2},
\]

\(N = \text{number of ions per c.c.}\)

From the matrix for \(S\), we can write

\[
S = -\frac{e\beta}{m_0p^2(\beta^2 - \omega^2)} \left( 1 + \frac{ie}{m_0cp\beta}[l \times \mathbf{H}] - \frac{e^2}{m_0^2c^2p^2\beta^2}(HH) \right),
\]
Propagation of electromagnetic waves through the atmosphere

where \((HH)\) is the dyadic product of the vector \(H\). Instead of using the components of the displacement vector \(\mathbf{p}\) we can use the polarization vector \(P\) defined by

\[
P = 4\pi Ne\mathbf{p} = 4\pi Ne \mathbf{S} \cdot \mathbf{E} = \sigma_h \cdot \mathbf{E},
\]

where \(\sigma_h\) is called 'the scattering tensor' by Darwin.

Different workers used different frameworks and as a result, the components of the vector \(P\) are expressed in different forms. \(XYZ\) (figure 1) is any general framework with \(OZ\) as vertical; this framework has been used by Hartree (1932) and Saha, Rai & Mathur (1937). \(OH\) is the direction of the magnetic field and \(ZX'O'H\) is the magnetic meridian with \(OX'\) as the horizontal projection of the magnetic line of force. \(OY'\) is perpendicular to the magnetic meridian. Let \(OZ'\) be the line in \(ZX'O'H\), the magnetic meridian, perpendicular to \(OH\). Appleton (1932) used the framework \(X'Y'Z\) in which, for propagation along \(OZ\), \(H_x' = \) component of the magnetic field transverse to the direction of propagation = \(H_T\), \(H_y' = 0\), \(H_z' = \) component along the direction of propagation. Saha & Banerjea (1945) used the principal axes \(HY'Z'\) of the tensor \(\mathbf{S}\) in which \(\omega_H = \omega_x\), \(\omega_y' = \omega_y \), \(\omega_z' = \omega_z\) and \(\angle ZOZ' = \delta\), the dip angle.

![Figure 1](image.png)

The components of polarization vector \(P\) were obtained explicitly in this form (4) by Saha, Rai & Mathur (1937). Hartree (1931) obtained the expression for the displacement vector in which the electrons are assumed to be bound by a quasi-elastic force proportional to \(k_1^2\); as the ions in the ionosphere are free, we put \(k_1^2 = 0\) and then Hartree's expression for the displacement vector is identical with (4). Appleton's results are obtained from (3) by putting \(\omega_x' = \omega_T\), \(\omega_y' = 0\), \(\omega_z' = \omega_L\).

\[
E_x' = -\frac{1}{r} [\beta P_x' + i\omega_L P_y'],
\]

\[
E_y' = -\frac{1}{r} [-i\omega_L P_x' + \beta P_y' + i\omega_T P_z'],
\]

\[
E_z' = -\frac{1}{r} [-i\omega_T P_y' + \beta P_z'],
\]

\((6')\)
B. K. Banerjea

Saha & Banerjea's results also follow from (3) by putting \( \omega_H = \omega, \omega_H' = \omega_x = 0 \):

\[
E_H = -\frac{\beta}{r} P_H, \quad (E_{y'} \pm i E_{z'}) = -\frac{\beta \pm \omega}{r} (P_{y'} \pm i P_{z'}). 
\]

(7)

2. Solution of the Maxwell equation

Next we proceed to set up the propagation equations for the electric and magnetic vectors of the electromagnetic wave from the Maxwell's equations, viz.

\[
\nabla \times H = \frac{1}{c} \dot{D}, \quad \nabla \times E = -\frac{1}{c} \dot{H},
\]

\[
\nabla_0 H = \nabla_0 D = 0, \quad D = E + P.
\]

(8)

Then

\[
\nabla \times \nabla \times E + \frac{1}{c^2} = 0,
\]

(9)

and

\[
\nabla \times \nabla \times H - \frac{1}{c} \nabla \times \dot{D} = 0.
\]

(10)

Also from (6), since \( P = 4\pi NeS E \), we get

\[
D = E + 4\pi NeS E = (1 + 4\pi NeS) \cdot E = K \cdot E,
\]

(11)

where \( K \) is the complex dielectric tensor and given by the matrix.

\[
K = \begin{vmatrix}
1 - 4\pi A(\beta^2 - \omega_z^2) & 4\pi A(\omega_x \omega_y + i \beta \omega_z) & 4\pi A(\omega_x \omega_z - i \beta \omega_y) \\
4\pi A(\omega_x \omega_y - i \beta \omega_z) & 1 - 4\pi A(\beta^2 - \omega_y^2) & 4\pi A(\omega_y \omega_z + i \beta \omega_x) \\
4\pi A(\omega_x \omega_z + i \beta \omega_y) & 4\pi A(\omega_y \omega_z - i \beta \omega_x) & 1 - 4\pi A(\beta^2 - \omega_x^2)
\end{vmatrix}
\]

(12)

\[
= \begin{vmatrix}
1 - \frac{r}{\beta} & 0 & 0 \\
0 & 1 - \frac{r \beta}{\beta^2 - \omega^2} & \frac{i r \omega}{\beta^2 - \omega^2} \\
0 & \frac{i r \omega}{\beta^2 - \omega^2} & 1 - \frac{r \beta}{\beta^2 - \omega^2}
\end{vmatrix}
\]

(12')

the former form (12) being with respect to the general co-ordinates, while the latter (12') with respect to the principal axes \( OH, OY', OZ' \), as used by Saha & Banerjea (1945).

The vectors \( E, H \) and \( D \) occurring in the Maxwell equations (9) and (10) are the field vectors within the ionosphere. The total electric field \( E \), used in (9), however, may be looked upon, as suggested by Darwin (1924), as the superimposition of the incident field \( E_i \) and the resultant scattered field \( E_s \).

Let us next consider incident wave given by the electric vector

\[
E_i = E_0 e^{i(pt - k_{\infty}r)},
\]

where \( N_0 \) is the wave-normal vector with components \( l, m, n; r \) the propagation vector, and \( k \) is the wave number \( p/c \). If we picture the ionosphere as a homogeneous
Propagation of electromagnetic waves through the atmosphere

refracting medium, separated from the region where \( \mu = 1 \), by a sharp boundary (or stratified layer), and consider the wave travelling from below without change of wave form, we can put the refracted wave as represented by

\[
\begin{align*}
E &= E_0 e^{i(\mu l - kN_0 r)}, \\
D &= D_0 e^{i(\mu l - kN_0 r)}, \\
H &= H_0 e^{i(\mu l - kN_0 r)},
\end{align*}
\]

(13)

where \( \mathbf{N} \) is a vector with components \( l, m \) and \( q \). For vertical propagation \( \mathbf{N}_0 \mathbf{r} = qz \) and \( q \) becomes equal to the complex refractive index.

Then, from (13),

\[
[\nabla \times \ ] = -i k [\mathbf{N} \times \ ],
\]

\[
[\nabla \times \nabla \times \ ] = k^2 [N^2 I - (\mathbf{N} \mathbf{N})].
\]

Thus the wave equation for the electric vector becomes

\[
[N^2 I - (\mathbf{N} \mathbf{N})] E = D,
\]

(14a)

or

\[
[(N^2 - 1) I - (\mathbf{N} \mathbf{N})] E = P,
\]

(14b)

or

\[
[K - N^2 I + (\mathbf{N} \mathbf{N})] E = 0.
\]

(14c)

Similarly for the magnetic vector

\[
[(N^2 - 1) I - (\mathbf{N} \mathbf{N})] \mathbf{H} = [I \times \mathbf{N}] \mathbf{P}.
\]

(15)

Equations (14) and (15) are the most general equations of propagation of the electromagnetic field vectors in a homogeneous ionosphere. Equations used by Hartree (1931), Appleton (1932), Saha, Rai & Mathur (1937), etc., can be obtained as special cases of (14) and (15).

Thus Hartree confines himself to a wave polarized in the \( XZ \) plane and inclined at an angle \( \theta \) with \( OZ \). Then the components of the vector \( \mathbf{N} \) occurring in (14) and (15) becomes \( \sin \theta \), \( 0 \), \( q \) and (14a) becomes

\[
\begin{align*}
D_x &= q^2 E_x - q \sin \theta E_z, \\
D_y &= (q^2 + \sin^2 \theta) E_y, \\
D_z &= -q \sin \theta E_x + \sin^2 \theta E_z,
\end{align*}
\]

(16)

which are identical with equations of Hartree’s (1931) work. Hartree did not work out the corresponding equation for the magnetic vector.

Appleton considered the case of vertical propagation of a linearly polarized wave and then the components of \( \mathbf{N} \) are \( O, O, q \), where \( q \) is the complex refractive index. Then (14b) becomes,

\[
\begin{align*}
(q^2 - 1) E_x &= P_x, \\
(q^2 - 1) E_y &= P_y, \\
E_z &= -P_z,
\end{align*}
\]

(17)
and (15) becomes
\[
\begin{aligned}
(q^2 - 1) H_x &= q P_y' \\
(q^2 - 1) H_y &= -q P_x' \\
H_z &= 0.
\end{aligned}
\tag{18}
\]

These equations are identical with equations (26) of Appleton (1932). Equations (17), again, can be obtained from (16) by putting \(\theta = 0\) when Hartree's case coincides with Appleton's.

In the Saha-Banerjea framework (1945) for vertical propagation, along \(OZ\) the components of \(\mathbf{N}\) along \(OH, OY'\) and \(OZ'\) are \(-q \sin \delta, O, q \cos \delta\), where \(\delta\) is the dip angle. Correspondingly, the wave equations become
\[
\begin{aligned}
D_H &= q^2 \sin^2 \delta E_H - q^2 \sin \delta \cos \delta E_{\varphi}' \\
D_{\varphi}' &= q^2 E_{\varphi}' \\
D_{\varphi} &= -q^2 \sin \delta \cos \delta E_H + q^2 \sin^2 \delta E_{\varphi}'.
\end{aligned}
\tag{19}
\]

and
\[
\begin{aligned}
(q^2 - 1) H_H &= q \cos \delta P_{\varphi}' \\
(q^2 - 1) H_{\varphi}' &= -q \cos \delta P_H - q \sin \delta P_{\varphi} \\
(q^2 - 1) H_{\varphi} &= -q \sin \delta P_{\varphi}'.
\end{aligned}
\tag{20}
\]

We can also evaluate the expression for the complex refractive index from equation (14c), which states that the determinant of the tensor \([K - \mathbf{N}^2] + (\mathbf{N}\mathbf{N})\] must vanish. Since the value of refractive index does not depend on the direction of propagation of the wave and since the value of a determinant is independent of the axes used, we confine ourselves to vertical propagation and to Saha-Banerjea framework, which is the principal framework and so four out of nine components of \(K\) vanish. Thus \(N_H = -q \sin \delta, N_{\varphi}' = 0, N_{\varphi} = q \cos \delta\), where \(q\) is the required complex refractive index. The refractive index equation becomes
\[
\begin{vmatrix}
1 - \frac{r}{\beta} - q^2 + q^2 \sin^2 \delta & 0 & -q^2 \sin \delta \cos \delta \\
0 & 1 - \frac{r \beta}{\beta^2 - \omega^2} - q^2 & \frac{ir\omega}{\beta^2 - \omega^2} \\
-q^2 \sin \delta \cos \delta & -\frac{ir\omega}{\beta^2 - \omega^2} & 1 - \frac{r \beta}{\beta^2 - \omega^2} - q^2 + q^2 \cos^2 \delta
\end{vmatrix} = 0
\]

or
\[
q^2 = 1 - \frac{2r}{2\beta + \frac{\omega^2 \cos^2 \delta}{(r - \beta)} \pm \sqrt{\left(\frac{\omega^4 \cos^4 \delta}{(r - \beta)^2} + 4\omega^2 \sin^2 \delta\right)}}
\tag{21}
\]

which is identical with Appleton's equation.
Polarization of the radio waves

The electromagnetic waves in the ionosphere are not strictly transverse with respect to the electric vector for \((\mathbf{E}, \mathbf{r}) = 0\) but they are transverse with respect to the magnetic vector as \((\mathbf{H}, \mathbf{r}) = 0\). Thus, it has become customary to express the nature of the polarization of the radio waves in terms of the magnetic vector. From (13) and (14) it can be easily shown that

\[
\frac{f}{H_y} = \frac{i}{\omega_L} \left( \frac{r}{1-q^2} - \beta \right) = \rho e^{-i\phi}, \quad \text{say.}
\]

Eliminating \((1-q^2)\) from this and equation (14) we get,

\[
f + \frac{1}{f} = \frac{i \omega_T}{\omega_L} \left( r - \beta \right).
\]

This is a quadratic equation in \(f\), and, if \(f_1, f_2\) be the roots,

\[
f_1 f_2 = 1, \quad \text{i.e.} \quad \rho_1 = 1/\rho_2, \quad \phi_1 = -\phi_2.
\]

In order to get the actual form of the polarization ellipse, we put \(H_x = \text{real part of } Ae^{i\omega t}\) and \(H_y = \text{real part of } \rho A e^{i(\omega t - \phi)}\). Eliminating the time factor \(\omega t\) we get

\[
H_x^2 - \frac{2H_x H_y \cos \phi}{\rho} + \frac{H_y^2}{\rho^2} = A^2 \sin^2 \phi.
\]

Since the discriminant of this quadratic form is negative, the vector whose components are \(H_x, H_y\) is an ellipse in the \(XY\) plane. The points of contact of the ellipse with the circumscribed rectangle are \((\pm A \rho, \pm A \cos \phi), (\pm A \cos \phi, \pm A \rho)\). In general the axes of the ellipse do not coincide with the co-ordinate axes but the two systems may be brought to coincidence by rotating the ellipse through an angle \(\psi\) (called the tilt of the polarization ellipse) about the \(Z\)-axis, such that the product of the components of the vector \(\mathbf{H}\) in the new co-ordinates vanish. This gives

\[
\tan 2\psi = -\frac{2\rho \cos \phi}{1 - \rho^2}.
\]
The second ellipse corresponding to the other root of the equation (22) is given by

$$\frac{H_x^2}{\rho_x^2} - \frac{2H_x H_y \cos \phi}{\rho} + H_y^2 = \frac{A}{\rho^2} \sin^2 \phi.$$  \hfill (24)

Since $H_x = Re A e^{i\omega t}$, $H_y = Re A e^{i(\omega t - \phi)}$, for $\phi$ positive the sense of rotation is clockwise and for $\phi$ negative the sense is anticlockwise.

Let us next find out the values of $\rho$ and $\phi$. From the relation

$$\frac{i}{\omega L} \frac{\omega_T^2}{\rho - \beta} = \rho e^{-i\phi} + \frac{1}{\rho} e^{i\phi},$$

we can obtain, by equating the real and imaginary parts,

$$(r - 1)(\rho + 1/\rho) \cos \phi + (v/p)(\rho - 1/\rho) \sin \phi = 0,$$

$$(v/p)(\rho + 1/\rho) \cos \phi - (r - 1)(\rho - 1/\rho) \sin \phi = \omega_T^2/\omega_L^2.$$  

From these two expressions it is possible to find out the values of $\rho$ and $\phi$, but the equations become somewhat unwieldy. Let us first take the case where friction $v$ can be neglected as this case leads to considerable simplifications. We have from above

$$\phi = \frac{1}{2} \pi,$$

$$\rho = \frac{\omega \cos^2 \delta}{2(1 - r) \sin \delta} \left\{ 1 + \sqrt{\left( 1 + \frac{4 \sin^2 \delta (1 - r)^2}{\omega^2 \cos^4 \delta} \right)} \right\}. \hfill (25)$$

This expression was obtained by Saha, Rai & Mathur (1937) and was discussed fully by them, with special reference to the polarization of waves of 100 m. length at selected stations ranging from the magnetic north to the magnetic south pole.

These relations have been verified by Appleton & Ratcliffe (1928) in England. They put $r = 0$ for the reflected wave and get

$$\rho_1 = 0.9633, \quad \rho_2 = -1.0373 \quad \text{for} \quad \lambda = 100\, \text{m}.$$  

Both the $O$-wave and the $x$-wave are therefore nearly circularly polarized and the sense of rotation for the $O$-wave is anti-clockwise showing $\phi$ negative as we would expect from (24).

For Australia, $\delta_L = \text{negative}$, we have $\rho_1 = 0.9732, \rho_2 = 1.0274$; both the $O$- and the $x$-waves are again circularly polarized as observed by Green (1932).

For equatorial region, there is only one observation, due to Berkner & Wells (1937) at Huancayo. They found $\rho_1 = 1/0.129, \rho_2 = 1/7.643$ with the $O$-wave linearly polarized along the magnetic north-south and the $X$-wave along the magnetic east-west.

In order to link up the different methods the following transcription table is helpful.
### Unification of Notation of the Different Workers

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Appleton and his school</th>
<th>Hartree</th>
<th>Saha and his school</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field vectors</td>
<td>$D, E, H, P$</td>
<td>$D, L, H$</td>
<td>$D, E, H, P$</td>
</tr>
<tr>
<td>Earth's field</td>
<td>$H$</td>
<td>$H$</td>
<td>$H$</td>
</tr>
<tr>
<td>Refractive index</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Pulsatance</td>
<td>$p$</td>
<td>$K_c$</td>
<td>$p$</td>
</tr>
<tr>
<td>Gyrofrequency, $\frac{eH}{m_0c}$</td>
<td>$P_a$</td>
<td>$K_0c$</td>
<td>$P_a$</td>
</tr>
<tr>
<td>Collision frequency</td>
<td>$\nu$</td>
<td>$2K_c\nu$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>Displacement of the ion</td>
<td>$(x, y, z)$</td>
<td>$\rho$</td>
<td>$(\xi, \eta, \zeta)$</td>
</tr>
<tr>
<td>Dipole moment</td>
<td>$Ne(x, y, z)$</td>
<td>$Ne\rho$</td>
<td>$Ne(\xi, \eta, \zeta)$</td>
</tr>
<tr>
<td>Complex refractive index</td>
<td>$\varepsilon$</td>
<td>$K$</td>
<td>$q$</td>
</tr>
<tr>
<td>Conductivity</td>
<td>$\sigma, \sigma_\perp$</td>
<td></td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Phase velocity</td>
<td>$v$</td>
<td>$v$</td>
<td>$v$</td>
</tr>
<tr>
<td>Group velocity</td>
<td>$w$</td>
<td>$w$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

**Abbreviations used:**

\[
\begin{align*}
\frac{4\pi Ne^2}{m_0} & = p_0^2 & \quad & \rho_0^2 & \quad & p_0^3 \\
\text{Lorentz factor} & = \alpha & \quad & \beta & \quad & \omega \\
\frac{m_0p^2}{4\pi Ne^2} & = \frac{p^2}{p_0^2} & \quad & -\alpha & \quad & \frac{1}{\sigma_0} - \beta & \quad & \frac{1}{r} \\
\frac{m_0p}{4\pi Ne^2 \left[ \frac{eH}{m_0c} \right]} & = \frac{pp_h}{p_0^2} & \quad & \gamma & \quad & \frac{\rho}{\sigma_0} & \quad & \frac{\omega}{r} \\
\frac{eH}{m_0c(p - iv)} & = \frac{p_h}{\alpha} & \quad & -\gamma & \quad & \tau & \quad & \omega \\
\frac{4\pi Ne^2}{m_0cp} & = \frac{p_h}{\alpha} & \quad & \frac{4\pi A}{\beta^2} & \quad & \beta \\
\frac{m_0p^2(1 - iv/p)}{4\pi Ne^2} & = \frac{1}{\alpha^2 - \gamma^2} & \quad & \xi & \quad & \frac{1}{\beta} \\
\frac{m_0p^2(1 - iv/p)}{4\pi Ne^2} & = \frac{1}{\alpha + \alpha} & \quad & \xi & \quad & \sigma_0 & \quad & \frac{\rho}{\sigma} \\
\frac{m_0c^4(-p^2 + ivp)}{4\pi Ne^2} & = \frac{e}{\alpha} & \quad & \alpha & \quad & \frac{e}{m_0p^2\beta} \\
\text{Scattering tensor} & = \sigma & \quad & \frac{r\beta A}{\beta^2 - \omega^2} \\
\frac{m_0p\nu}{4\pi Ne^2} & = \beta & \quad & \frac{\delta}{r} \\
\end{align*}
\]
In order to explain the return of the radio waves from the ionosphere different workers have used different criteria:

(i) Appleton & Hartree take that the wave gets reflected when the refractive index vanishes and obtain the well-known relation \( p^2 = p_0^2 \) and \( p^{-2} = p_0^2 \pm pp_h \) as the conditions for reflexions.

(ii) Booker (1936) assumes that the wave is reflected when the Poynting vector \( \mathbf{S} \) becomes perpendicular to the wave normal \( \mathbf{N} \), i.e. for vertical propagation, the wave is reflected when the energy vector becomes horizontal.

(iii) In order to explain the fourth condition of reflexion, observed by Bajpai & Pant (1937) at Allahabad, Rai (1937) assumes that the vanishing of the group velocity should be regarded as the proper criterion for reflexion. He obtained in addition to Appleton-Hartree conditions, a fourth relation \( p_0^2 = p^2 \frac{p_0^2 - p_H^2}{p^2 - p_L^2} \approx p^2 - p_T^2 \) (for \( p_L \ll p_T \)) which explains the results of Bajpai & Pant.

(iv) Bose (1938) suggested that the wave gets reflected when the magnetic vector \( \mathbf{H} \) is zero. This can take place, either when \( \mathbf{E} \) is parallel to \( \mathbf{N} \) or \( \nu/c \approx \infty \). The first condition is, as will be shown presently, equivalent to Appleton’s condition.

All the above criteria are assumptions, and though probably they give us a rough approximation to the actual process of wave propagation in the ionosphere, they cannot be regarded as strictly mathematical solutions to the Maxwellian equations, not even for a homogeneous absorbing ionosphere. Further, in an absorbing medium as the ionosphere is, \( \mu \) can never become zero.

It is, however, shown below that the above four assumptions, though apparently different and probably intended by their authors to represent different criteria, are actually identical, when friction is neglected.

From the Maxwell’s field equations, for a homogeneous ionosphere

\[
\frac{\partial \mathbf{H}}{\partial t} = -c \mathbf{\nabla} \times [\mathbf{E}_0 e^{i\nu(t - \mathbf{N} \cdot \mathbf{v})}],
\]

where \( \mathbf{N} \) wave normal, \( \mathbf{r} \) propagation vector. Therefore

\[
\mathbf{H} = H_0 e^{i\nu(t - \mathbf{N} \cdot \mathbf{v})}, \quad H_0 = i\rho c/\nu [\mathbf{N} \times \mathbf{E}_0].
\]

Thus the magnetic vector is normal to the plane defined by \( \mathbf{N} \) and \( \mathbf{E} \). Again

\[
\frac{\partial \mathbf{D}}{\partial t} = c \mathbf{\nabla} \cdot \mathbf{H}, \quad \text{i.e.} \quad \mathbf{D} = D_0 e^{i\nu(t - \mathbf{N} \cdot \mathbf{v})},
\]

where

\[
D_0 = [\mathbf{H} \times \mathbf{N}] = c/\nu [-\mathbf{N}(\mathbf{N} \mathbf{E}_0) + \mathbf{E}_0].
\]

Thus \( \mathbf{D} \) is normal to the plane defined by \( \mathbf{H}_0 \), and \( \mathbf{N} \) and is coplanar with the vectors \( \mathbf{N} \) and \( \mathbf{E} \). Hence, as in crystals, the wave is transverse in \( \mathbf{D} \) and \( \mathbf{H} \). The vector \( \mathbf{E} \) makes an angle \( \Psi \) with \( \mathbf{D} \) such that

\[
\cos \Psi = \frac{v}{c} \frac{\mathbf{D} \cdot \mathbf{E}}{|\mathbf{D}| |\mathbf{E}|} = \frac{v}{c} \left[ 1 - \frac{4\pi \sigma}{p} \right].
\]
Propagation of electromagnetic waves through the atmosphere

The Poynting vector \( \mathbf{S} = c/4\pi [\mathbf{E} \times \mathbf{H}] \) is evidently coplanar with \( \mathbf{D} \), \( \mathbf{E} \) and \( \mathbf{N} \) and perpendicular to \( \mathbf{E} \), i.e. makes the same angle \( \Psi \) with \( \mathbf{N} \). The actual orientations of the field vectors are as shown in figure 3.

![Figure 3](http://rspa.royalsocietypublishing.org/)

Expression for \( \cos \Psi \) shows that all the conditions reduce to the single condition \( \Psi = \frac{1}{2} \pi \), for then \( \mu^2 = 0 \) (Appleton-Hartree condition), \( \mathbf{E} \) becomes parallel to \( \mathbf{N} \) as suggested by Bose and \( \mathbf{S} \) becomes perpendicular to \( \mathbf{N} \) (Booker's condition).

Thus the different postulates about reflexion of waves are identical.

Nature of the Problem at Present

The above discussions have unified the different works on the problem of the propagation of the electromagnetic wave in the ionosphere as treated from the standpoint of the ray method. The shortcomings of this method have already been pointed out.

Recently, several workers have attacked the problem from the standpoint of wave treatment, though the inferences drawn therefrom are not substantially conclusive. Hartree (1932) attempted to solve the wave equation by assuming a linear ion-barrier, which does not at all tally with the actual case. Appleton, Naismith & Ingram (1939) have quoted a result of Booker's, who has given a solution of the wave equation for a parabolic layer, valid for frequencies near the critical value, but invalid for other waves. Wilkes (1940), Rawyer (1939), Rydbeck (1940) and others have attempted to solve the wave equation with different hypothetical ion-barriers which do not fit the actual case. Saha & Rai (1937) attempted to solve for a general barrier, using B.K.W. approximate method and the shortcomings of this method are pointed out by the authors themselves. The complete forms of the wave equation have been deduced by Saha & Banerjea (1945) by using the principal axes \( H'Y'Z'(XYZ) \), and they have the form

\[
\nabla^2 E_x + \frac{p^2}{c^2} \left( 1 - \frac{p_0^2}{p^2 - i\nu p} \right) E_x = 0,
\]

\[
\nabla^2 (E_y \pm i E_z) + \frac{p^2}{c^2} \left( 1 - \frac{p_0^2}{p^2 - i\nu p \pm \nu p_h} \right) (E_y \pm i E_z) = 0.
\]

Here \( E_x, E_y, E_z \) have been written for \( E_H, E_y', E_z' \) for the sake of simplification of symbols.
From these equations, we see clearly that on entrance into the ionosphere, the
three components $E_x, E_y + iE_z, E_y - iE_z$ travel with different velocities and absorp-
tion, depending on $p_0^2, \nu, p_h$.

The wave equations are of the type, commonly known as the Riccatti equation,
which often reduces to integrable equations like Bessel, Legendre, hypergeometric,
Airy's integral, Whittaker's integral for special functional expressions for the
variable refractive index. Some of these special forms have been used by the authors
mentioned above, though the general cases have still not been attempted. In fact,
the general case of the Riccatti's equation cannot be solved analytically but it can
often be reduced to approximate integrable forms, as in B.K.W. approximate
processes. The analytical justification of such approximations are still lacking as is
evident from the discussions of Langer (1937). The alternative method is to solve
the problem by the method of integral equation, as the problem is much too similar
to Liouville's problem in integral equations.

To fit the actual ionospheric case, however, other less accurate but simpler
methods are available. For the E region, the ion-barrier is given as a function of
height by Chapman (1931) and Pannekoek (1929), and the results though originated
from quite different starting points are, as shown by Saha & Rai (1937), concordant.
The nature of this function has also been tested experimentally to fit the E-region.
So the Riccatti equation can be put in this form, and its solution can be obtained
analytically for different ranges, though no functional solution holds good for the
entire range. Then the different solutions can be made to fit each other by the common
method of matching. In this way a less analytic but more useful solution can be
obtained, which when properly adjusted to fit the boundary conditions will link up
the strength of the received signal with the circuital constants and the wattage
transmitted.

The alternative numerical method is to solve the equation with Chapman barrier
with the help of Bush's differential analyser and then to fit a group of analytical
functions with the actual solutions. By this method also we can arrive at expressions
like reflexion coefficient, etc., which are directly observable. Attempts are made
along both lines by the author.

The work was undertaken at the suggestion of Professor M. N. Saha, F.R.S., to
whom I express my sincere thanks for constant help and guidance. I have been the
recipient of a Research Scholarship from the Radio Research Committee of the
Council of Scientific and Industrial Research, and wish to express my thanks to the
Council for facility to work and permission to publish this paper.
References

Bruevitch 1934 *Nature*, 133, 175.