A circular inclusion with circumferentially inhomogeneous interface in antiplane shear

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A general method is developed for the rigorous solution of a problem associated with a circular inclusion embedded within an infinite matrix in antiplane shear. The bonding at the inclusion–matrix interface is assumed to be imperfect. Most significant is the fact that the imperfection in the interface is assumed to be circumferentially inhomogeneous.

Using analytic continuation, the basic boundary-value problem for two analytic functions is reduced to a first-order differential equation for a single analytic function and the closed-form solution is obtained.

The method is illustrated using several specific examples. The results from these examples are compared to the corresponding results when the imperfection in the interface is homogeneous. These comparisons illustrate how the circumferential variation of the parameter describing the imperfection has a pronounced effect on the stresses induced within the inclusion.

1. Introduction

Problems involving elastic inclusions with imperfect bonding at the inclusion–matrix interface (imperfect interface) are receiving an increasing amount of attention in the literature (see, for example, Benveniste 1984; Aboudi 1987; Sideridis 1988; Achenbach & Zhu 1989, 1990; Hashin 1990, 1991*a*, *b*, 1992; Pagano & Tandon 1990; Teng 1992; Jasiuk & Kouider 1993; Jayaraman *et al.* 1993; Jun & Jasiuk 1993; Qu 1993*a*, *b*; Gao 1995; Tandon & Pagano 1996). Interest in these problems is motivated mainly by a desire to study interface damage in composites (for example, debonding, sliding and/or cracking across an interface) and its subsequent effect on the effective properties of composites.

One of the more widely used models of an imperfect interface is based on the assumption that tractions are continuous but displacements are discontinuous across the interface. More precisely, jumps in the displacement components are assumed to be proportional (in terms of 'spring-factor-type' interface parameters) to their respective interface traction components. Under these assumptions, Hashin (1991*b*) has examined the case of a spherical inclusion imperfectly bonded to a three-dimensional matrix and found that, in contrast to the case of perfect bonding, under a remote uniform stress field, the state of stress inside the inclusion is no longer uniform. The analogous result for plane deformation has been established by Gao (1995). In both these works, in an effort to simplify the analysis, the spring-factor-type parameters

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characterizing the displacement jumps are assumed to be constant. The latter conditions imply that interface imperfections are uniform along the entire length of the material interface. This effectively rules out the more general (and realistic) case of inhomogeneous interface damage in which the extent of bonding at the inclusion–matrix interface varies along the interface itself.

To the authors' knowledge, a rigorous solution of the problem of an elastic inclusion with the more general circumferentially inhomogeneous interface remains absent from the literature (the authors have noted, among others, the investigations of Sendeckyj (1974) and Karihaloo & Viswanathan (1985), in which the problem of an elastic inclusion with partial debonding of its interface is considered).

In the present work we derive rigorous solutions of the problems associated with a circular elastic inclusion embedded within an infinite matrix in antiplane shear when the interface is circumferentially inhomogeneous.

The formulation of the basic boundary-value problem describing the antiplane deformation of an elastic inclusion with imperfect interface is presented in § 1. The case of a circular inclusion with homogeneous imperfect interface is discussed in § 2. Here, we find that, in contrast to the results obtained in Hashin (1991*b*) and Gao (1995), in the case of antiplane deformation, under a remote uniform antiplane stress field, the state of stress inside a circular inclusion with a homogeneous imperfect interface (of the above-mentioned type) remains uniform. In fact, we prove that the same conclusion is true in the more general case involving a three-phase circular inclusion in antiplane shear. These simple results are significant in that they distinguish the case of antiplane shear deformation from those of three-dimensional and plane deformations in problems involving elastic inclusions with (homogeneous) imperfect interface.

In § 4, we address the problem of a circular inclusion with circumferentially inhomogeneous interface. Unfortunately, in this case, the conventional power series method does not lead to a closed-form solution. In fact, we obtain a coupled infinite system of algebraic equations for the unknown coefficients. Instead, we use analytic continuation to reduce the basic complex boundary-value problem involving two analytic functions, to a first-order differential equation for a single analytic function defined inside the inclusion. This method leads to a rigorous closed-form solution of the problem.

In §§ 5 and 6, we illustrate our results using several specific examples. The explicit results from these examples are used to draw comparisons with the case of a homogeneous imperfect interface indicating how the circumferential variation of the parameter describing the imperfection has a significant effect on the stress field within the inclusion.

2. Formulation

Consider a domain in R^2, infinite in extent, containing a single internal elastic inclusion, with elastic properties different from the surrounding matrix. The linearly elastic materials occupying the matrix and the inclusion are assumed to be homogeneous and isotropic with associated shear moduli μ_1 and μ_2, respectively. At infinity, the prescribed deformation is that of simple shear so that the deformation $u(x, y)$ in the matrix satisfies

$$u(x, y) = ax - by + O(1), \quad x^2 + y^2 \to \infty,$$

where a and b are given constants (remote stress parameters) and (x, y) is a generic point in R^2. We represent the matrix by the domain S_1 and assume that the inclusion occupies a circular region S_2 with centre at the origin and radius R. The inclusion–matrix interface will be denoted by the curve Γ. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively, and $u(x, y)$ will denote the elastic (antiplane) deformation at the point (x, y).

It is prescribed that the circular inclusion is imperfectly bonded to the matrix along Γ by the 'spring-layer-type' interface referred to in § 1. The interface condition on Γ is therefore given by

$$\beta[u_1 - (u_2 + u^*)] = \mu_2 \frac{\partial u_2}{\partial n} = \mu_1 \frac{\partial u_1}{\partial n}, \quad \text{on } \Gamma, \tag{2.1}$$

where n is the outward unit normal to Γ, β is the imperfect interface parameter and $u^*(x, y)$ represents the additional displacement induced within the inclusion by a uniform (stress-free) eigen-strain specified below. In accordance with Hashin (1991*b*), we note that if $\beta = 0$, the condition (2.1) reduces to the case of a traction-free interface while if β is infinite, (2.1) corresponds to a perfectly bonded interface. Consequently, the following boundary value problem describes the antiplane deformation of a circular inclusion with imperfect interface of the form (2.1) (see Ru & Schiavone 1996):

$$\left.\begin{aligned} &\nabla^2 u_1 = 0, \quad \text{in } S_1, \quad \nabla^2 u_2 = 0, \quad \text{in } S_2, \\ &\beta(u_1 - u_2) = \mu_2 \frac{\partial u_2}{\partial n} + \beta u^*(x, y), \quad \mu_1 \frac{\partial u_1}{\partial n} = \mu_2 \frac{\partial u_2}{\partial n}, \quad \text{on } \Gamma, \\ &u_1(x, y) \cong ax - by + O(1), \quad x^2 + y^2 \to \infty. \end{aligned}\right\} \tag{2.2}$$

Denote by $v_i(x, y)$ the harmonic functions conjugate to $u_i(x, y)$. Since the external loading is self-equilibrated, $v_i(x, y)$ are single-valued and uniquely determined within an integration constant and the corresponding complex potentials $\phi_1(z)$ and $\phi_2(z)$ are analytic within S_1 and S_2, respectively. Thus,

$$2u_i(z) = \phi_i(z) + \overline{\phi_i(z)}, \quad \sigma_{13} - i\sigma_{23} = \mu_i \phi_i'(z), \quad z \in S_i \quad (i = 1, 2). \tag{2.3}$$

Noting that

$$2\frac{\partial u_2}{\partial n} = \phi_2'(z)\mathrm{e}^{\mathrm{i}n(z)} + \overline{\phi_2'(z)}\mathrm{e}^{-\mathrm{i}n(z)}, \quad z \in \Gamma, \tag{2.4}$$

where $\mathrm{e}^{\mathrm{i}n(z)}$ represents (in complex form) the outward normal to Γ at z, the boundary value problem (2.2) can be written in the following form

$$\left.\begin{aligned} &\phi_1(z) = \delta\phi_2(z) + (1 - \delta)\overline{\phi_2(z)} + \alpha[\phi_2'(z)\mathrm{e}^{\mathrm{i}n(z)} + \overline{\phi_2'(z)}\mathrm{e}^{-\mathrm{i}n(z)}] + u^*(z), \quad z \in \Gamma, \\ &\phi_1(z) \cong Az + O(1), \quad |z| \to \infty. \end{aligned}\right\} \tag{2.5}$$

Here

$$A \equiv a + \mathrm{i}b, \quad \alpha \equiv \frac{\mu_2}{2\beta} \geqslant 0, \quad \delta \equiv \frac{\mu_1 + \mu_2}{2\mu_1} > \frac{1}{2}, \quad u^* = \omega z + \overline{\omega z} \tag{2.6}$$

and ω is a known (complex) constant determined by the uniform eigen-strain given in the circular inclusion. Without loss of generality, we have assumed that the origin of coordinates has been chosen such that the rigid-body displacement at infinity is zero.

3. Circular inclusion with homogeneous imperfect interface

To examine the effect of the circumferential inhomogeneity of an imperfect interface on the stress field inside a circular inclusion, we first find the corresponding solution in the case of a homogeneous imperfect interface. Consider then, a circular inclusion with homogeneous imperfect interface characterized by the constant parameter α (or β).

(a) A circular inclusion

For a circular inclusion, we have

$$R\mathrm{e}^{\mathrm{i}n(z)} = z, \quad z \in \Gamma,$$

so that the interface condition (2.5) can be written as

$$\phi_1(z)+(\delta-1)\overline{\phi_2}\left(\frac{R^2}{z}\right)-\alpha\overline{\phi_2'}\left(\frac{R^2}{z}\right)\frac{R}{z}-\frac{\overline{\omega}}{z}R^2 = \delta\phi_2(z)+\alpha\phi_2'\frac{z}{R}+\omega z, \quad z \in \Gamma. \tag{3.1}$$

Using analytic continuation (see, for example, Muskhelishvili 1963; England 1971), the right-hand side of (3.1) is analytic in S_2 and the left-hand side of (3.1) is analytic in S_1, except at infinity where the left-hand side of (3.1) has the asymptotic behavior

$$Az + (\delta-1)\overline{\phi_2(0)}, \quad z \to \infty.$$

It follows from Liouville's theorem that in S_2,

$$\delta\phi_2(z) + \alpha\phi_2'\frac{z}{R} + \omega z = Az + (\delta-1)\overline{\phi_2(0)}, \quad z \in S_2,$$

which leads to

$$\phi_2(z) = \left(\frac{A-\omega}{\delta+(\alpha/R)}\right)z, \quad z \in S_2, \tag{3.2}$$

which, from (2.3), implies that the stress field inside the inclusion is uniform. This (unexpected) result is in sharp contrast to the results obtained by Hashin (1991*b*) and Gao (1995) for the corresponding problems in three-dimensional and plane elasticity, respectively, where, in each case, it was shown that in the case of a homogeneous imperfect interface, the stress inside the inclusion is not uniform.

In S_1, we have

$$\phi_1(z) = Az + \frac{R^2}{z}\left[\frac{((\alpha/R)+1-\delta)\overline{A}+(2\delta-1)\overline{\omega}}{\delta+(\alpha/R)}\right], \quad z \in S_1, \tag{3.3}$$

which enables us to calculate the stress field in the surrounding matrix. It should be noted that (3.2) and (3.3) can also be derived using the method of power series.

(b) A three-phase circular inclusion

The results obtained in § 3 *a* for a single circular inclusion are easily extended to the case of a three-phase circular inclusion with imperfect interface leading to a much stronger result. (The solution of the three-phase elastic inclusion problem provides the 'fundamental solution' for the generalized self-consistent method (see, for example, Christensen & Lo 1979; Luo & Weng 1989; Hashin 1990; Jun & Jusiuk 1993) in the mechanics of composite materials.) To see this, consider the following.

Suppose there is an intermediate annular region S_0 (with shear modulus μ_0 and outer radius R_1) between the circular region S_2 and the matrix S_1. Assume that S_0

is perfectly bonded to S_1 but imperfectly bonded to S_2 with a constant interface parameter β. Define the following quantities.

$$\delta_1 \equiv \frac{\mu_1+\mu_0}{2\mu_0} > \frac{1}{2}, \quad \delta_2 \equiv \frac{\mu_0+\mu_2}{2\mu_0} > \frac{1}{2}, \quad \alpha \equiv \frac{\mu_2}{2\beta}. \tag{3.4}$$

The corresponding antiplane problem requires that we find three analytic functions $\phi_i(z)$ $(i=0,1,2)$ in the domains S_i, respectively, satisfying the following conditions

$$\left.\begin{aligned} &\phi_0(z) = \delta_1\phi_1(z) + (1-\delta_1)\overline{\phi_1(z)}, \quad |z| = R_1,\\ &\phi_0(z) = \delta_2\phi_2(z) + (1-\delta_2)\overline{\phi_2(z)} + \alpha\left[\phi_2'(z)\frac{z}{R} + \overline{\phi_2'(z)}\frac{R}{z}\right] + u^*, \quad |z| = R,\\ &\phi_1(z) \cong Az + o(1), \quad |z| \to \infty. \end{aligned}\right\} \tag{3.5}$$

To solve this problem, let

$$\phi_1(z) = Az + \sum_{n=1}^{\infty} X_n z^{-n}, \quad \phi_2(z) = Y_0 + \sum_{n=1}^{\infty} Y_n z^n, \tag{3.6}$$

where X_n, Y_n $(n = 1, 2, \ldots)$ and Y_0 are some (undetermined) complex numbers. From the first interface condition (3.5), we obtain $\phi_0(z)$ in terms of $\phi_1(z)$ as follows:

$$\phi_0(z) = \delta_1\left[Az + \sum_{n=1}^{\infty} X_n z^{-n}\right] + (1-\delta_1)\left[\frac{\overline{A}R_1^2}{z} + \sum_{n=1}^{\infty}\frac{\overline{X_n}z^n}{R_1^{2n}}\right]. \tag{3.7}$$

Substituting (3.6) and (3.7) into the second interface condition (3.5), we find that

$$Y_0 = 0, \quad X_n = 0, \quad Y_n = 0, \quad n > 1. \tag{3.8}$$

The remaining two non-zero constants X_1 and Y_1 are determined by the conditions

$$\left.\begin{aligned} \delta_1 A + (1-\delta_1)\frac{\overline{X_1}}{R_1^2} &= \left(\delta_2 + \frac{\alpha}{R}\right)Y_1 + \omega,\\ \delta_1 X_1 + (1-\delta_1)R_1^2\overline{A} &= [(1-\delta_2)R + \alpha]\overline{Y_1}R + R^2\overline{\omega}. \end{aligned}\right\} \tag{3.9}$$

It follows that the stress field inside a three-phase circular inclusion with homogeneous imperfect interface is uniform. In fact, this constant stress field is given by

$$\frac{\sigma_{13} - \mathrm{i}\sigma_{23}}{\mu_2} = Y_1 = \frac{(2\delta_1 - 1)A + \left[(1-\delta_1)\dfrac{R^2}{R_1^2} - \delta_1\right]\omega}{(\delta_1+\delta_2-1)\dfrac{R^2}{R_1^2} + \delta_1\delta_2\left(1 - \dfrac{R^2}{R_1^2}\right) + \alpha\dfrac{R}{R_1^2}\left[\delta_1\left(1 + \dfrac{R_1^2}{R^2}\right) - 1\right]}. \tag{3.10}$$

Note that (3.10) reduces to (3.2) when $\delta_1 = 1$.

4. A circular inclusion with circumferentially inhomogeneous interface

The idea of an imperfect interface has been developed mainly to account for interface damage in composites (for example, debonding, sliding and/or cracking across an interface). Most of the existing analytical models are based on the assumption that the fibre–matrix interface has been damaged uniformly along the interface. However, in practice, an interface usually exhibits circumferentially inhomogeneous damage. One such example is the partially debonded elastic inclusion (see, for example, Karihaloo & Viswanathan 1985). In this section we present a general method for the rigorous solution of the problem corresponding to a circular inclusion with circumferentially inhomogeneous interface subjected to antiplane shear.

(*a*) *Circumferentially inhomogeneous imperfect interface*

Consider the circumferentially inhomogeneous imperfect interface described by the following (variable) parameter $\alpha(\theta)$:

$$\alpha(\theta) = \alpha\left(\frac{1+f(\theta)}{1+g(\theta)}\right), \quad \alpha > 0, \quad -f(\theta) < 1, \quad -g(\theta) < 1. \tag{4.1}$$

Here, $f(\theta)$ and $g(\theta)$ are two real periodic (period 2π) functions prescribed on the circular interface Γ. Note that as $f(\theta)$ approaches -1, $\alpha(\theta) = 0$ (perfect bonding). Similarly, as $g(\theta)$ approaches -1, $\alpha(\theta)$ tends to infinity (complete debonding). Hence, (4.1) is suitable as a model of an inhomogeneous imperfect interface since it allows for both perfect bonding and complete debonding along the interface. Several cases in which $g(\theta) \equiv 0$ on Γ will be discussed in § 5. Other examples in which $f(\theta) \neq 0$ and $g(\theta) \neq 0$ will be considered in § 6.

Consider the case when the functions $f(\theta)$ and $g(\theta)$ are given by the finite series

$$\left.\begin{aligned} f(\theta) &= \sum_{k=1}^{m}[a_k \sin[k\theta] + b_k \cos[k\theta]], \quad a_m^2 + b_m^2 \neq 0, \quad \text{unless } f(\theta) \equiv 0, \\ g(\theta) &= \sum_{k=1}^{n}[c_k \sin[k\theta] + d_k \cos[k\theta]], \quad c_n^2 + d_n^2 \neq 0, \quad \text{unless } g(\theta) \equiv 0, \end{aligned}\right\} \tag{4.2}$$

where m and n are two non-negative integers, and a_k, b_k $(k = 1, 2, \ldots, m)$, and c_k, d_k $(k = 1, 2, \ldots, n)$ are given real numbers. Let

$$\left.\begin{aligned} f(z) &= \frac{1}{2}\sum_{k=1}^{m}\left[\mathrm{i}a_k\left(\frac{R^k}{z^k} - \frac{z^k}{R^k}\right) + b_k\left(\frac{z^k}{R^k} + \frac{R^k}{z^k}\right)\right], \\ g(z) &= \frac{1}{2}\sum_{k=1}^{n}\left[\mathrm{i}c_k\left(\frac{R^k}{z^k} - \frac{z^k}{R^k}\right) + d_k\left(\frac{R^k}{z^k} + \frac{z^k}{R^k}\right)\right]. \end{aligned}\right\} \tag{4.3}$$

Clearly,

$$f(\theta) = f(z), \quad g(\theta) = g(z), \quad z \in \Gamma \quad (z = R\mathrm{e}^{\mathrm{i}\theta}).$$

Next, we state a simple result regarding the roots of the following polynomial in (z/R):

$$\left(\frac{z}{R}\right)^m [1 + f(z)]. \tag{4.4}$$

If z^*/R is a root of (4.4), then it is not difficult to see that $R/\overline{z^*}$ is also a root of (4.4). Since $f(\theta) + 1 > 0$ on Γ (see (4.1)), the polynomial (4.4) has no roots on Γ. Hence, of the $2m$ roots of the polynomial (4.4), m are located inside the circle Γ and the remaining m lie outside Γ.

(*b*) *Elimination of* $\phi_1(z)$

For an arbitrary variable interface parameter $\alpha(\theta)$, the boundary value problem (2.5) for two analytic functions $\phi_1(z)$ and $\phi_2(z)$ can be reduced to a simpler boundary-value problem for a single analytic function $\phi_2(z)$. To see this, note that since $\alpha(\theta)$ and $u^*(z)$ in (2.5) are real,

$$\phi_1(z) + (2\delta - 1)\overline{\phi_2}\left(\frac{R^2}{z}\right) = \overline{\phi_1}\left(\frac{R^2}{z}\right) + (2\delta - 1)\phi_2(z), \quad z \in \Gamma. \tag{4.5}$$

Using analytic continuation, $\phi_1(z)$ can be expressed in terms of $\phi_2(z)$ as follows:

$$\phi_1(z) = Az + \frac{\overline{A}R^2}{z} + (1-2\delta)\overline{\phi_2}\left(\frac{R^2}{z}\right) - (1-2\delta)\overline{\phi_2}(0), \quad (1-2\delta)[\phi_2(0) - \overline{\phi_2}(0)] = 0. \tag{4.6}$$

Hence, the problem (2.5) reduces to one in which we seek a single analytic function $\phi_2(z)$ in S_2 satisfying the following boundary condition.

$$Az + \frac{\overline{A}R^2}{z} - \delta\overline{\phi_2}\left(\frac{R^2}{z}\right) - \delta\phi_2(z) + (2\delta-1)\overline{\phi_2}(0) - u^* = \alpha(\theta)[\phi_2'\mathrm{e}^{\mathrm{i}\theta} + \overline{\phi_2'(z)}\mathrm{e}^{-\mathrm{i}\theta}], \quad z \in \Gamma. \tag{4.7}$$

Since $\phi_2(z)$ is analytic in the circular region S_2, it admits the Taylor expansion

$$\phi_2(z) = \sum_{n=0}^{\infty} B_n z^n, \quad B_0 = \overline{B_0}, \quad |z| \leqslant R. \tag{4.8}$$

Unfortunately, since the parameter $\alpha(\theta)$ varies with θ, the conventional power series method leads to a coupled infinite system of algebraic equations for the undetermined coefficients B_k ($k = 0, 1, 2, \ldots$) which means that we cannot determine the exact expressions for any of the coefficients B_k, not even the coefficient B_1 (which is directly related to the average stresses within the circular inclusion). To overcome this difficulty, using analytic continuation (see Muskhelishvili 1963; England 1971), we reduce (4.7) to a first-order differential equation for $\phi_2(z)$ in the circular region S_2. In this way, a closed-form solution can be obtained for the general case of a circumferentially inhomogeneous interface described by the model (4.1). This leads to, in particular, the exact expressions for the average stresses within the circular inclusion (these expressions are calculated for several specific examples in §§ 5 and 6).

(*c*) *Reduction to a differential equation for* $\phi_2(z)$

Substituting (4.1) into the boundary condition (4.7) leads to

$$[1 + g(z)]\left[(A - \omega)z + \frac{(\overline{A} - \overline{\omega})R^2}{z} - \delta\overline{\phi_2}\left(\frac{R^2}{z}\right) + (2\delta - 1)\overline{\phi_2}(0)\right]$$
$$-\alpha[1 + f(z)]\overline{\phi_2'}\left(\frac{R^2}{z}\right)\frac{R}{z} = \alpha[1 + f(z)]\phi_2'(z)\frac{z}{R} + \delta[1 + g(z)]\phi_2(z), \quad z \in \Gamma. \tag{4.9}$$

Using analytic continuation, we assert that the functions

$$\overline{\phi_2}\left(\frac{R^2}{z}\right), \quad \overline{\phi_2'}\left(\frac{R^2}{z}\right), \quad |z| > R$$

are analytic in S_1. Hence, the right-hand side of (4.9) is analytic in S_2 except at the origin where it has the singular terms

$$\frac{1}{2}\delta\sum_{k=1}^{n}(d_k + \mathrm{i}c_k)\frac{R^k}{z^k}\sum_{j=0}^{k-1}B_j z^j + \frac{1}{2}\alpha\sum_{k=1}^{m}(b_k + \mathrm{i}a_k)\frac{R^{k-1}}{z^{k-1}}\sum_{j=0}^{k-1}jB_j z^{j-1}.$$

The left-hand side of (4.9) is analytic in S_1 except at infinity where it approaches

$$-\frac{\alpha}{2R}\sum_{k=1}^{m}(b_k-\mathrm{i}a_k)\frac{z^k}{R^k}\sum_{j=0}^{k}j\overline{B_j}\left(\frac{R^2}{z}\right)^j+(A-\omega)z\left[1+\frac{1}{2}\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\right]$$
$$+(A-\omega)\tfrac{1}{2}R(d_1+\mathrm{i}c_1)+\tfrac{1}{2}(\overline{A}-\overline{\omega})\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^{k-1}}{R^{k-2}}-\delta B_0$$
$$+(2\delta-1)B_0\left[1+\frac{1}{2}\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\right]-\frac{1}{2}\delta\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\sum_{j=0}^{k}\overline{B_j}\left(\frac{R^2}{z}\right)^j.$$

Denote the sum of the above two expressions by $S(z)$, namely

$$S(z)\equiv\tfrac{1}{2}\delta\sum_{k=1}^{n}(d_k+\mathrm{i}c_k)\frac{R^k}{z^k}\sum_{j=0}^{k-1}B_jz^j+\tfrac{1}{2}\alpha\sum_{k=1}^{m}(b_k+\mathrm{i}a_k)\frac{R^{k-1}}{z^{k-1}}\sum_{j=0}^{k-1}jB_jz^{j-1}$$
$$-\frac{\alpha}{2R}\sum_{k=1}^{m}(b_k-\mathrm{i}a_k)\frac{z^k}{R^k}\sum_{j=0}^{k}j\overline{B_j}\left(\frac{R^2}{z}\right)^j+(A-\omega)z\left[1+\frac{1}{2}\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\right]$$
$$+(A-\omega)\tfrac{1}{2}R(d_1+\mathrm{i}c_1)+\tfrac{1}{2}(\overline{A}-\overline{\omega})\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^{k-1}}{R^{k-2}}-\delta B_0$$
$$+(2\delta-1)B_0\left[1+\frac{1}{2}\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\right]-\tfrac{1}{2}\delta\sum_{k=1}^{n}(d_k-\mathrm{i}c_k)\frac{z^k}{R^k}\sum_{j=0}^{k}\overline{B_j}\left(\frac{R^2}{z}\right)^j,\quad(4.10)$$

and consider the function $\Phi(z)$ defined by

$$\Phi(z)\equiv\begin{cases}[1+g(z)]\left[(A-\omega)z+\dfrac{(\overline{A}-\overline{\omega})R^2}{z}-\delta\overline{\phi_2}\left(\dfrac{R^2}{z}\right)+(2\delta-1)\overline{\phi_2}(0)\right]\\ -\alpha[1+f(z)]\overline{\phi_2'}\left(\dfrac{R^2}{z}\right)\dfrac{R}{z}-S(z),\quad\left|\dfrac{z}{R}\right|>1;\\ \alpha[1+f(z)]\phi_2'(z)\dfrac{z}{R}+\delta[1+g(z)]\phi_2(z)-S(z),\quad\left|\dfrac{z}{R}\right|<1.\end{cases}\quad(4.11)$$

Defined in this way, $\Phi(z)$ is analytic in both S_2 (excluding the origin) and S_1 (excluding the point at infinity) and continuous across Γ. In addition, it approaches zero as z approaches both the origin and the point at infinity, identifying the latter as removable singularities. It follows that $\Phi(z)$ given by (4.11) is analytic on the whole plane including the point at infinity where it tends to zero. From Liouville's theorem, we conclude that $\Phi(z)$ is identically zero. Thus, from (4.11), we have

$$a[1+f(z)]\phi_2'(z)\frac{z}{R}+\delta[1+g(z)]\phi_2(z)=S(z),\quad|z|\leqslant R,\quad(4.12)$$

and

$$[1+g(z)]\left[(A-\omega)z+\frac{(\overline{A}-\overline{\omega})R^2}{z}-\delta\overline{\phi_2}\left(\frac{R^2}{z}\right)+(2\delta-1)\overline{\phi_2}(0)\right]$$
$$-\alpha[1+f(z)]\overline{\phi_2'}\left(\frac{R^2}{z}\right)\frac{R}{z}=S(z),\quad|z|\geqslant R.\quad(4.13)$$

The function $\phi_2(z)$ determined from (4.12) must be compatible with that obtained from (4.13). In fact, it can be shown that the following condition is necessary and sufficient for this required compatibility:

$$\begin{aligned} B_0 = \tfrac{1}{2}R[(A-\omega)(d_1+\mathrm{i}c_1)+\overline{A}-\overline{\omega}(d_1-\mathrm{i}c_1)] \\ -\frac{1}{2}\delta\sum_{k=1}^{n}[(d_k+\mathrm{i}c_k)B_k+(d_k-\mathrm{i}c_k)\overline{B_k}]R^k \\ -\frac{\alpha}{2R}\sum_{k=1}^{m}[(b_k+\mathrm{i}a_k)B_k+(b_k-\mathrm{i}a_k)\overline{B_k}]kR^k. \end{aligned} \tag{4.14}$$

In addition, since the undetermined constants B_k $(k = 0,1,2,\dots,\mathrm{major}(m,n))$ appear in the right-hand side of (4.12), any admissible solution $\phi_2(z)$ of (4.12) must satisfy the consistency conditions

$$\frac{\phi_2^{(k)}(0)}{k!} = B_k, \quad k = 0,1,2,\dots,\mathrm{major}(m,n). \tag{4.15}$$

There are two cases to consider as follows.

Case 1: $n \geqslant m$. Assume that the solution of (4.12) is analytic in S_2. Substituting its Taylor expansion into (4.12) and comparing coefficients of all non-positive powers of z (from z^0 to z^{-n}), we obtain $n+1$ equations. In view of the compatibility condition (4.14), these $n+1$ equations imply that

$$\frac{\phi_2^{(k)}(0)}{k!} = B_k, \quad k = 0,1,2,\dots,n.$$

That is, when $n \geqslant m$, any analytic solution of (4.12) with B_k $(k = 0,1,2,\dots,n)$ satisfying (4.14) will automatically satisfy the consistency conditions (4.15).

Case 2: $m > n$. Proceeding as in case 1 and comparing coefficients of all non-positive powers of z (this time from z^0 to $z^{-(m-1)}$), we obtain m equations. As above, it follows that any analytic solution of (4.12) with B_k $(k = 0,1,2,\dots,m)$ satisfying (4.14) will satisfy the $(m+1)$ consistency conditions (4.15) if the only if the following (single) consistency condition is satisfied:

$$\phi_2(0) = B_0 \tag{4.16}$$

(*d*) *The general solution of (4.12)*

The differential equation (4.12) for $\phi_2(z)$ can be written in the form

$$\phi_2'(z) + \frac{\delta}{\alpha}\left[\frac{1+g(z)}{(z/R)[1+f(z)]}\right]\phi_2(z) = \frac{S(z)}{\alpha(z/R)[1+f(z)]}, \quad \left|\frac{z}{R}\right| \leqslant 1. \tag{4.17}$$

The general solution of the corresponding homogeneous equation is given by

$$\phi_2(z) = C\exp(Q(z)), \quad Q(z) \equiv -\frac{\delta}{\alpha}\int\frac{1+g(t)}{(t/R)[1+f(t)]}\,\mathrm{d}t, \quad \left|\frac{z}{R}\right| \leqslant 1, \tag{4.18}$$

where C is an arbitrary constant of integration. The general solution of the non-homogeneous equation (4.17) is now obtained using variation of parameters. In fact, we find that

$$\phi_2(z) = C(z)\exp(Q(z)), \quad C(z) \equiv \int_{z_0}^{z}\frac{S(t)\exp(-Q(t))}{a-(t/R)[1+f(t)]}\,\mathrm{d}t + C_0, \tag{4.19}$$

where z_0 is some fixed point in S_2 and C_0 is an arbitrary constant of integration. The presence of irregular points such as poles, essential singular points and/or branch points of the multivalued functions appearing in (4.18) and (4.19), means that the solution $\phi_2(z)$, given by (4.18), is, in general, defined only in the (simply-connected) appropriately cut region S_2. Accordingly, $\phi_2(z)$ given by (4.19) is not, in general, analytic in the uncut region S_2. For example, $\phi_2(z)$ given by (4.19) may be discontinuous across branch cuts and/or may become unbounded at some or all isolated irregular points. Hence, to ensure analyticity of $\phi_2(z)$ in the uncut region S_2, the following two conditions must be verified.

(i) $\phi_2(z)$ given by (4.19) is bounded at all irregular points.

(ii) $\phi_2(z)$ given by (4.19) is continuous across all branch cuts.

These analyticity conditions, together with (4.14) and (4.16), will determine the unknown constants B_k ($k = 0, 1, 2, \ldots, \mathrm{major}(m, n)$) appearing on the right-hand side of (4.12).

Consider each of the two cases mentioned above.

Case 1: $n \geqslant m$. First note that the denominator of the rational function

$$-\frac{\delta}{\alpha}\left[\frac{1+g(z)}{(z/R)(1+f(z))}\right] = -\frac{\delta}{\alpha}\left[\frac{(z/R)^n(1+g(z))}{(z/R)^{n+1}(1+f(z))}\right], \tag{4.20}$$

is of degree $m+n+1$ while the numerator is of degree $2n$. In particular, of the $m+n+1$ roots of the denominator (see § 4 *a*), $n+1$ are located inside S_2 while the remaining m roots lie outside S_2. Consequently, the rational function (4.20) can be decomposed into the sum of $m+n+1$ partial fractions if $m = n$, or into the sum of $m+n+1$ partial fractions and a polynomial of degree $n-m-1$ if $n > m$. In this way, $Q(z)$ given by (4.18) can be calculated explicitly. Furthermore, the non-homogeneous term on the right-hand side of (4.17) can be decomposed into the sum of a polynomial of degree $n-m$ and the $m+n+1$ partial fractions, of which $n+1$ partial fractions are singular within S_2. Hence, when $n \geqslant m$, the solution (4.19) has $n+1$ singular points and the $n+2$ undetermined constants, C_0 and $B_k (k = 0, 1, 2, \ldots, n)$, are chosen to satisfy the $n+1$ analyticity conditions and the compatibility condition (4.14).

Case 2: $m > n$. In this case the denominator of the rational function

$$-\frac{\delta}{\alpha}\left[\frac{1+g(z)}{(z/R)[1+f(z)]}\right] = -\frac{\delta}{\alpha}\left[\frac{(z/R)^{m-1}[1+g(z)]}{(z/R)^m[1+f(z)]}\right], \tag{4.21}$$

is of degree $2m$ while the numerator is of degree $n+m-1$. Of the $2m$ roots of the denominator, m are located inside S_2. Consequently, (4.21) can be decomposed into the sum of $2m$ partial fractions and $Q(z)$ given by (4.18) can be calculated explicitly. Furthermore, the non-homogeneous term on the right-hand side of (4.17) can be decomposed into the sum of $2m$ partial fractions of which m are singular within S_2. Hence, when $m > n$, the solution (4.19) has m singular points and the $m+2$ undetermined constants, C_0 and B_k ($k = 0, 1, 2, \ldots, m$), should be chosen to satisfy the m analyticity conditions, the compatibility condition (4.14) and the consistency condition (4.16).

In conclusion, the closed-form solution is given by (4.19) and the unknown constants C_0 and B_k ($k = 0, 1, 2, \ldots, \mathrm{major}(m, n)$) are determined by the analyticity of the solution (4.19) and the corresponding supplementary conditions.

(*e*) *A particular class of circumferentially inhomogeneous interface*

In this section we determine the constants C_0 and B_k ($k = 0, 1, 2, \ldots,$ major (m, n)), when the interface parameter takes the particular form

$$\alpha(\theta) = \alpha\left[\frac{1 + \alpha_1\cos(m\theta)}{1 + \alpha_2\cos(n\theta)}\right], \quad 0 < \alpha, \quad -1 < \alpha_1 < 1, \quad -1 < \alpha_2 < 1. \tag{4.22}$$

For this class of interface, the polynomial (4.4) takes the form

$$\frac{2}{\alpha_1}\left(\frac{z}{R}\right)^m + \left(\frac{z}{R}\right)^{2m} + 1 = 0. \tag{4.23}$$

Let the m roots of (4.23) located within the unit circle (see § 4 *a*) be denoted by

$$\rho_1, \rho_2, \ldots \rho_m.$$

These m roots are thus determined by

$$\rho^m = \rho^*, \tag{4.24}$$

where

$$\rho^* = \begin{cases} \alpha_1 > 0, & \sqrt{\dfrac{1}{\alpha_1^2} - 1} - \dfrac{1}{\alpha_1} < 0, \\ \alpha_1 < 0, & -\sqrt{\dfrac{1}{\alpha_1^2} - 1} - \dfrac{1}{\alpha_1} > 0. \end{cases} \tag{4.25}$$

The remaining m roots of (4.23) (located outside the unit circle) are given by

$$\frac{1}{\rho_1}, \frac{1}{\rho_2}, \ldots, \frac{1}{\rho_m}.$$

In the following sections, we discuss, in detail, particular examples of (4.22).

5. Examples: the inhomogeneous interface (4.22) when $\alpha_2 = 0$

Let $\alpha_2 = 0$ in (4.22) and consider the class of inhomogeneous interface defined by

$$\alpha(\theta) = \alpha[1 + \alpha_1\cos(m\theta)], \quad 0 < \alpha, \quad -1 < \alpha_1 < 1. \tag{5.1}$$

(*a*) *The case* $m = 1$

Let us begin with the simplest case of (5.1): $m = 1$. From (4.19), the solution in this case is given by

$$\phi_2(z) = \frac{2}{\alpha\alpha_1}\left(\frac{z}{R} - \rho^*\right)^{\gamma_1}\left(\frac{z}{R} - \frac{1}{\rho^*}\right)^{-\gamma_1}\int_{R\rho^*}^{z} S(t)\left(\frac{t}{R} - \rho^*\right)^{-(1+\gamma)}\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^{\gamma_1 - 1} \mathrm{d}t, \tag{5.2}$$

where

$$S(z) = (A - \omega)z + (\delta - 1)B_0 - \tfrac{1}{2}\alpha\alpha_1\overline{B}_1, \quad \gamma_1 \equiv \frac{\delta R(\rho^{*2} + 1)}{\alpha(\rho^{*2} - 1)} < 0. \tag{5.3}$$

Note that, in (5.2), the integrand has a branch point at $z = R\rho^*$ in S_2. Accordingly, we pick a branch of the integrand which has a branch cut parallel to the real axis. Also, since $\gamma_1 < 0$, the integral in (5.2) is convergent at the lower limit $z = R\rho^*$ and, $C_0 = 0$ guarantees that $\phi_2(z)$ is bounded at $z = R\rho^*$. It remains to prove that

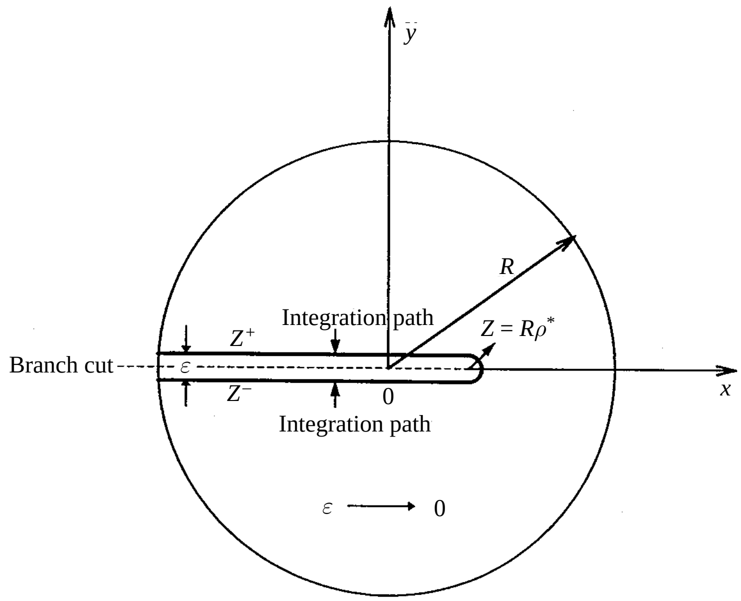


Figure 1. Continuity of $\phi_2(z)$ across branch cut: $\phi_2(z^+) = \phi_2(z^-)$

$\phi_2(z)$, given by (5.2), is continuous across the branch cut. To do this, consider the difference $\phi_2(z^+) - \phi_2(z^-)$, where $\phi_2(z^\pm)$ denote the values of $\phi_2(z)$ as z approaches the branch cut from above and below, respectively. It is clear that, integrating along the integration paths as shown in figure 1, because of the compensating nature of the terms outside the integral, we obtain $\phi_2(z^+) = \phi_2(z^-)$ implying that $\phi_2(z)$ is continuous across the branch cut.

Hence, the unknown constants B_0 and B_1 appearing in (5.2) are determined by the compatibility condition (4.14) and the consistency condition (4.16) which give

$$B_0 + \tfrac{1}{2}\alpha\alpha_1(B_1 + \overline{B}_1) = 0, \tag{5.4}$$

$$B_0 = \frac{2}{\alpha\alpha_1}(-\rho^*)^{\gamma_1}\left(-\frac{1}{\rho^*}\right)^{-\gamma_1}\int_{R\rho^*}^{0} S(t)\left(\frac{t}{R} - \rho^*\right)^{-(1+\gamma_1)}\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^{\gamma_1 - 1} \mathrm{d}t. \tag{5.5}$$

In fact, the exact expression of B_1 can be determined by

$$B_1 I_0 + \frac{2(A-\omega)}{\alpha\alpha_1} I_1 = (\delta I_0 - 1)(B_1 + \overline{B}_1), \tag{5.6}$$

where

$$I_0 = \frac{2R}{\alpha\alpha_1}\int_{\rho^*}^{0}\left(1 - \frac{t}{\rho^*}\right)^{-(1+\gamma_1)}(1 - t\rho^*)^{\gamma_1 - 1}\,\mathrm{d}t, \tag{5.7}$$

$$I_1 = \frac{2R^2}{\alpha\alpha_1}\int_{\rho^*}^{0} t\left(1 - \frac{t}{\rho^*}\right)^{-(1+\gamma_1)}(1 - t\rho^*)^{\gamma_1 - 1}\,\mathrm{d}t. \tag{5.8}$$

Once B_1 is found, B_0 is obtained from (5.4) and the closed-form solution is then given by (5.2).

For example, if $\gamma_1 = -1$, we find that

$$B_1 = \frac{4}{\delta\alpha_1^2}\left(\frac{1-\rho^{*2}}{1+\rho^{*2}}\right)^2 (A-\omega)\left[\frac{1}{1-\rho^{*2}} + \frac{\ln[1-\rho^{*2}]}{\rho^{*2}}\right], \quad \gamma_1 = -1. \tag{5.9}$$

It is not difficult to show that as α_1 tends to zero, this result reduces to (3.2) for a homogeneous imperfect interface. Furthermore, using the relation (2.3), the exact expression for the average stresses within the circular inclusion (when $\gamma_1 = -1$) is given by

$$[\overline{\sigma}_{13} - \mathrm{i}\overline{\sigma}_{23}] = h_1(\rho^*)(\overline{\sigma}_{13} - \mathrm{i}\overline{\sigma}_{23})_0, \quad h_1(\rho^*) \equiv \frac{2}{\rho^{*2}}\left[1 + \frac{(1-\rho^{*2})\ln[1-\rho^{*2}]}{\rho^{*2}}\right]. \tag{5.10}$$

Here, '$\overline{\sigma}$' represents the average stress taken within the circular inclusion and the subscript '0' denotes the corresponding stress in the case of a homogeneous interface ($\alpha_1 = 0$) under otherwise identical conditions. The graph of the function $h_1(\rho^*)$ is shown in figure 2. It is clear that (5.10) provides a useful comparison between the average stresses within the circular inclusion when the interface is inhomogeneous and the corresponding average stresses in the case of a homogeneous interface. Moreover, in view of the fact that, in each case, α is related to the parameter α_1 of the inhomogeneous interface by (4.25) and (5.3) (for example, here $\gamma_1 = -1$, so that (5.3) leads to

$$\frac{\alpha}{R} = \delta\left[\frac{1+\rho^{*2}}{1-\rho^{*2}}\right],$$

which, in view of (4.25), gives a relation between α and α_1), (5.10) indicates that the circumferential variation of the interface parameter, $\alpha(\theta)$, has a significant effect on the average stresses within the inclusion (see figure 2).

(*b*) *The general case, $m > 1$*

Next, we consider the more general case $m > 1$. In this case, the solution is given by

$$\phi_2(z) = \frac{2}{\alpha\alpha_1}\left[\left(\frac{z}{R}\right)^m - \rho^*\right]^{\gamma_1/m}\left[\left(\frac{z}{R}\right)^m - \frac{1}{\rho^*}\right]^{-\gamma_1/m}$$
$$\times \int_{R\rho_1}^{z}\left(\frac{t}{R}\right)^{m-1} S(t)\left[\left(\frac{t}{R}\right)^m - \rho^*\right]^{-(1+(\gamma_1/m))}\left[\left(\frac{t}{R}\right)^m - \frac{1}{\rho^*}\right]^{(\gamma_1/m)-1} \mathrm{d}t, \tag{5.11}$$

where

$$\left(\frac{z}{R}\right)^{m-1} S(z) = \frac{\alpha\alpha_1}{2R}\left[\frac{R}{z}\sum_{k=1}^{m-1} kB_k z^k - \frac{z^{2m-1}}{R^{2m-1}}\sum_{k=1}^{m} k\overline{B}_k\left(\frac{R^2}{z}\right)^k\right]$$
$$+ (A-\omega)R\left(\frac{z}{R}\right)^m + (\delta-1)B_0\left(\frac{z}{R}\right)^{m-1}. \tag{5.12}$$

Proceeding as in § 5 *a*, branch cuts are made from each of the m branch points of the multi-valued functions appearing in (5.11). Each branch cut is parallel to the real axis and there is no overlap. Since $\gamma_1 < 0$, the integral in (5.11) converges at each branch point and $\phi_2(z)$ given by (5.11) is continuous across all branch cuts. In addition, the boundedness of $\phi_2(z)$ at $z = R\rho_1$ implies $C_0 = 0$. Now, B_k $(k = 0, 1, 2, \ldots, m)$ are determined by the condition (4.14)

$$B_0 + m\alpha\alpha_1 \left(\frac{B_m + \overline{B}_m}{2}\right) R^{m-1} = 0, \tag{5.13}$$

the consistency condition (4.16)

$$B_0 = \frac{2}{\alpha\alpha_1}[\rho^*]^{\gamma_1/m}\left[-\frac{1}{\rho^*}\right]^{-\gamma_1/m} \times \int_{R\rho_1}^{0} \left(\frac{t}{R}\right)^{m-1} S(t) \left[\left(\frac{t}{R}\right)^m - \rho^*\right]^{-(1+(\gamma_1/m))} \left[\left(\frac{t}{R}\right)^m - \frac{1}{\rho^*}\right]^{\gamma_1/m-1} \mathrm{d}t, \tag{5.14}$$

and the $m-1$ boundedness conditions of $\phi_2(z)$ given by (5.11) at the remaining $m-1$ branch points $z = R\rho_k$ $(k = 2, \dots, m)$:

$$\int_{R\rho_1}^{R\rho_k} \left(\frac{t}{R}\right)^{m-1} S(t) \left[\left(\frac{t}{R}\right)^m - \rho^*\right]^{-(1+(\gamma_1/m))} \left[\left(\frac{t}{R}\right)^m - \frac{1}{\rho^*}\right]^{(\gamma_1/m)-1} \mathrm{d}t = 0, \quad k = 2, \dots, m. \tag{5.15}$$

Using (5.13)–(5.15), the undetermined coefficients B_k $(k = 0, 1, 2, \dots, m)$ can be expressed in terms of definite integrals of known power functions. Hence, the closed-form solution $\phi_2(z)$ is obtained from (5.11) for this more general case of the circumferentially inhomogeneous interface (5.1).

For example, consider the case $m = 2$. The only three unknown constants B_0, B_1 and B_2 are determined by the equations

$$\left.\begin{aligned} B_0 + \alpha\alpha_1(B_2 + \overline{B}_2)R = 0, \\ B_0 = \tfrac{1}{2}\alpha\alpha_1 B_1 I_{00} + [(\delta-1)B_0 - \alpha\alpha_1 R\overline{B}_2]I_{01} + [(A-\omega)R - \tfrac{1}{2}\alpha\alpha_1\overline{B}_1]I_{02}, \\ \tfrac{1}{2}\alpha\alpha_2 B_1 I_{10} = [(\delta-1)B_0 - \alpha\alpha_1 R\overline{B}_2]I_{11} + [(A-\omega)R - \tfrac{1}{2}\alpha\alpha_1\overline{B}_1]I_{12} = 0, \end{aligned}\right\} \tag{5.16}$$

where

$$\left.\begin{aligned} I_{0k} &= \frac{2}{\alpha\alpha_1}[-\rho^*]^{\gamma_1/2}\left[-\frac{1}{\rho^*}\right]^{-\gamma_1/2} \\ &\quad \times \int_{R\rho_1}^{0} \left(\frac{t}{R}\right)^{k} \left[\left(\frac{t}{R}\right)^2 - \rho^*\right]^{-(1+(\gamma_1/2))} \left[\left(\frac{t}{R}\right)^2 - \frac{1}{\rho^*}\right]^{(\gamma_1/2)-1} \mathrm{d}t, \quad k = 0, 1, 2, \\ I_{1k} &= \int_{R\rho_1}^{R\rho_2} \left(\frac{t}{R}\right)^{k} \left[\left(\frac{t}{R}\right)^2 - \rho^*\right]^{-(1+(\gamma_1/2))} \\ &\quad \times \left[\left(\frac{t}{R}\right)^2 - \frac{1}{\rho^*}\right]^{(\gamma_1/2)-1} \mathrm{d}t, \quad k = 0, 1, 2. \end{aligned}\right\} \tag{5.17}$$

Further, for $m = 2$, note that

$$\rho_2 = -\rho_1, \quad \rho_1^2 = \rho_2^2 = \rho^*,$$

so that

$$\frac{I_{02}}{I_{00}} = \frac{I_{12}}{I_{10}}. \tag{5.18}$$

Hence, $B_0 = B_2 = 0$ and B_1 is given by

$$\frac{\alpha\alpha_1}{2R}B_1 = \frac{(\overline{A} - \overline{\omega}) + (A - \omega)I_{10}/I_{12}}{1 - (I_{10}/I_{12})^2}. \tag{5.19}$$

In particular, if $m = 2$ and $\gamma_1 = -2$, we have

$$\frac{I_{10}}{I_{12}} = \frac{\displaystyle\int_0^{R\rho_1} \left[\left(\frac{z}{R}\right)^2 - \frac{1}{\rho^*}\right]^{-2} \mathrm{d}z}{\displaystyle\int_0^{R\rho_1} (zR)^2 \left[\left(\frac{z}{R}\right)^2 - \frac{1}{\rho^*}\right]^{-2} \mathrm{d}z},$$

which leads to

$$\frac{I_{10}}{I_{12}} = \rho^* \frac{\left[\dfrac{2\rho^*}{1-\rho^{*2}} - \ln\left[\dfrac{1-\rho^*}{1+\rho^*}\right]\right]}{\left[\dfrac{2\rho^*}{1-\rho^{*2}} + \ln\left[\dfrac{1-\rho^*}{1+\rho^*}\right]\right]}. \tag{5.20}$$

The constant B_1 is now obtained by substituting (5.20) into (5.19). The average stresses within the circular inclusion can now be calculated explicitly.

6. Examples: the inhomogeneous interface (4.22) when $m = n \geqslant 1$

In this section, we consider another subclass of the inhomogeneous interface (4.22) given by

$$\alpha(\theta) = \alpha\left[\frac{1 + \alpha_1 \cos(m\theta)}{1 + \alpha_2 \cos(m\theta)}\right], \quad 0 < \alpha, \quad -1 < \alpha_1 < 1, \quad -1 < \alpha_2 < 1. \tag{6.1}$$

From (4.19), the solution in this case is given by

$$\left.\begin{aligned} \phi_2(z) &= C(z)\exp(Q(z)), \\ C(z) &= \frac{2}{\alpha\alpha_1}\int_{z_0}^{z} \frac{(t/R)^m S(t)\exp^{-Q(t)}}{(t/R)\left[(t/R)^m - \rho^*\right]\left[(t/R)^m - (1/\rho^*)\right]}\,\mathrm{d}t + C_0, \end{aligned}\right\} \tag{6.2}$$

where

$$\exp(Q(z)) = \left(\frac{z}{R}\right)^{\gamma}\left[\left(\frac{z}{R}\right)^m - \rho^*\right]^{\kappa(\gamma/m)}\left[\left(\frac{z}{R}\right)^m - \frac{1}{\rho^*}\right]^{-\kappa(\gamma/m)}, \tag{6.3}$$

$$\gamma \equiv \frac{-\alpha_2\delta R}{\alpha\alpha_1}, \quad \kappa \equiv \frac{1+\rho^{*2}}{1-\rho^{*2}}\left(\frac{\alpha_1}{\alpha_2} - 1\right). \tag{6.4}$$

It will be shown that the method of determining the unknown constants for this subclass depends on the signs of the two parameters (6.4). For this reason, we shall distinguish the following three cases

$$\text{(i) } \gamma < 0, \quad \kappa \leqslant 0, \quad \text{(ii) } \gamma < 0, \quad \kappa > 0, \quad \text{(iii) } \gamma > 0, \quad \kappa < 0. \tag{6.5}$$

(*a*) *The simple case $m = n = 1$*

Let us begin with the simplest case $m = n = 1$. We consider each of (i), (ii) and (iii) in (6.5).

(i) $\gamma < 0$ *and* $\kappa \leqslant 0$

Here, the solution is given by

$$\phi_2(z) = \frac{2}{\alpha\alpha_1}\left(\frac{z}{R}\right)^{\gamma}\left(\frac{z}{R}-\rho^*\right)^{\kappa\gamma}\left(\frac{z}{R}-\frac{1}{\rho^*}\right)^{-\kappa\gamma} \times \int_0^z \frac{t}{R}S(t)\left(\frac{t}{R}\right)^{-(1+\gamma)}\left(\frac{t}{R}-\rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{t}{R}-\frac{1}{\rho^*}\right)^{\kappa\gamma-1}\mathrm{d}t, \quad (6.6)$$

where the two non-overlapping branch cuts for the multi-valued functions

$$\left(\frac{z}{R}\right)^{\gamma}, \quad \left(\frac{z}{R}-\rho^*\right)^{\kappa\gamma}, \quad |z| < R,$$

are taken along the real axis from $z = 0$ and $z = R\rho^*$, respectively. Since $\gamma < 0$, the integral in (6.6) is convergent at $z = 0$ and $\phi_2(z)$ given by (6.6) is continuous across the branch cuts starting at $z = 0$. In addition, we impose $C_0 = 0$ to ensure the boundedness of $\phi_2(z)$ at $z = 0$.

Next, we consider the branch point $z = R\rho^*$. Divide the integral in (6.6) into two parts

$$\int_0^z \left[\frac{t}{R}S(t)\left(\frac{t}{R}\right)^{-(1+\gamma)}\left(\frac{t}{R}-\rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{t}{R}-\frac{1}{\rho^*}\right)^{\kappa\gamma-1}\right]\mathrm{d}t = \left(\int_0^{z_0}+\int_{z_0}^{z}\right)\left[\frac{t}{R}S(t)\left(\frac{t}{R}\right)^{-(1+\gamma)}\left(\frac{t}{R}-\rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{t}{R}-\frac{1}{\rho^*}\right)^{\kappa\gamma-1}\right]\mathrm{d}t, \quad (6.7)$$

where z_0 is an arbitrary non-zero point in the cut region S_2. Performing an integration by parts on the second integral on the right-hand side of (6.7), reveals that its contribution to $\phi_2(z)$ is bounded as z approaches the branch point $z = R\rho^*$. In addition, the first integral on the right-hand side of (6.7) tends toward zero as z approaches $R\rho^*$. Consequently, $\phi_2(z)$, given by (6.6), is bounded at the branch point $z = R\rho^*$.

To ensure the analyticity of $\phi_2(z)$ in S_2, it remains to verify the continuity of $\phi_2(z)$ (given by (6.6)) across the branch cut emanating from $z = R\rho^*$. To do this, consider the integral defined by

$$\Psi(z) = \int_0^z \frac{t}{R}S(t)\left[\frac{z/R}{t/R}\right]^{(1+\gamma)}\left[\frac{((z/R)-\rho^*)}{((t/R)-\rho^*)}\right]^{(1+\kappa\gamma)}\left(\frac{t}{R}-\frac{1}{\rho^*}\right)^{\kappa\gamma-1}\mathrm{d}t, \quad z \in S_2. \quad (6.8)$$

We require that $\Psi(z^+) = \Psi(z^-)$, where $\Psi(z^{\pm})$ denote the values of $\Psi(z)$ as z approaches the branch cut from the upper and lower half-plane, respectively. Note that here, since the integrand in (6.8) has a singularity of order higher than one at $z = R\rho^*$, no integration path is allowed to pass through the branch point $z = R\rho^*$. Hence, the required condition ensuring that $\Psi(z^+) = \Psi(z^-)$ takes the form

$$\int_{\Gamma^*} \frac{t}{R}S(t)\left[\frac{z/R}{t/R}\right]^{(1+\gamma)}\left[\frac{((z/R)-\rho^*)}{((t/R)-\rho^*)}\right]^{(1+\kappa\gamma)} \times \left(\frac{t}{R}-\frac{1}{\rho^*}\right)^{\kappa\gamma-1}\mathrm{d}t = 0, z = \begin{cases} z^+, & \mathrm{Im}[t] > 0, \\ z^-, & \mathrm{Im}[t] < 0, \end{cases} \quad (6.9)$$

where z is an arbitrarily chosen point on the branch cut from $z = R\rho^*$ in S_2 and Γ^*

is an arbitrary closed curve contained entirely within S_2, enclosing the branch point $z = R\rho^*$ and intersecting the real axis only at the arbitrarily chosen point z and at the origin $z = 0$.

In conclusion, the only two unknown constants B_0 and B_1 are determined by the compatibility condition (4.14) and the above supplementary condition (6.9). For example, if we choose the parameters α and α_2 in such a way that

$$\kappa = -\tfrac{1}{2}, \quad \gamma = -2,$$

and leave α_1 arbitrary within the range (see (A 5) in the appendix)

$$-\tfrac{9}{15} < \alpha_1 < \tfrac{9}{15},$$

then the condition (6.9) implies that the residue of the integrand in (6.9) at the pole $z = R\rho^*$ should be zero. This requires that

$$2\alpha_2 R(A-\omega)\rho^{*3} + 3\rho^{*2}[(A-\omega)R + (\delta-1)B_0\tfrac{1}{2}\alpha_2] + 2\rho^*[\delta B_0 + \tfrac{3}{4}B_1\delta R\alpha_2] + \tfrac{1}{2}\delta\alpha_2 B_0 = 0.$$

The constants B_0 and B_1 can now be determined from this condition and the compatibility condition (4.14). Once B_0 and B_1 are determined, the closed-form solution is given by (6.6).

(ii) $\gamma < 0$ *and* $\kappa > 0$

Here, the solution $\phi_2(z)$ is given by

$$\phi_2(z) = \frac{2}{\alpha\alpha_1}\left(\frac{z}{R}\right)^{\gamma}\left(\frac{z}{R} - \rho^*\right)^{\kappa\gamma}\left(\frac{z}{R} - \frac{1}{\rho^*}\right)^{-\kappa\gamma} \times \int_0^z \frac{t}{R}S(t)\left(\frac{t}{R}\right)^{-(1+\gamma)}\left(\frac{t}{R} - \rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^{\kappa\gamma-1} \mathrm{d}t. \quad (6.10)$$

Note that the integral in (6.10) is convergent at both the lower limit $z = 0$ and at the branch point $z = R\rho^*$. Moreover, $\phi_2(z)$ given by (6.10) is continuous across all branch cuts and $C_0 = 0$ ensures the boundedness of $\phi_2(z)$ at $z = 0$. Also, to ensure the boundedness of $\phi_2(z)$ as z tends to $z = R\rho^*$, we have necessarily (see Muskhelishvili 1963; England 1971)

$$\int_0^{R\rho^*} \frac{z}{R}S(z)\left(\frac{z}{R}\right)^{-(1+\gamma)}\left(\frac{z}{R} - \rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{z}{R} - \frac{1}{\rho^*}\right)^{\kappa\gamma-1} \mathrm{d}z = 0. \quad (6.11)$$

The two constants B_0 and B_1 are uniquely determined by the condition (6.11) and the compatibility condition (4.14). Once the constants B_0 and B_1 are obtained, the closed-form solution for $\phi_2(z)$ is given by (6.10).

(iii) $\gamma > 0$ *and* $\kappa < 0$

The solution in this case is given by

$$\phi_2(z) = \frac{2}{\alpha\alpha_1}\left(\frac{z}{R}\right)^{\gamma}\left(\frac{z}{R} - \rho^*\right)^{\kappa\gamma}\left(\frac{z}{R} - \frac{1}{\rho^*}\right)^{-\kappa\gamma} \times \int_{R\rho^*}^z \frac{t}{R}S(t)\left(\frac{t}{R}\right)^{-(1+\gamma)}\left(\frac{t}{R} - \rho^*\right)^{-(1+\kappa\gamma)}\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^{\kappa\gamma-1} \mathrm{d}t. \quad (6.12)$$

Choosing $C_0 = 0$ means that this expression for $\phi_2(z)$ is bounded at both $z = 0$ and $z = R\rho^*$. Furthermore, $\phi_2(z)$, given by (6.12), is continuous across the branch cut taken from $z = R\rho^*$. However, as in § 6 *a* (i), an additional continuity condition must be imposed to ensure the continuity of $\phi_2(z)$ across the branch cut taken from the branch point $z = 0$. As in § 6 *a* (i), the two constants B_0 and B_1 are then determined by this continuity condition and the compatibility condition (4.14).

(*b*) *The general case:* $m = n > 1$

The methods described in § 6 *a* for $m = n = 1$ can be extended to the case $m = n > 1$ for all three of the cases (i), (ii) and (iii) defined by (6.5). For simplicity, we discuss only the case involving a particular combination of the interface parameters.

For the subclass (6.1), we have three independent interface parameters α, α_1 and α_2. In the appendix, we show that α and α_2 can be chosen as functions of α_1 such that

$$\gamma = \text{a given negative integer } < -m, \quad \kappa\gamma = m, \tag{6.13}$$

with α_1 arbitrary in the range (see (A 5) in the appendix)

$$\rho^{*2} < \left(\frac{(m/\gamma)+1}{(m/\gamma)-1}\right)^2. \tag{6.14}$$

For this combination of the interface parameters, the solution is given by

$$\phi_2(z) = C(z)\left(\frac{z}{R}\right)^{\gamma}\frac{[(z/R)^m - \rho^*]}{[(z/R)^m - (1/\rho^*)]}, \quad C(z) = \frac{2}{\alpha\alpha_1}\int_0^z \frac{(t/R)^m S(t)(t/R)^{-(1+\gamma)}}{[(t/R)^m - \rho^*]^2}\,\mathrm{d}t, \tag{6.15}$$

where, since the integrand in (6.15) has the m poles $z = R\rho_k$ $(k = 1, 2, \ldots, m)$, $\phi_2(z)$ given by (6.15) is single-valued in S_2 only when the residues of the integrand function

$$\frac{(z/R)^m S(z)(z/R)^{-(1+\gamma)}}{\displaystyle\prod_{k=1}^{m}[(z/R)-\rho_k]^2}, \tag{6.16}$$

at each of the m poles $z = R\rho_k$ $(k = 1, 2, \ldots, m)$, are zero.

For example, if $m = 2$ and $\gamma = -4$, we find, from (4.10), that

$$\begin{aligned}\left(\frac{z}{R}\right)^5 S(z) = {} & (A-\omega)\tfrac{1}{2}R\alpha_2\left(\frac{z}{R}\right)^8 + (\delta-1)B_0\tfrac{1}{2}\alpha_2\left(\frac{z}{R}\right)^7 + [(A-\omega)R \\ & + (\overline{A}-\overline{\omega})R\tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\overline{B}_1(\alpha\alpha_1 + \delta R\alpha_2)]\left(\frac{z}{R}\right)^6 + [(\delta-1)B_0 \\ & - R\overline{B}_2(\alpha\alpha_1 + \tfrac{1}{2}\delta R\alpha_2)]\left(\frac{z}{R}\right)^5 + \tfrac{1}{2}B_1(\alpha\alpha_1 + \delta R\alpha_2)\left(\frac{z}{R}\right)^4 + \tfrac{1}{2}\delta B_0\alpha_2\left(\frac{z}{R}\right)^3,\end{aligned}$$

and the admissible range of α_1 is limited by (6.14) which gives

$$\rho^{*2} < \tfrac{1}{9}, \quad \tfrac{9}{15} < \alpha_1 < \tfrac{9}{15}.$$

Setting

$$P\left(\frac{z}{R}\right) = \sum_{k=3}^{8} p_k\left(\frac{z}{R}\right)^k \equiv \left(\frac{z}{R}\right)^5 S(z), \tag{6.17}$$

the condition (6.16) requires that the residues of

$$\frac{P(t)}{(t-\rho_1)^2(t-\rho_2)^2}, \tag{6.18}$$

at the two poles ρ_1 and ρ_2, must be zero. This means that

$$P'(\rho_k)\rho_k = P(\rho_k), \quad k = 1, 2. \tag{6.19}$$

Since $\rho_1 = -\rho_2$, the condition (6.19) is equivalent to

$$\sum_{k=3,5,7} k p_k \rho_1^k = \sum_{k=3,5,7} p_k \rho_1^k, \tag{6.20}$$

$$\sum_{k=4,6,8} k p_k \rho_1^k = \sum_{k=4,6,8} p_k \rho_1^k. \tag{6.21}$$

The constants B_0, B_1 and B_2 are now determined by the two conditions (6.20), (6.21) and the compatibility condition (4.14) which, in this case reduces to

$$B_0 + (B_2 + \overline{B}_2)\tfrac{1}{2}R[\delta R\alpha_2 + 2\alpha\alpha_1] = 0. \tag{6.22}$$

It is not difficult to see that $B_0 = B_2 = 0$ and that B_1 is given by

$$\left.\begin{aligned} B_1 &= \frac{2(A-\omega)[47\rho^{*2} - 15] + 10(\overline{A} - \overline{\omega})(\rho^* - \rho^{*3})}{5\delta(25\rho^{*2} - 9)}, \\ m &= n = 2, \quad \gamma = -4, \quad \kappa = -\tfrac{1}{2}, \quad \rho^{*2} < \tfrac{1}{9}. \end{aligned}\right\} \tag{6.23}$$

Consequently, the average stresses within the circular inclusion are given by

$$\overline{\sigma}_{13} = h_2(\rho^*)[\overline{\sigma}_{13}]_0, \tag{6.24}$$

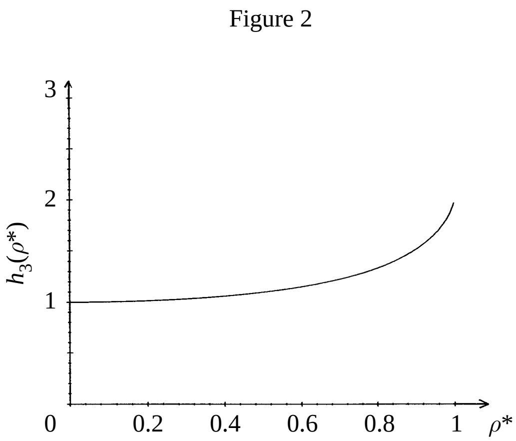
$$h_2(\rho^*) \equiv \left(\frac{3 + 7\rho^{*2}}{1 + 3\rho^{*2}}\right) \frac{[47\rho^{*2} - 15 + 5\rho^* - 5\rho^{*3}]}{5(25\rho^{*2} - 9)}, \quad \rho^{*2} < \tfrac{1}{9},$$

$$\overline{\sigma}_{23} = h_3(\rho^*)[\overline{\sigma}_{13}]_0, \quad h_3(\rho^*) = h_2(-\rho^*), \quad \rho^{*2} < \tfrac{1}{9}, \tag{6.25}$$

where, as before, $\overline{\sigma}$ indicates the average stress within the inclusion, the subscript '0' denotes the corresponding stress in the case of a homogeneous interface ($\rho^* = 0$, or equivalently, $\alpha_1 = \alpha_2 = 0$), and $h_2(\rho^*)$ and $h_3(\rho^*)$ are shown in figure 3. The formulae (6.24) and (6.25) relate the average stresses within the circular inclusion when the interface is inhomogeneous to the corresponding average stresses in the case of a homogeneous interface under otherwise identical conditions. Hence, (6.24) and (6.25) describe the net effect of circumferential inhomogeneity in the interface parameter on the average stresses within the inclusion. This is illustrated in figure 3 where, even in the restricted range $-0.3 < \rho^* < 0.3$, the effect of the inhomogeneity in the interface parameter is seen to have a significant effect on the average stresses within the inclusion.

7. Conclusions

Interface imperfections in a composite material are almost always inhomogeneous along the length of the material interface. Despite this fact, in an effort to simplify the analysis, the majority of theoretical investigations dealing with inclusion problems and composites with imperfect bonding at an interface, have assumed that any imperfections are uniform along the entire length of the material interface. In this paper, we have developed and solved (rigorously) a model of a circular inclusion in antiplane shear in which the imperfect interface is allowed to be circumferentially inhomogeneous. In doing so, we have described the effect of circumferential inhomogeneity on the stress field induced inside a circular inclusion in antiplane deformation.



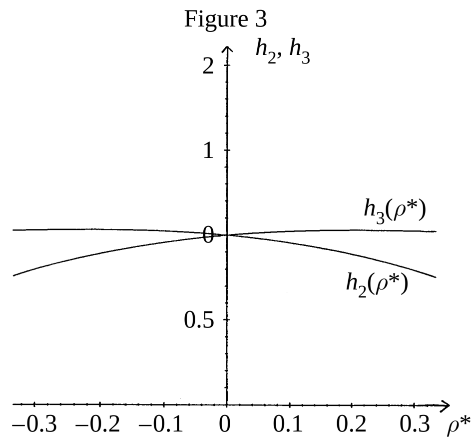


Figure 2. The function $h_1(\rho^*)$.

Figure 3. The functions $h_2(\rho^*)$ and $h_3(\rho^*) = h_2(-\rho^*)$

Our method is necessitated by the fact that the circumferential variation of the parameter describing the imperfect interface leads to the failure of the conventional power series method in obtaining a closed-form solution. Instead, analytic continuation is used to obtain the rigorous solution in which any unknown constants are determined from analyticity requirements and certain other supplementary conditions.

To illustrate the method, in §§ 5 and 6, two classes of circumferentially inhomogeneous interface are considered and explicit results are presented for several examples of the circumferential variation of the interface parameter. These results demonstrate conclusively how the circumferential variation of the interface parameter has a significant effect on the stresses and the average stresses induced within the inclusion. In fact, it is clear from the results in this paper, that replacing the circumferentially inhomogeneous interface parameter by its average value over the entire interface will lead to a significant error in even the average stresses within the circular inclusion.

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Appendix

For the circumferentially inhomogeneous interface described by (6.1), there are three independent interface parameters α, α_1 and α_2. Here we derive a necessary and sufficient condition under which it is possible to choose α and α_2 as functions of α_1 such that the left-hand sides of the following two inequalities (see (6.4)):

$$\gamma < 0, \quad \kappa\gamma > 0 \tag{A 1}$$

take the prescribed negative and positive constants, respectively, when α_1 varies within a certain admissible range.

To this end, from (6.4), α and α_2 can be expressed in terms of α_1 as follows

$$\frac{2}{\alpha_2} = (\kappa - 1)\rho^* - \frac{(\kappa + 1)}{\rho^*}, \quad \frac{\alpha}{R} = \tfrac{1}{4}\delta\left(\frac{1 + \rho^{*2}}{(1 - \kappa)\rho^{*2} + 1 + \kappa}\right). \tag{A 2}$$

Recall that α and α_2 are required to satisfy the restrictions

$$\alpha > 0, \quad |\alpha_2| < 1, \tag{A 3}$$

from which it follows that

$$\frac{\rho^*}{\alpha_2} < 0, \quad \left[(\kappa - 1)\rho^* - \frac{(1+\kappa)}{\rho^*}\right]^2 > 4,$$

or, equivalently

$$\frac{1+\kappa}{(\kappa-1)} < \rho^{*2} < \left(\frac{1+\kappa}{\kappa-1}\right)^2.$$

Hence, the necessary and sufficient condition for the existence of such a non-zero admissible range of α_1 is

$$0 > \kappa > -1. \tag{A 4}$$

In this case (namely, the prescribed constant κ satisfies (A 4)), the admissible range of α_1 is limited by

$$\rho^{*2} < \left(\frac{\kappa+1}{\kappa-1}\right)^2, \tag{A 5}$$

where ρ^* is determined by α_1 from (4.25). It is seen from (4.25) and (A 2), that $\rho^* = 0$ (or equivalently, $\alpha_1 = 0$) leads to $\alpha_2 = 0$. Hence, the imperfect interface described by such a combination of interface parameters corresponds to a homogeneous imperfect interface if and only if $\rho^* = 0$.

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