The form of the divergencies in quantum electrodynamics

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The results of Landau and his collaborators on the asymptotic form of the propagators for high energies and the dependence of the renormalization constants on the cut-off are re-derived, starting from the functional equations of Gell-Mann & Low. It is proved further that, in electrodynamics, the cut-off cannot be made arbitrarily large, without the 'bare-particle' charge becoming imaginary.

Introduction

Landau, Abrikisov & Halatnikov (1954*a*) have recently shown a way of finding some important properties of renormalized quantum electrodynamics. They observe that the perturbation series for any of the renormalization constants has the form (in terms of the charge, e, and $L = \ln(P^2/m^2)$, where P is a high-energy cut-off)

$$a_{11}e^2L + e^4(a_{21}L + a_{22}L^2) + \dots. \tag{1}$$

This is because a graph of order $2n$ cannot (to take the most complicated case) contain more than n divergent parts nested inside one another; and each successive integration in such a nest can only increase the power of the logarithm by 1. Rewriting (1),

$$L(b_{11}e^2 + b_{12}e^4 + \dots) + L^2(b_{22}e^4 + b_{23}e^6 + \dots) + \dots. \tag{2}$$

The basic idea is to assume that, whereas $e^2 = \frac{1}{137}$, $Le^2 \gg e^2$; and then to neglect all terms $O(e^2)$. Then (2) becomes

$$b_{11}e^2L + b_{22}e^4L^2 + b_{33}e^6L^3 + \dots, \tag{3}$$

and so all the renormalization constants are functions of (e^2L) alone.

On this assumption, Landau *et al.* reduce Schwinger's (1951) functional-differential equations to a set of three ordinary simultaneous integral equations for the divergent (unrenormalized) Green functions. Approximate methods (Landau *et al.* 1954*b, c*) for solving these equations yield the dependence of the renormalization constants on Le^2, and the asymptotic form of the Green functions for high energies.

Gell-Mann & Low (1954) have derived functional equations which must be satisfied by the asymptotic Green functions in a renormalizable theory (the relevance of this paper has been noted by Dyson (1955) in his reviews of the work of Landau *et al.*). It is shown in the next section that these equations together with the above assumptions give the results of Landau *et al.* very directly.

There is an important respect in which the derivation given here falls short of that of Landau *et al.* In order to use Gell-Mann & Low's equations, one must *assume* the theory to be renormalizable; whereas Landau *et al.* *show* it to be renormalizable in their zero-order approximation. It should also be emphasized that the cut-off is here introduced purely as a postulate.

The properties of a model field theory of Lee (1954) have aroused speculation about the ultimate consistency of the renormalization method. In the final section it is shown that electrodynamics does indeed have the conjectured feature: that the cut-off must not exceed a certain critical value, if the 'bare-particle' Hamiltonian is to be Hermitian.

The propagators

Because of gauge-invariance, electrodynamics has peculiarities which obscure some of the features of the method. Therefore, we consider first, as a mathematical model, a pseudoscalar meson theory with *small* coupling constant.

Denote by g or e the 'experimental' coupling constant or charge, and by g_B or e_B the 'bare-particle' ones. Suppose, for large energy-momenta (in the notation of Dyson (1949), with m the nucleon mass)

$$\left.\begin{aligned} S_{Fc}(p) &\sim (\gamma p)^{-1} s(p^2/m^2, g^2), \\ \Delta_{Fc}(p) &\sim (p^2)^{-1} (p^2/m^2, g^2), \\ \Gamma_{5c}(p,p) &\sim \gamma_5 \gamma(p^2/m^2, g^2). \end{aligned}\right\} \tag{4}$$

Then, in terms of the cut-off P, Gell-Mann & Low's functional equations are

$$\left.\begin{aligned} s(p^2/m^2, g^2)/s(P^2/m^2, g^2) &= f_2(p^2/P^2, g_B^2), \\ \delta(p^2/m^2, g^2)/\delta(P^2/m^2, g^2) &= f_3(p^2/P^2, g_B^2), \\ \gamma(p^2/m^2, g^2)/\gamma(P^2/m^2, g^2) &= f_1(p^2/P^2, g_B^2), \\ g_B^2 = g^2[s(P^2/m^2, g^2)\,\gamma(P^2/m^2, &g^2)]^2\,\delta(P^2/m^2, g^2). \end{aligned}\right\} \tag{5}$$

Now, writing $L = \ln(P^2/m^2)$, $l = \ln(p^2/m^2)$, and making the assumption explained in the Introduction,* one may put

$$s(P^2/m^2, g^2) = s(Lg^2), \quad \text{etc.}$$

Abbreviating $lg^2 = \lambda$, $Lg^2 = \Lambda$, equations (5) become

$$s(\lambda)/s(\Lambda) = f_2[(\lambda-\Lambda)/g^2, g^2c(\Lambda)], \quad \text{etc.,} \tag{6}$$

where $g_B^2 = g^2c(\Lambda)$. These equations we proceed to solve.

Equation (6) must have the form

$$s(\lambda)/s(\Lambda) = f_2[(\lambda-\Lambda)\,c(\Lambda)], \quad \text{etc.}$$

Introducing inverse functions, this becomes

$$x/X = f_2\{[s^{-1}(x) - s^{-1}(X)]\,C(X)\}, \quad \text{etc.,}$$

where $C(X) = c[s^{-1}(X)]$. Inverting this equation and differentiating with respect to x,

$$(1/X)f_2^{-1\prime}(x/X) = C(X)\,s^{-1\prime}(x), \quad \text{etc.}$$

* Landau *et al.* actually use the assumption formulated in terms of g_B, whereas we are able to use g throughout.

This is a standard form of functional equation, with general solution

$$s^{-1\prime}(x) = Ax^k, \quad \text{etc.} \tag{7}$$

Equation (7) gives for the original functions (normalizing to $s(0) = \delta(0) = \gamma(0) = 1$) *either*

$$\left.\begin{aligned} s(\lambda) &= (1-\alpha\lambda)^{n_2}, \\ \delta(\lambda) &= (1-\alpha\lambda)^{n_3}, \\ \gamma(\lambda) &= (1-\alpha\lambda)^{n_1}, \\ g_B^2/g^2 = c(\Lambda) &= (1-\alpha\Lambda)^{-1}, \end{aligned}\right\} \tag{8a}$$

together with $$f_i(z) = (1-\alpha z)^{n_i};$$

or $$s(\lambda) = \exp(\alpha_2\lambda), \quad \delta(\lambda) = \exp(\alpha_3\lambda), \quad \gamma(\lambda) = \exp(\alpha_1\lambda), \quad c(\Lambda) = 1, \tag{8b}$$

together with $$f_i(z) = \exp(\alpha_i z).$$

The alternative (8*b*) corresponds to the special case $k = -1$ in (7). The last of equations (5) implies

$$2n_1 + 2n_2 + n_3 = -1, \tag{9a}$$

or $$2\alpha_1 + 2\alpha_2 + \alpha_3 = 0. \tag{9b}$$

Finally, Gell-Mann & Low's relations

$$Z_3^{-1} = \delta(P^2, g^2) = \delta(\Lambda), \quad \text{etc.},$$

give (for the alternative (8*a*))

$$Z_i^{-1} = [1 - g^2 \ln(P^2/m^2)]^{n_i}. \tag{10}$$

One may note that the exponent in g_B^2/g^2, in contrast to the exponent in the Green functions, is already fixed at this stage.

The constants α and n_i may be determined by comparison with second order perturbation theory. It is known, in fact, that (in charge-symmetric theory)

$$S_{Fc}(p) \sim (\gamma p)^{-1}[1 + (3/8\pi)\ln(p^2/m^2)g^2 + \dots],$$

$$\Delta_{Fc}(p) \sim (p^2)^{-1}[1 + (1/\pi)\ln(p^2/m^2)g^2 + \dots],$$

$$\Gamma_{5c}(p, p) \sim \gamma_5[1 - (1/4\pi)\ln(p^2/m^2)g^2 + \dots].$$

Comparing these with the expansions of (8*a*) and using (9*a*),

$$\alpha = 5/4\pi, \quad n_2 = -\tfrac{3}{10}, \quad n_3 = -\tfrac{4}{5}, \quad n_1 = \tfrac{1}{5};$$

and the alternative (8*b*), (9*b*) is ruled out. These final results are just those of Abrikosov, Galanin & Halatnikov (1954).

Now consider electrodynamics. With $D_{Fc}(p) \sim (p^2)^{-1} d(p^2, e^2)$ instead of Δ_{Fc} and m now the electron mass, there are equations quite analogous to (5) with solution analogous to (8*a*). Because of Ward's identity, $n_1 + n_2 = 0$, and (9*a*) becomes

$$n_3 = -1;$$

and hence $$e_B^2 = e^2 d(P^2/m^2, e^2), \tag{11}$$

$$d(p^2) = [1 - \alpha e^2 \ln(p^2/m^2)]^{-1}. \tag{12}$$

There is also the alternative (8*b*), which takes the form

$$s(\lambda) = \exp(\alpha\lambda), \quad \gamma(\lambda) = \exp(-\alpha\lambda), \quad d(\lambda) = 1, \quad e_B^2 = e^2. \tag{13}$$

The known perturbation result is

$$D_{Fc}(p) \sim (p^2)^{-1}\,[1 + (1/3\pi)\,e^2 \ln(p^2/m^2) + \ldots]$$

and comparison with (12) yields

$$\alpha = 1/3\pi \tag{14}$$

and rules out alternative (13), which does not seem to have any physical significance. Further, it is known that, if calculations are made in the Lorentz gauge, then Z_2 contains no second-order divergence; therefore comparison (using equations analogous to (10)) gives $n_1 = n_2 = 0$. This completes the derivation of the results obtained by Landau and his collaborators.

Charge renormalization

One of the more important results of the previous section is

$$Z_3^{-1} = e_B^2/e^2 = [1 - (e^2/3\pi)\ln(P^2/m^2)]^{-1}, \tag{15}$$

which follows from (11) and (12). In Lee's (1954) model field theory, there is an exact equation for charge renormalization identical (except for a numerical factor) with (15); and consequently an upper limit is placed upon the cut-off, P, if the theory is to be Hermitian (Pauli & Källén 1955).

The question arises whether (15) imposes an analogous restriction on the cut-off in electrodynamics.* Equation (15), by itself, does no such thing; for it is easy to construct functions $Z_3^{-1}(P^2, e^2)$, which reduce to (15) in the approximation of the Introduction, and yet which are positive for all sufficiently large P^2. However, for the case of electrodynamics, Gell-Mann & Low (1954) have given an exact functional form for Z_3^{-1}, which will be shown to be of use here. In our notation, the equation is

$$e_B^2 = F[L + \phi(e^2)]. \tag{16}$$

With $\Lambda = Le^2$, (15) and (16) imply

$$(1/e^2)\,F[\Lambda/e^2 + \phi(e^2)] \to (1 - \alpha\Lambda)^{-1}, \tag{17}$$

as $e^2 \to 0$ but Λ remains finite. Hence, ϕ must have the form

$$\phi(e^2) = \beta/e^2 + \psi(e^2),$$

where $e^2\psi(e^2) \to 0$ as $e^2 \to 0$. Then (17) becomes

$$(1/e^2)\,F[(\Lambda + \beta)/e^2] \to (1 - \alpha\Lambda)^{-1},$$

which requires $\beta = -1/\alpha$ and

$$F(z) = -1/(\alpha z) + G(z),$$

* This possibility has been stressed by W. Pauli, for instance at the Pisa Conference, 1955. The author is indebted to Professor Peierls for first telling him of Professor Pauli's views.

where $zG(z) \to 0$ as $z \to \infty$. Thus we have proved, returning to (16), that

$$e_B^2 \sim -3\pi[\ln(P^2/m^2)]^{-1},$$

the conjectured result.

This proof is not rigorous. Assumptions of good behaviour have been made: for instance, that functions are analytic in e^2 at $e^2 = 0$, and that $e^2 = \frac{1}{137}$ is not a pole of $\phi(e^2)$.

We conclude with a separate observation. From equation (16), Gell-Mann & Low conclude that, instead of tending to zero as $L \to \infty$, e_B^2 may tend to a finite limit, *independent* of e^2. We have already seen that the second alternative does not, in fact, obtain; but it is of interest to see that it can be ruled out on the basis of equation (16) alone. For, we must have

$$(1/e^2)\, F[L + \phi(e^2)] \to 1$$

as $e^2 \to 0$, for L finite. This demands $\phi(e^2) \to \infty$ as $e^2 \to 0$, and $F(z) \to 0$ as $z \to \infty$; the required result.

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