On the scattering of fast neutrons by protons

BY W. H. RAMSEY, H. H. WILLS Physical Laboratory, University of Bristol

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The features of the scattering of fast neutrons by protons are calculated using the Möller-Rosenfeld version of the meson theory of nuclear forces. The experimental results of Occhialini & Powell are used to check the predicted angular distribution of the scattered particles and to determine the mass of the meson; the meson mass indicated is about 215 electronic masses, which agrees with the mass of cosmic ray mesons. The total scattering cross-section predicted by the theory agrees with the empirical results.

INTRODUCTION

A detailed knowledge of the interaction between the neutron and the proton is of great importance for the theory of nuclear forces. As the deuteron has only one stable state, no very far-reaching conclusions about the interaction can be drawn from the discrete spectrum. The main attention must, therefore, be directed to the careful investigation of the scattering of neutrons by protons. Measurements yield a total cross-section for scattering and an angular distribution of the scattered particles which are characteristic of the energy of the incident neutrons.

In the present paper the version of the meson theory of nuclear forces, usually known as the Möller-Rosenfeld theory, will be used to make detailed theoretical predictions about the scattering cross-section and its dependence on the angle of scattering and on the energy of the incident particles. This theory contains three adjustable parameters, two of which will be chosen so that the energies of the stable and virtual states of the deuteron are given correctly; the third parameter is the mass of the meson. The experimental results of Occhialini & Powell (1947) will be used to check the predicted angular distribution of the scattered neutrons and to determine the mass of the meson.

Interest in the scattering of fast neutrons by protons was increased as a result of experiments by Amaldi, Bocciarelli, Ferretti & Trabacchi (1942) which suggested that the scattering is predominantly forward. Hulthén (1943, 1944a) has shown that this is at variance with meson theories which are symmetrical as regards electric charge, including the Möller-Rosenfeld theory, and that important modifications of present ideas on the nature of nuclear forces would have to be made if the conclusions of these experiments were to be confirmed by later investigators. It is therefore important that neutron-proton scattering should be further investigated experimentally and the results viewed in the light of current meson theories.

Some of the results of this investigation have already been published by Fröhlich, Ramsey & Sneddon (1947) in connexion with the Physical Society Conference at Cambridge.

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INTERACTION BETWEEN NUCLEONS

The version of meson theory due to Møller & Rosenfeld (1940), which involves electrically charged and neutral mesons in a symmetrical way, will be used to describe the interaction between two nucleons. The objections to theories which do not allow both charged and neutral mesons are well known. Theories postulating charged mesons only cannot account for the equality, or approximate equality, of neutron-proton and proton-proton forces. On the other hand, theories postulating neutral mesons only cannot lead to the saturation property of nuclear forces exhibited by heavy nuclei, nor can the mesons be identified with the charged particles found in cosmic radiation. The Møller-Rosenfeld theory is the most consistent and least ambiguous of all the symmetrical theories; in the static approximation no arbitrary 'cutting off' of the interaction is required to avoid divergencies arising from terms of the dipole-dipole type. Fröhlich, Huang & Sneddon (1947) have used this theory with some success to calculate the binding energies of the lightest nuclei, and Rozental (1941, 1945) has shown that a satisfactory description of $\beta$-disintegration can be based on it. It has also been shown by Hulthén (1943) that the non-static part of the interaction in this theory can account for the observed value and sign of the quadrupole moment of the deuteron.

The static part of the interaction between two nucleons is given by

\[ V = \frac{G}{r} e^{-\lambda r}, \quad (1a) \]

\[ G = (\tau_1, \tau_2) \left\{ g^2 + f^2 (\sigma_1, \sigma_2) \right\}, \quad (1b) \]

where $r$ is the distance between the nucleons and where $\sigma_i, \tau_i$ are the usual vector operators associated with the spin and isotopic spin respectively of nucleon $i$. The length $\lambda^{-1}$, which measures the range of the force, is related to the meson mass $m$ by the equation

\[ \lambda = mc^2/h. \quad (2) \]

The quantities $f$ and $g$ are universal constants with the dimensions of an electric charge. Rosenfeld (1945) has shown that part of the non-static interaction in the original paper by Møller & Rosenfeld (1940) is of the same mathematical form as the static interaction and can be included in it by merely altering the values of the charge constants. The charges $f$ and $g$ appearing in equation (1b) are connected with the charges $f_i$ and $g_i$ appearing in the paper by Møller & Rosenfeld (1940) by the equations

\[ 4\pi f^2 = g^2_2 - \frac{m}{M} f_1 f_2, \]

\[ 4\pi g^2 = g^2_1 - \frac{m}{M} \left( \frac{2g_1 g_2 - f_1 f_2}{3} \right), \]

where $M$ denotes the mass of a nucleon (the difference in the masses of the neutron and proton is neglected throughout this paper). The remaining part of the non-static
interaction is comparatively small and will be neglected; the work of Hulthén (1943, 1944a) shows that this is justified at the energies considered.

The value of the operator $G$ defined by equation (1b) depends only on the symmetry properties of the wave function describing the two nucleons. The operator $(\sigma_1, \sigma_2)$ has the value +1 or −3 according as the wave function is symmetrical or antisymmetrical in the spin variables; the value of the operator $(\tau_1, \tau_2)$ is determined analogously. Consider the wave function representing the two nucleons in the system of co-ordinates in which the centre of mass is at rest; suppose that the spatial part of this wave function is of the form

$$\psi_l(r) = \frac{1}{r} R_l(r) Y_l^m(\theta, \phi),$$

where $Y_l^m(\theta, \phi)$ is the usual surface harmonic. All the relevant symmetry properties of the complete wave function are determined if the value of $l$ is specified and if it is indicated whether the state is a singlet or a triplet. The symmetry with respect to the isotopic spin variables is then determined by the fact that the complete wave function must be antisymmetrical with respect to the interchange of the two nucleons, in accordance with the Pauli principle. The eigenvalues of the operator $G$ in the different states are given in table 1 for the neutron-proton interaction.

<table>
<thead>
<tr>
<th>Table 1. Static interaction constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$ even</td>
</tr>
<tr>
<td>singlet $G = -(3f^2 - g^2)$</td>
</tr>
<tr>
<td>triplet $G = -3(f^2 + g^2)$</td>
</tr>
<tr>
<td>$l$ odd</td>
</tr>
<tr>
<td>singlet $G = 3(3f^2 - g^2)$</td>
</tr>
<tr>
<td>triplet $G = (f^2 + g^2)$</td>
</tr>
</tbody>
</table>

As $f^2 \sim 0.07hc$ and $g^2 \sim 0.03hc$ (see table 3) it follows that, in the Møller-Rosenfeld theory, the interaction for states with odd values of $l$ is strongly repulsive if the spins of the neutron and proton form a singlet, and only weakly repulsive if they form a triplet. For neutrons of low energy (say below 5 MeV), the anisotropy in the scattering is mainly due to the P-states; as the interaction is repulsive for these states, the intensity of the scattered wave is greater in the backward direction than in the forward direction in the system of co-ordinates in which the centre of mass is at rest. The higher orders (D, F, G, etc.) become more important as the energy of the incident neutrons is increased; nevertheless, the feature of backward scattering remains if the Møller-Rosenfeld description of nuclear forces is correct.

The set of interaction constants given in table 1 is characteristic of the Møller-Rosenfeld theory. Other theories give different sets of constants; for example, in the case of ‘ordinary’ forces the interaction is not influenced by the quantum state of the particles, and in the mixed neutral meson theory the interaction depends on the spin state but not on the value of $l$. The set of constants given in table 1 is even more characteristic of the theory than is the radial dependence of equation (1a), and should be, if possible, tested directly by experiment.

It will be assumed that the wave function which describes the behaviour of the particles satisfies a Schrödinger wave equation. The difficulties of setting up and
solving a relativistic wave equation which can be applied to nuclear particles are very great, and it does not seem necessary to take this course. The kinetic energy of an 18 MeV neutron is only 0.02 MeV, and so the relativistic correction at this energy will be very small. Moreover, it would be inconsistent to use a relativistic wave equation and to neglect the non-static terms in the interaction.

For a wave function of the form given in equation (3) the problem reduces to one of finding the appropriate solution of the differential equation

\[
\left[ \frac{\hbar^2}{M} \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + E - \frac{G}{r} e^{-\lambda r} \right] R_\ell(r) = 0.
\]  

(4)

It is convenient to introduce the dimensionless quantities

\[ x = \lambda r, \quad \mu = m/M, \quad a = E/\mu^2 Me^2, \quad b = -G/\mu c, \]

(5)

so that equation (4) takes the form

\[ L R_\ell(x) \equiv \left[ \frac{d^2}{dx^2} - \frac{l(l+1)}{x^2} + a + b \frac{1}{x} e^{-x} \right] R_\ell(x) = 0. \]  

(4a)

THE DEUTERON

The ground state of the deuteron is a \(^3S\) state which, according to Wiedenbeck & Marhoefer (1945), has a binding energy of \(2.185 \pm 0.006\) MeV. A virtual \(^1S\) state also is known from the scattering of very slow neutrons by protons. Hanstein (1941) found a cross-section of \(21 \pm 1 \times 10^{-24}\) cm.\(^2\) for the scattering of neutrons of energy 1 eV by protons; this corresponds to an energy of about 0.06 MeV for the virtual state, the precise value depending on the meson mass assumed. For a fixed value of \(\lambda\) these experimental results have been used to determine the charge constants \(f\) and \(g\) appearing in the interaction (1); this leaves the range of the force or the meson mass as the only undetermined parameter.

Hulthén (1944b, 1945) has shown that, for the ground state, satisfactory results can be obtained using a wave function of the form

\[ R_0(x) = (1 - e^{-x}) (1 + \beta e^{-x}) e^{-\sqrt{\alpha} x}, \]

(6)

and that for \(S\) waves of the continuous spectrum a suitable wave function is

\[ R_0(x) = \cos \eta_0 \sin x \sqrt{\alpha} + (1 - e^{-x}) (1 + \beta e^{-x}) \sin \eta_0 \cos x \sqrt{\alpha}, \]

(7)

where the phase shift \(\eta_0\) is related to the energy parameter \(\alpha\) in such a way as to give the observed cross-section for the scattering of slow neutrons; in equation (7) only \(\alpha \sim 0\) is of interest. In each case the parameter \(\beta\) is to be determined by a variation of the integral

\[ L = \int_0^\infty R_0(x) LR_0(x) \, dx, \]

where \(L\) is the operator defined by equation (4a) with \(l = 0\). The value of the constant \(b\) is then given by the relation

\[ L = 0. \]
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The results of these calculations are given in table 2. In table 3 the charge constants are shown as a function of the mass of the meson ($m_e$ denotes the mass of the electron).

**Table 2**

<table>
<thead>
<tr>
<th>meson mass</th>
<th>$^3S$ state</th>
<th>$^1S$ state</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta$</td>
<td>$b$</td>
</tr>
<tr>
<td>$200m_e$</td>
<td>0.5889</td>
<td>2.65</td>
</tr>
<tr>
<td>$225m_e$</td>
<td>0.5841</td>
<td>2.54</td>
</tr>
<tr>
<td>$250m_e$</td>
<td>0.5801</td>
<td>2.46</td>
</tr>
</tbody>
</table>

**Table 3. Static charge constants**

<table>
<thead>
<tr>
<th>meson mass</th>
<th>$f^c_hc$</th>
<th>$g^c_hc$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$200m_e$</td>
<td>0.0653</td>
<td>0.0309</td>
</tr>
<tr>
<td>$225m_e$</td>
<td>0.0729</td>
<td>0.0310</td>
</tr>
<tr>
<td>$250m_e$</td>
<td>0.0806</td>
<td>0.0310</td>
</tr>
</tbody>
</table>

**Determination of phase shifts and application of Born approximation**

Calculation of the scattering to be expected involves finding the solution (4a) which vanishes at the origin and which has the asymptotic form

$$R_l(x) \sim \sin \left( x \sqrt{a} - \frac{1}{2}l\pi + \eta_l \right),$$

where $\eta_l$ is the phase shift introduced by the interaction between the nucleons. The characteristics of the scattering are completely determined by the phase shifts $\eta_l$ (cf. Mott & Massey 1933). In equations (4) and (5) the energy of the incident neutrons in the laboratory system of co-ordinates is $2E$.

The simplest procedure to determine the phase shifts is to use the following approximation:

$$\eta_l = b \frac{1}{\sqrt{a}} \int_0^\infty e^{-x} J_{l+\frac{1}{2}}(x\sqrt{a})^2 \, dx$$

$$= \frac{b}{2\sqrt{a}} Q_l \left( 1 + \frac{1}{2a} \right), \quad (8)$$

where $Q_n(z)$ is the Legendre function of the second kind of order $n$ (cf. Mott & Massey 1933; Watson 1944). If formula (8) is used throughout and if only first powers of the phase shifts are retained in the expression for the amplitude of the scattered wave, the scattering is given by the Born approximation (cf. equations (11)). Equation (8) is sufficiently accurate only for large values of $l$. For the values of $a$ and $b$ considered the phase shifts given by equation (8) were wrong by 2 to 7% in the case of $^3P$, $^3D$ and $^1D$ states, and by 32 to 37% in the case of $^1P$ states. The phase shifts for $S$ states were underestimated by as much as 60%. This approximation consistently overestimates the phase shifts for the states in which the interaction is repulsive, and it consistently underestimates the phase shifts for the states in which the interaction is attractive; the net result is a serious exaggeration of the anisotropy of the scattering.
A much better approximation has been developed by Pais (1946). A suitable solution of equation (4a) when there is no interaction between the nucleons is

\[ R_\ell(x) = \sqrt{\frac{1}{2} \pi x \sqrt{a}} J_{\ell + \frac{1}{2}}(x \sqrt{a}). \]

Pais takes the effect of the interaction between the nucleons into account by using a solution of the type

\[ R_\ell(x) = \sqrt{\frac{1}{2} \pi x \sqrt{a}} J_{\ell + \frac{1}{2}}(x \sqrt{a}), \quad \eta = -\frac{1}{2} \pi \Lambda, \]

where \( \Lambda \) is to be determined from equation (4a) by the relation

\[ \int_0^\infty R_\ell(x) L R_\ell(x) \, dx = 0. \]

This leads directly to Pais’s formula

\[ \frac{2l + \Lambda + 1}{2l + 2\Lambda + 1} \Lambda = -\frac{b}{\pi \sqrt{a}} Q_{l+\Lambda} \left(1 + \frac{1}{2a}\right), \tag{9} \]

where \( Q_{n}(z) \) is again a Legendre function of the second kind. This formula may be expected to give very accurate results provided that the interaction between the nucleons is small compared with the centrifugal term \( l(l+1)/x^2 \). It breaks down completely in the case of S states, and it is not sufficiently accurate in the case of \(^1P\) states. Table 4 compares the results of Pais’s formula with those found by numerical integration.* Pais’s formula gives phase shifts wrong by 7 to 10% in these cases. Nevertheless, the improvement over the Born formula (8) can be gauged from the fact that the corresponding errors with that approximation were 32 to 37%.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( b )</th>
<th>Numerical Integration</th>
<th>Pais’s Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.64</td>
<td>-5.09</td>
<td>-0.307</td>
<td>-0.328</td>
</tr>
<tr>
<td>0.595</td>
<td>-5.09</td>
<td>-0.293</td>
<td>-0.3145</td>
</tr>
<tr>
<td>0.47</td>
<td>-5.12</td>
<td>-0.251</td>
<td>-0.272</td>
</tr>
<tr>
<td>0.38</td>
<td>-5.13</td>
<td>-0.215</td>
<td>-0.236</td>
</tr>
<tr>
<td>0.32</td>
<td>-5.12</td>
<td>-0.188</td>
<td>-0.206</td>
</tr>
</tbody>
</table>

* The numerical integrations were carried out for charge constants corresponding to the virtual \(^3\)S state of the deuteron having zero energy (cf. Fröhlich, Ramsey & Sneddon 1947). For the purpose of assessing the accuracy of Pais’s formula the results of the actual numerical integrations have been given in Table 4.

Phase shifts found by numerical integration have been modified to correspond to the charge constants given in Table 3 on the assumption that the phase shift can be expressed as a power series in \( \beta \):

\[ \eta = c_1 b + c_2 b^2 + c_3 b^3, \]

where \( c_1, c_2 \) and \( c_3 \) are constants and \( c_1 \) corresponds to the Born formula (8). The error introduced by this alteration should be very small.
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The phase shifts of all $S$ and $^1P$ states were found by a step-by-step numerical integration. Hulthén (1943) has pointed out that the accuracy of a numerical integration can be estimated from the identity

$$\sin \eta_i = b \frac{1}{\xi^2} \int_0^\infty \left( \frac{4 \pi x}{\sqrt{a}} \right)^{-1} e^{-x} J_{\xi+1}(x \sqrt{a}) R_i(x) \, dx,$$

by using the value of $R_i(x)$ found by the numerical integration. This was done for the third and fifth cases of table 4, and the values $-0.2513$ and $-0.1877$ respectively were found for the phase shifts; these differ by less than $1\%$ from the results of numerical integration.

The phase shifts found in the various cases considered are given in table 5; these refer to the values of the charge constants given in table 3. The method used to derive each value is indicated; numerical integration, the Pais approximation (9) and the Born formula (8) are denoted by $(N)$, $(P)$ and $(B)$ respectively.

<table>
<thead>
<tr>
<th>energy</th>
<th>$18.05\text{ MeV}$</th>
<th>$13.25\text{ MeV}$</th>
<th>$13.25\text{ MeV}$</th>
<th>$13.25\text{ MeV}$</th>
<th>$9.02\text{ MeV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>meson mass</td>
<td>$225m_e$</td>
<td>$200m_e$</td>
<td>$225m_e$</td>
<td>$250m_e$</td>
<td>$225m_e$</td>
</tr>
<tr>
<td>$^1S$</td>
<td>0.859 $(N)$</td>
<td>0.868 $(N)$</td>
<td>0.910 $(N)$</td>
<td>0.956 $(N)$</td>
<td>0.965 $(N)$</td>
</tr>
<tr>
<td>$^1P$</td>
<td>-0.282 $(N)$</td>
<td>-0.267 $(N)$</td>
<td>-0.229 $(N)$</td>
<td>-0.198 $(N)$</td>
<td>-0.172 $(N)$</td>
</tr>
<tr>
<td>$^1D$</td>
<td>0.0315 $(P)$</td>
<td>0.0328 $(P)$</td>
<td>0.0216 $(P)$</td>
<td>0.0163 $(P)$</td>
<td>0.0125 $(P)$</td>
</tr>
<tr>
<td>$^1F$</td>
<td>-0.0240 $(P)$</td>
<td>-0.0210 $(P)$</td>
<td>-0.0140 $(P)$</td>
<td>-0.00931 $(P)$</td>
<td>-0.00641 $(P)$</td>
</tr>
<tr>
<td>$^1G$</td>
<td>0.00215 $(B)$</td>
<td>0.00187 $(B)$</td>
<td>0.00108 $(B)$</td>
<td>0.00063 $(B)$</td>
<td>0.00039 $(B)$</td>
</tr>
<tr>
<td>$^1H$</td>
<td>-0.00125 $(B)$</td>
<td>-0.00151 $(B)$</td>
<td>-0.00076 $(B)$</td>
<td>-0.00039 $(B)$</td>
<td>-0.00022 $(B)$</td>
</tr>
<tr>
<td>$^3S$</td>
<td>1.578 $(N)$</td>
<td>1.684 $(N)$</td>
<td>1.694 $(N)$</td>
<td>1.711 $(N)$</td>
<td>1.835 $(N)$</td>
</tr>
<tr>
<td>$^3P$</td>
<td>-0.0664 $(P)$</td>
<td>-0.0655 $(P)$</td>
<td>-0.0543 $(P)$</td>
<td>-0.0449 $(P)$</td>
<td>-0.0404 $(P)$</td>
</tr>
<tr>
<td>$^3D$</td>
<td>0.0533 $(P)$</td>
<td>0.0508 $(P)$</td>
<td>0.0362 $(P)$</td>
<td>0.0262 $(P)$</td>
<td>0.0209 $(P)$</td>
</tr>
<tr>
<td>$^3F$</td>
<td>-0.0450 $(P)$</td>
<td>-0.0412 $(P)$</td>
<td>-0.00261 $(B)$</td>
<td>-0.00166 $(B)$</td>
<td>-0.00119 $(B)$</td>
</tr>
<tr>
<td>$^3G$</td>
<td>0.0036 $(B)$</td>
<td>0.00327 $(B)$</td>
<td>0.00179 $(B)$</td>
<td>0.00101 $(B)$</td>
<td>0.00064 $(B)$</td>
</tr>
<tr>
<td>$^3H$</td>
<td>-0.00023 $(B)$</td>
<td>-0.00029 $(B)$</td>
<td>-0.00014 $(B)$</td>
<td>-0.00007 $(B)$</td>
<td>-0.00004 $(B)$</td>
</tr>
</tbody>
</table>

As an alternative to the treatment by the phase-shift method, the Born approximation may be used to calculate the features of the scattering. Using the system of co-ordinates in which the centre of mass is at rest, the Schrödinger equation appropriate to the problem is

$$(\nabla^2 + k^2) \Psi = \frac{M}{\hbar^2} \textbf{v} \Psi, \quad k^2 = \frac{2E}{\hbar^2}, \quad (10)$$

where $\Psi$ is the complete wave function representing the behaviour of the nucleons, $2E$ is the energy of the incident neutrons and $\textbf{v}$ is the interaction operator (1). The singlet and triplet states form two non-combining sets and so may be treated separately. In the Born approximation it is assumed that the wave function $\Psi$ on the right-hand side of equation (10) may be replaced by the corresponding solution of the free equation. This procedure is justified only if the total cross-section for scattering is small. If the beam of neutrons is incident in the $z$-direction, in the case
of the singlet states the right-hand side of equation (10) becomes in this approximation

$$\frac{M}{\hbar^2} \nabla \Psi \simeq \frac{M}{\hbar^2} V\{S_A \cos kz + iS_S \sin kz\}$$

$$= -\frac{M}{\hbar^2} (3f^2 - g^2) \frac{1}{r} e^{-\lambda r} \{S_A \cos kz - 3iS_S \sin kz\}, \quad (10a)$$

making use of the eigenvalues of the operator $V$ given in table 1; here $S_S$, $S_A$ respectively are symmetrical and antisymmetrical partial wave functions depending only on the spin and isotopic spin variables. Inserting $(10a)$ into equation (10), multiplying by $(S_A^* + S_S^*)$ and summing over the spin and isotopic spin variables, the equation reduces to

$$\left(\nabla^2 + k^2\right) \psi(r) = -\frac{M}{\hbar^2} (3f^2 - g^2) \{\cos kz - 3i \sin kz\} \frac{1}{r} e^{-\lambda r},$$

where $\psi(r)$ is the spatial part of the wave function $\Psi$. This equation can be solved by the standard method (cf. Mott & Massey 1933). The treatment of the triplet states is similar. This calculation gives the following amplitudes of the scattered waves at angle $\theta$:

$$F_S(\theta) = -\frac{M}{\hbar^2} (3f^2 - g^2) \left(\frac{1}{\lambda^2 + 4k^2 \sin^2 \frac{1}{2} \theta} - \frac{2}{\lambda^2 + 4k^2 \cos^2 \frac{1}{2} \theta}\right),$$

$$F_T(\theta) = \frac{M}{\hbar^2} (f^2 + g^2) \left(\frac{1}{\lambda^2 + 4k^2 \sin^2 \frac{1}{2} \theta} + \frac{2}{\lambda^2 + 4k^2 \cos^2 \frac{1}{2} \theta}\right), \quad (11)$$

where $F_S(\theta)$, $F_T(\theta)$ refer to the singlet and triplet states respectively. The cross-section for scattering through an angle $\theta$ in the centre of mass system of co-ordinates is

$$I(\theta) = \frac{1}{4} |F_S(\theta)|^2 + \frac{3}{4} |F_T(\theta)|^2.$$

### Angular distribution and total cross-section

In this section the results of the calculations by the phase-shift method will be presented and compared with the available experimental material.

Throughout the calculations six orders (that is, S, P, D, F, G and H states) have been taken into account. Fröhlich, Ramsey & Sneddon (1947) have pointed out the necessity of this in calculating the angular distribution of the scattered neutrons; figure 3 of that paper, which shows the contribution of the different orders in the case of $13\frac{1}{4}$ MeV neutrons and a meson mass of $200m_e$, clearly demonstrates the importance of the states with large angular momenta. This state of affairs is exaggerated in the $18\cdot05$ MeV case considered here. The higher orders, on the other hand, contribute little to the total cross-section; 92% of the total cross-section for the scattering of $18\cdot05$ MeV neutrons is due to the $S$-states and the predominance is even more pronounced at lower energies.

The cross-section $I(\theta)$ for scattering through an angle $\theta$ in the system of co-ordinates in which the centre of mass is at rest, and the total cross-section for scattering can be derived from the phase shifts given in table 5 by a standard procedure
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(cf. Mott & Massey 1933). The angular distribution of the scattered wave is the most characteristic feature of the scattering and its analysis is the most powerful method of testing a theory of nuclear forces by experiments on scattering. All meson theories which involve electrically charged and electrically neutral mesons in a symmetrical way lead to a distribution of scattered neutrons which is predominantly backward, that is, $I(\pi) > I(0)$. On the other hand, meson theories postulating neutral mesons only lead to scattering predominantly forward, that is $I(0) > I(\pi)$. Theories which allow only charged mesons lead to an angular distribution even more predominantly backward than do the symmetrical theories. It is therefore of the greatest importance that the scattering cross-section $I(\theta)$ be found accurately experimentally over the entire range $0 \leq \theta \leq \pi$, and compared with the predictions of the various contending theories. However, the experimental material at present available is not sufficient to allow so detailed a test, and attempts (e.g. Amaldi et al. 1942) have been made to assign a number which measures the departure of the distribution from isotropy. The usual measure $\gamma$, which will be referred to as the anisotropy ratio, is defined by the relation

$$\gamma = \frac{I(\pi)}{I(\frac{1}{2}\pi)}. \quad (12)$$

In the next section it will be shown that this ratio is not the most characteristic index which can be associated with the angular distribution; a more suitable number $\rho$, which will be referred to as the asymmetry ratio, is defined by

$$\rho = \frac{I(\pi)}{I(0)}. \quad (13)$$

<table>
<thead>
<tr>
<th>energy (MeV)</th>
<th>meson mass</th>
<th>anisotropy ratio</th>
<th>asymmetry ratio</th>
<th>total cross-section (cm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.05</td>
<td>225$m_e$</td>
<td>1.90</td>
<td>1.69</td>
<td>$5.61 \times 10^{-25}$</td>
</tr>
<tr>
<td>13.25</td>
<td>200$m_e$</td>
<td>1.70</td>
<td>1.59</td>
<td>$7.53 \times 10^{-25}$</td>
</tr>
<tr>
<td>13.25</td>
<td>225$m_e$</td>
<td>1.45</td>
<td>1.41</td>
<td>$7.43 \times 10^{-25}$</td>
</tr>
<tr>
<td>13.25</td>
<td>250$m_e$</td>
<td>1.29</td>
<td>1.29</td>
<td>$7.37 \times 10^{-25}$</td>
</tr>
<tr>
<td>9.02</td>
<td>225$m_e$</td>
<td>1.17</td>
<td>1.19</td>
<td>$10.34 \times 10^{-25}$</td>
</tr>
</tbody>
</table>

Table 6. Characteristics of scattering

The results of the calculations are summarized in table 6 and in figures 1 to 4. In figures 1 and 2 the calculated angular distributions for the scattering of 9.02 and 13.25 MeV neutrons are compared with the experimental values found by Occhialini & Powell (1947) for 9.2 and 13.4 MeV neutrons respectively. These workers, who used a photographic method to detect the recoil protons, did not measure absolute cross-sections, and in each case their values have been normalized to correspond to the theoretical total cross-section for a meson mass of 225$m_e$; this procedure is justified, as the theory gives the total cross-section correctly (see figure 3). In figures 1 and 2 only the more recent observations which were made under refined conditions have been included. The standard error associated with each empirical value allows only for the statistical fluctuation in the number of tracks, and not for any systematic
error in the method; there are, in all, 2562 tracks for the 9·2 MeV case and 415 for the 13·4 MeV case. Provided that there is no large systematic error which has been overlooked by the investigators, these experiments leave little doubt that the anisotropy ratio is greater than unity; there is less than one chance in a thousand that these experimental values are a fluctuation of an angular distribution which has an anisotropy ratio less than unity. It is apparent from table 6 and from figure 2

![Figure 1](image-url)  
**Figure 1.** The calculated variation of the scattering cross-section for 9-02 MeV neutrons with the angle of scattering in the case of a meson mass of 225m_e. The experimental results shown are those of Occhialini & Powell (1947) for the scattering of 9-2 MeV neutrons by protons.

that the anisotropy ratio γ varies rapidly with the meson mass; if the quantity (γ−1) were known within 10% for the scattering of 13-25 MeV neutrons, the uncertainty in the meson mass which must be used in the theory would be about 3%. The meson mass which is indicated by the experiments of Occhialini & Powell is 211 ± 10m_e in the 9-2 MeV case and 250 ± 30m_e in the 13-4 MeV case, or, taking both sets of observations together, 215 ± 10m_e; the standard errors again take only statistical fluctuations into account. These are in good agreement with the value 210 to 220m_e found by Fröhlich, Huang & Sneddon (1947) from the binding energies of the lightest nuclei, and also with the mass of the mesons found in cosmic radiation which is about 200m_e. However, these experimental results fall far short of the detailed
test to which the theory should be subjected, especially in the case of the higher energy, and they cannot be reconciled with the experiments of Amaldi et al. (1942), who obtained an anisotropy ratio less than unity using a counter technique to detect the recoil protons. This discrepancy emphasizes the need for further work on the scattering of fast neutrons by protons by both the photographic and counter methods.

![Graph showing scattering cross-section vs angle of scattering](image)

**Figure 2.** The calculated variation of the scattering cross-section for 13·25 MeV neutrons with the angle of scattering for meson masses of 200, 225 and 250 electronic masses. The experimental results shown are those of Occhialini & Powell (1947) for the scattering of 13·4 MeV neutrons by protons.

Figure 3 shows the dependence of the total cross-section for scattering on the energy of the incident neutrons for a meson mass of 225 $m_e$. The cross-sections determined experimentally by various workers are also shown, together with the errors estimated by the investigators. The agreement is fairly good and constitutes a test of the theory. From the values given in table 6 for 13·25 MeV neutrons it can be seen that the total cross-section, unlike the angular distribution, is highly insensitive
to the meson mass assumed; decreasing the meson mass by 20% increases the total cross-section by only 2%. Thus, if the theory gave a wrong value for the total cross-section it would not be possible to rectify this by altering the assumed mass of the meson. Also, it does not follow that a theory whose arbitrary constants have been adjusted to give the states of the deuteron correctly will also give the total cross-section correctly; Rarita & Schwinger (1941) have shown that their neutral theory gives total cross-sections wrong by 50% in the 15 MeV region under these conditions. However, the agreement indicated in figure 3 is not as decisive as might appear. The variation of the total cross-section with energy is dominated by the usual reciprocal of the energy term, and so if the total cross-section agrees with experiment at one energy it will agree with experiment over a wide range of energies. The test therefore effectively consists in comparing one number predicted by theory with the empirical value; nevertheless, this is as good a test of the theory as, say, comparing the calculated with the observed binding energy of the α-particle.

**Scattering of Very High Energy Neutrons by Protons**

At medium and high energies the total cross-section for scattering is approximately inversely proportional to the energy. Hence the Born approximation may be used to survey the characteristics of the scattering at energies much greater than those considered in the preceding section. Hulthén (1943) has shown that this approximation also gives the characteristics of the scattering qualitatively even at medium energies (10 to 20 MeV), though not much weight can be placed on its
quantitative predictions in this region; for example, for the scattering of 18.05 MeV neutrons and for a meson mass of 225\text{m}_e, formulae (11) lead to an anisotropy ratio of 4.3 compared with the actual value of 1.90 as given by the phase-shift method.

From formulae (11) the following expressions are obtained for the anisotropy ratio $\gamma$ and the asymmetry ratio $\rho$ in the Born approximation:

$$\gamma = \frac{(1 + 2a)^2 (3f^2 - g^2)^2 (1 + 8a)^2 + 3(f^2 + g^2)^2 (3 + 8a)^2}{(3f^2 - g^2)^2 + 27(f^2 + g^2)^2},$$

$$\rho = \frac{(3f^2 - g^2)^2 (1 + 8a)^2 + 3(f^2 + g^2)^2 (3 + 8a)^2}{(3f^2 - g^2)^2 (1 - 4a)^2 + 3(f^2 + g^2)^2 (3 + 4a)^2},$$

where the parameter $a$, defined by relations (5), is proportional to the energy of the incident neutrons. For very large energies the anisotropy ratio tends to infinity as the square of the energy and the asymmetry ratio has the asymptotic value $+4$. The behaviour of these ratios as a function of the energy is shown in figure 4 in the case of a meson mass of 225\text{m}_e. In the region below 20 MeV the curves have been constructed from the results of the phase-shift calculations; the broken curves are extrapolations which exhibit the general features of relations (14). To a good approximation these ratios depend on the mass of the meson through the parameter $a$ only; if the energy scale is altered by a factor $(225\text{m}_e/m)^2$ the curves will be approximately true for a meson mass $m$.

To study the significance of the ratios (14), let $G_S$, $G_T$ be the values of the operator $G$ of equation (1a) for the singlet and triplet states respectively when $l$ is even, and let the corresponding values for odd $l$ be $(1 + \lambda_S) G_S$ and $(1 + \lambda_T) G_T$. The factors

![Graph](image_url)

**Figure 4.** The variation of the anisotropy ratio $\gamma$ and the asymmetry ratio $\rho$ with energy.
\( \lambda_s \) and \( \lambda_T \) and the difference \((G_T - G_S)\) measure the departure of the nuclear forces from 'ordinary' forces which do not depend on the quantum state of the particles. In this notation the anisotropy and asymmetry ratios are

\[
\begin{align*}
\gamma &= \left( \frac{1 + 2a}{1 + 4a} \right)^2 \left( \frac{G_S^2(1 - 2\lambda_s a)^2 + 3G_T^2(1 - 2\lambda_T a)^2}{G_S^2 + 3G_T^2} \right), \\
\rho &= \frac{G_S^2(1 - 2\lambda_s a)^2 + 3G_T^2(1 - 2\lambda_T a)^2}{G_S^2(1 + 2(2 + \lambda_s) a)^2 + 3G_T^2(1 + 2(2 + \lambda_T) a)^2}.
\end{align*}
\] (15)

Thus the anisotropy ratio \( \gamma \) tends to a finite limit as the energy increases only if \( \lambda_s = \lambda_T = 0 \), and then the limit is \( +\frac{1}{4} \); 'ordinary' forces and the mixed neutral meson theory of Møller & Rosenfeld (1940) are included in this case. In all other cases \( \gamma \) tends to infinity as the square of the energy. It will be noted that a large anisotropy ratio does not imply that the scattering is predominantly backward; at large energies \( \gamma \) tends to infinity as the energy increases irrespective of the signs of \( \lambda_s \) and \( \lambda_T \). A large anisotropy ratio does imply a departure from 'ordinary' forces in that either \( \lambda_s \) or \( \lambda_T \) is not zero. A number characteristic of the theory can be introduced, namely,

\[
\left( \frac{\gamma}{a^2} \right)_{\infty} = \frac{\lambda_s^2 G_S^2 + 3\lambda_T^2 G_T^2}{G_S^2 + 3G_T^2},
\] (16)

where \( (\gamma/a^2)_{\infty} \) denotes the limiting value of \( (\gamma/a^2) \) as the energy tends to infinity. This limiting ratio involves the mass of the meson or the range of the forces directly through the parameter \( a \), and indirectly through the interaction constants \( G_S \) and \( G_T \) if these have been chosen to give the states of the deuteron correctly.

As the energy tends to infinity, the asymmetry ratio \( \rho \) tends to a limit which is characteristic of the theory on hand:

\[
\rho_{\infty} = \frac{\lambda_s^2 G_S^2 + 3\lambda_T^2 G_T^2}{(2 + \lambda_s)^2 G_S^2 + 3(2 + \lambda_T)^2 G_T^2}.
\]

In the most important cases this ratio reduces to a number which depends neither on the range of the force nor on the interaction constants \( G_S \) and \( G_T \), but only on the factors \( \lambda_s \) and \( \lambda_T \). In the symmetrical theory of Møller & Rosenfeld, which is the one treated in detail in this paper, \( \lambda_s = -4 \) and \( \lambda_T = -\frac{1}{3} \), and so \( \rho_{\infty} = +4 \). For 'ordinary' forces and for the mixed neutral meson theory \( \lambda_s \) and \( \lambda_T \) are zero, and so \( \rho_{\infty} \) vanishes. For the mixed charged theory \( \rho_{\infty} \) is infinite as \( \lambda_s = \lambda_T = -2 \). In these (and other) special cases \( \rho_{\infty} \) does not depend on the range of the force; in fact, it does not involve the form of the radial dependence of the interaction at all. In other cases \( \rho_{\infty} \) involves the range of the force only through the interaction constants \( G_S \) and \( G_T \) if these have been chosen to give the states of the deuteron correctly. Hence this limiting ratio may be used to test the eigenvalues of the operator \( G \) which are given in table 1 without referring to any specific form of radial dependence of the interaction. A fairly good indication of the value of \( \rho_{\infty} \) would be given by the scattering of 50 MeV neutrons, provided that the effect of the non-static interaction proves to be small at this energy. The second relation which is necessary to determine both
On the scattering of fast neutrons by protons

\( \lambda_S \) and \( \lambda_T \) can be obtained from equation (16) or from proton-proton scattering, but this will refer to a specific meson mass. Having established the set of eigenvalues of the operator \( G \), the radial dependence of the interaction can be investigated by a detailed study of the cross-section for scattering as a function of both the angle of scattering and the energy.

*Note added in proof (June 1947).* Laughlin & Kruger (1947) have investigated the scattering of 12 to 13 MeV neutrons by protons using a high pressure cloud chamber. The results indicate scattering of the ‘backward’ type which is in agreement with repulsive \( P \) states as in the Möller-Rosenfeld theory. Best agreement with theory is obtained if a meson mass of 265\( m_e \) is used; in view of the large statistical uncertainties and the possible systematic errors in the experimental methods, this is in satisfactory agreement with the estimates of the meson mass based on the results of the experiments of Occhialini & Powell (1947). It should also be pointed out that recent experiments on proton-proton scattering by Wilson (1947) indicate that the interaction is weakly repulsive in the \( \frac{1}{2} S \) state, which is in accord with the Möller-Rosenfeld theory.

In conclusion, the writer would like to express his thanks to Dr H. Fröhlich for acquainting him with the problem and for the many valuable suggestions given; he would also like to thank Professor N. F. Mott and Dr L. Hulthén for profitable discussions and Miss M. J. Lyttleton for assistance with computations.

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