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High-frequency surface wave excitation at a curved impedance boundary

By B. J. Smith and R. H. Tew
Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, UK

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We consider the situation when an arbitrary open, convex boundary, subject to an impedance condition, is insonified by a high-frequency time-harmonic plane wave. In particular, the inner diffraction structure close to the point where a propagating surface mode is excited is examined in detail, using the methods of multiple scales and asymptotic matching. This provides a complete description of the excitation process and of the outer ray form of the radiative surface mode for an abitrary, convex body. The calculation shows clearly the ideas behind the methodology and demonstrates how the 'launching' (or diffraction) coefficient for the amplitude of the surface mode can be calculated. It also helps to bring to the fore aspects of $\mathcal{O}(1)$ amplitude variations away from the point of excitation which can be interpreted as an $\mathcal{O}(1/k)$ correction to the phase of the radiated field.

Keywords: free-mode propagation; surface rays; diffraction by convex bodies

1. Introduction

One of the main results in a paper by Tew & Ockendon (1992) was a method for constructing the wavefield local to the point of excitation of a free-mode—by which we mean an outgoing eigensolution of the governing field equation (the Helmholtz equation, in fact) subject to a specified boundary condition—caused by a prescribed critically incident field in the high-frequency limit. This was done for the case of curved two-dimensional wavefronts impinging upon a flat impedance boundary. Our principal aim here is to modify this analysis to account for convex boundary curvature and to provide a full description of the excitation and propagation of the free-mode that is excited in this case.

The general feature of the analysis for the flat case was that in the neighbourhood of the critical point the radiated field was a plane wave modulated by a slowly varying amplitude, which was expressed in terms of a Fresnel integral. This slow variation had the expected specularly reflected field as part of its outer asymptotic limit, along with another contribution which yielded the radiation associated with the free-mode. This solution could then be used to supply the diffraction coefficients missing from the 'outer' ray theories of Keller & Karal (1960) and Grimshaw (1968) appropriate to this case.

In fact the method supplied more information than this, in that when it was applied to problems involving *complex* surface wave excitation, it also gave the 'region of existence' of the surface ray in the complex extension of the boundary. This in turn allowed the portion of the physical domain into which the free-mode penetrates to be predicted, and this is vital knowledge in order to get a complete description of the associated radiation patterns. Examples of this in the context of fluid-loading problems are given by Tew (1994, 1995) and Smith & Tew (1995).

In these previous analyses, we note that the slow variation in the amplitude of the local wavefield was a direct consequence of the small departure from planarity of the incoming wavefronts on the inner diffraction scale considered.

While it was never stated, it was assumed that the same analysis and solutions would arise if the gradual curvature was not in the incident wavefronts but in the boundary instead. This turns out not to be the case, though we find that similar general ideas can be carried over.

To be more specific, the curvature of the boundary means that we must solve the Helmholtz equation in curvilinear coordinates—arclength along, and normal distance from, the boundary is a natural choice—and whilst we are still free to introduce slow scales derived from the boundary curvature and then apply multiple scales methods, the field equations that we now get have *non-constant* coefficients. Also, the associated secularity conditions are more complicated than those that occur in the flat boundary case.

These non-trivial differences are highlighted here in the context of the problem of plane wave incidence on a curved, wavebearing, impedance boundary. The aim is to exemplify the modifications required using this paradigm, which has the advantage that it allows the important features to be identified clearly and permits immediate extension to harder problems involving fluid-loaded structures such as membranes, plates and elastic solids. Such extensions have been carried out and the details amount to a reworking of the analysis presented here but with the underlying structures somewhat clouded by a much greater degree of algebraic complexity. This is the reason for our presenting the current simpler problem, which we deliberately describe in concise and condensed form. We refer to the work of Smith (1995) and Rogoff (1996) for the details of these other situations.

The analysis is presented in terms of acoustic wave propagation through a compressible fluid, and is developed in terms of a velocity potential function. However, the results are more widely applicable than this, since they could just as easily apply to components of electric or magnetic fields (see Senior & Volakis (1995) for details).

2. General considerations

We begin the analysis by noting some features of the problem which would carry over directly to more general problems. First, we assume that the geometry is such that the fluid occupies the region D exterior to an open, convex boundary ∂D described parametrically in terms of arclength s in the form

$$x = x_0(s), \quad y = y_0(s). \tag{2.1}$$

We choose arclength to increase with x (i.e. in a clockwise direction) and define the angle of curvature $\psi(s)$ to be the angle between the positive x-axis and the tangent in the direction of increasing s at a point on ∂D.

This definition implies that the local tangent $\boldsymbol{t}(s)$ and normal $\boldsymbol{n}(s)$ are given directly by

$$\boldsymbol{t}(s) = (\cos\psi(s), \sin\psi(s)) \tag{2.2}$$

and

$$\boldsymbol{n}(s) = (-\sin\psi(s), \cos\psi(s)), \tag{2.3}$$

respectively. The radius of curvature $R(s)$ is given by

$$R(s) = -1/\psi'(s), \tag{2.4}$$

the minus sign occurring because of the choice of sense of increasing s.

The incoming plane wave is taken to be

$$\phi^{\text{inc}}(x,t) = \mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}, \quad k = \omega/c, \tag{2.5}$$

where c is the acoustic wavespeed. This term is subtracted from the total acoustic potential $\Phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}$ in D, allowing the problem to be posed in terms of the diffracted potential $\phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}$ given by

$$\Phi(x,y) = \mathrm{e}^{\mathrm{i}kx} + \phi(x,y), \qquad (x,y) \in D. \tag{2.6}$$

The time-harmonic factor $\mathrm{e}^{-\mathrm{i}\omega t}$ is henceforth implied but omitted.

For completeness we quote that the field equation to be solved for ϕ is the Helmholtz equation which, in terms of arclength s and normal distance n measured from ∂D into the fluid, is

$$\left\{\frac{1}{(1-n\psi')^2}\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} + \frac{n\psi''}{(1-n\psi')^3}\frac{\partial}{\partial s} - \frac{\psi'}{(1-n\psi')}\frac{\partial}{\partial n} + k^2\right\}\phi = 0. \tag{2.7}$$

Also, if we apply a ray ansatz to (2.7) in the form

$$\phi(x,y) \sim k^{-\sigma}\sum_{n=0}^{\infty}\frac{A_n(x,y)}{(ik)^n}\mathrm{e}^{iku(x,y)}, \tag{2.8}$$

as $k \to \infty$, then we obtain the usual eikonal equation

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u = 1 \tag{2.9}$$

and the recursive system of transport equations

$$A_n\nabla^2 u + 2\boldsymbol{\nabla} A_n \cdot \boldsymbol{\nabla} u + \nabla^2 A_{n-1} = 0, \quad n = 0,1,2,3,\ldots, \tag{2.10}$$

with $A_{-1}(x,y) \equiv 0$. Equation (2.9) can then be solved by characteristic methods to give u parametrically in terms of ray coordinates (ρ,τ)—which we shall define presently—in the form

$$u(\rho,\tau) = u_0(\rho) + \tau \tag{2.11}$$

along the rays, which are the characteristics of (2.9), defined by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(x(\rho,\tau), y(\rho,\tau)) = (p_0(\rho)q_0(\rho)) \qquad (x(\rho,0), y(\rho,0)) = (x_0(\rho), y_0(\rho)), \tag{2.12}$$

where

$$\boldsymbol{\nabla} u = (p_0(\rho), q_0(\rho)) \tag{2.13}$$

and $u = u_0(\rho)$ is the initial data for u prescribed on ∂D.

Given these results, the leading-order transport equation ((2.10) with $n = 0$) can be solved for $A_0(\rho,\tau)$ to give

$$A_0(\rho,\tau) = A_0(\rho,0)\left\{\frac{q_0(\rho)x_0'(\rho) - p_0(\rho)y_0'(\rho)}{\tau[q_0(\rho)p_0'(\rho) - p_0(\rho)q_0'(\rho)] + q_0(\rho)x_0'(\rho) - p_0(\rho)y_0'(\rho)}\right\}^{1/2}. \tag{2.14}$$

Further details of the ray equations and the solutions for the eikonal function u and leading-order amplitude A_0 are given in Bleistein (1984) and Keller & Lewis (1995).

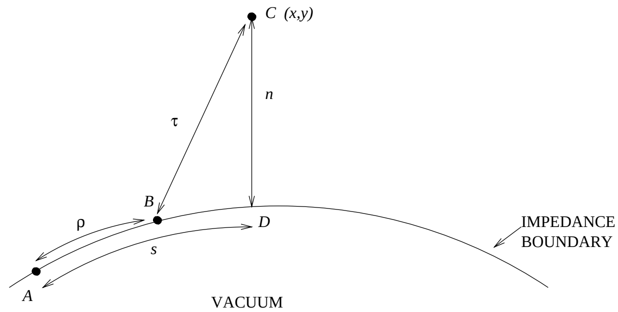

Figure 1. The geometry and coordinate systems. A is the (arbitrary) zero for arclength on the boundary. The reflected ray passing through a general point C with Cartesian coordinates (x, y) emanates from the boundary point B an arclength ρ from A. The ray distance from B to C is τ. The shortest distance from C to the boundary is n and the corresponding boundary point is D, an arclength s from A.

One final note concerning the differences in the definitions of the two arclength parameters ρ and s is perhaps in order. For an observation point C in D, the value of ρ is the arclength measured around to the point on ∂D at which the ray passing through C is emitted from ∂D. The parameter τ $(= 0$ on $\partial D)$ is then the distance along the ray from ∂D to P. Hence, the (ρ, τ) and (s, n) coordinate systems are different. This is shown in figure 1.

3. Impedance boundary value problem

The boundary condition chosen for the total potential Φ is

$$\frac{\partial \Phi}{\partial n} - \mathrm{i}k \sin\alpha\, \Phi = 0, \quad n = 0, \tag{3.1}$$

where $\alpha \in (0, \frac{1}{2}\pi)$. We take care to note that this boundary condition is a careful choice and ensures that the planar analogue of such a boundary can support a purely propagating supersonic free-mode. From (2.6), the scattered potential satisfies the Helmholtz equation (2.7) subject to

$$\frac{\partial \phi}{\partial n} - \mathrm{i}k \sin\alpha\phi = \mathrm{i}k(\sin\psi(s) + \sin\alpha)\mathrm{e}^{\mathrm{i}kx_0(s)}, \quad n = 0, \tag{3.2}$$

together with the stipulation that ϕ is outgoing and/or diminishing in amplitude with increasing distance from ∂D into D. The scattered field ϕ comprises a specularly reflected and a diffracted component. As will be demonstrated, the latter can be further decomposed into a propagating free-mode and a creeping field excited as a result of tangential ray incidence (see Keller & Lewis (1995) for details). We can now demonstrate the existence of both of these fields in naive terms from certain singularities in the specular field, the construction of which now follows.

(a) The specular field

If we decompose the incident field into a family of rays propagating parallel to the positive x-axis, then the reflected ray induced at the boundary point $(x_0(\rho), y_0(\rho))$ has an initial phase

$$u_0(\rho) = x_0(\rho). \tag{3.3}$$

The ray data

$$(p_0(\rho), q_0(\rho)) = (\cos 2\psi(\rho), \sin 2\psi(\rho)) \tag{3.4}$$

then follow by differentiation of (3.3) with respect to ρ on ∂D. More specifically, since $u_0'(\rho) = p_0(\rho)x_0'(\rho) + q_0(\rho)y_0'(\rho)$ follows from a straightforward application of the chain rule, we can use (3.3) and the fact that $(x_0'(\rho), y_0'(\rho)) = (\cos\psi(\rho), \sin\psi(\rho))$ from (2.2) to obtain

$$\cos\psi(\rho) = \cos(\psi(\rho) - \Theta(\rho)), \tag{3.5}$$

where $p_0(\rho) = \cos\Theta(\rho)$, $q_0(\rho) = \sin\Theta(\rho)$. Hence, either $\Theta(\rho) \equiv 0$ or $\Theta(\rho) = 2\psi(\rho)$ and the latter gives us (3.4). We discount the other possibility $\boldsymbol{p}_0(\rho) = (1, 0)$ since it does not yield an outgoing solution (in fact, it reproduces the incident field).

This allows us to integrate the ray equations (2.12) in the usual fashion to give the parametric form of the reflected rays as

$$x(\rho,\tau) = x_0(\rho) + \tau\cos 2\psi(\rho) \tag{3.5 a}$$

$$y(\rho,\tau) = y_0(\rho) + \tau\sin 2\psi(\rho) \tag{3.5 b}$$

along which

$$u(\rho,\tau) = x_0(\rho) + \tau. \tag{3.6}$$

These equations imply that the reflected rays leave the boundary at an angle $2\psi(\rho)$ to the x-axis and hence the incoming and reflected rays are bisected by the local normal to the boundary, as is well known to be the case.

We complete the calculation of the specular field by determining the leading-order amplitude $A_0(\rho,\tau)$ from (2.14). Of the two factors in (2.14), the second is straightforward to calculate using the expressions for $p_0(\rho)$ and $q_0(\rho)$ in (3.4) and noting that $(x_0'(\rho), y_0'(\rho)) = \boldsymbol{t}(\rho)$, where $\boldsymbol{t}$ is defined in (2.2). The first factor, $A_0(\rho, 0)$, can be calculated by noting that the leading-order term obtained by substituting the ray ansatz (2.8) into (3.2) is

$$A_0(\rho,0)\left(\frac{\partial u}{\partial n} - \sin\alpha\right) = \sin\psi(\rho) + \sin\alpha. \tag{3.7}$$

Since

$$\frac{\partial u}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{\nabla} u$$

and both $\boldsymbol{n}$ (given by (2.3)) and $\boldsymbol{\nabla} u$ (given by (3.4)) are known, we can deduce from this the value of $A_0(\rho, 0)$. Piecing together these results gives us an expression for $A_0(\rho,\tau)$ in the form

$$A_0(\rho,\tau) = \left(\frac{\sin\psi(\rho) + \sin\alpha}{\sin\psi(\rho) - \sin\alpha}\right)\left(\frac{\sin\psi(\rho)}{\sin\psi(\rho) - 2\tau\psi'(\rho)}\right)^{1/2}, \tag{3.8}$$

showing that as the reflected rays spread with increasing distance τ from ∂D, they incur an algebraic decay of order $\tau^{-1/2}$. Of course, if the boundary is flat then $\psi'(\rho) \equiv 0$ and

$$A_0(\rho,\tau) = \frac{\sin\psi(\rho) + \sin\alpha}{\sin\psi(\rho) - \sin\alpha}. \tag{3.9}$$

This is just the plane wave reflection coefficient for this particular data and this is consistent with the fact that for the curved boundary, the local reflection problem away from points of diffraction is that of plane wave incidence at a flat boundary. There are two such points of diffraction on ∂D where (3.8) is invalid. One is where $\psi(\rho) = \alpha$ and the other is such that $\psi(\rho) = 0$. The second of these corresponds to the case of tangential ray incidence and creeping ray initiation and we discuss this case no further—it is the other critical boundary point that is of interest to us here and is now the subject of an inner diffraction analysis.

(b) *Local diffraction structure near the critical boundary point*

Near this critical point the ray ansatz (2.8) breaks down and the full Helmholtz balance (2.7) must be solved. Returning to (s, n) coordinates, we are able to show that if $s = s_0$ is the value of the arclength at the critical point, so that $\psi(s_0) = \alpha$, then the appropriate inner scalings are

$$s = s_0 + k^{-1}\hat{s}, \quad n = k^{-1}\hat{n} \tag{3.10}$$

where $\hat{s}$ and $\hat{n}$ are both $\mathcal{O}(1)$. Under these circumstances, (2.7) and (3.2) are replaced by the approximations

$$\left\{\left(1 - \frac{4\hat{n}\delta}{\sin\alpha}\right)\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + \frac{2\delta}{\sin\alpha}\frac{\partial}{\partial\hat{n}} + 1\right\}\hat{\phi} = 0, \quad \hat{n} > 0, \tag{3.11}$$

$$\frac{\partial\hat{\phi}}{\partial\hat{n}} - \mathrm{i}\sin\alpha\hat{\phi} = 2\mathrm{i}\left(\sin\alpha - \frac{\delta\hat{s}\cos\alpha}{\sin\alpha}\right)\mathrm{e}^{\mathrm{i}kx_0(s_0) + \mathrm{i}\hat{s}\cos\alpha + \mathrm{i}\delta\hat{s}^2}, \quad n = 0, \tag{3.12}$$

where $\hat{\phi}$ is the inner potential, $R_0 = (-\psi'(s_0))^{-1}$ is the radius of curvature of the boundary at the critical point and where we have identified the small parameter

$$\delta = \frac{\sin\alpha}{2kR_0} \ll 1. \tag{3.13}$$

The derivation of (3.11) and (3.12) relies upon local Taylor expansions for $x_0(s)$ and $\psi(s)$, which in turn requires a smooth boundary with a slowly varying curvature. We now introduce slow scales

$$(S, N) = \delta^{1/2}(\hat{s}, \hat{n}) \tag{3.14}$$

and seek an expansion

$$\hat{\phi} \sim \delta^{-1/2}\hat{\phi}_0 + \hat{\phi}_1 + \delta^{1/2}\hat{\phi}_2 + \cdots. \tag{3.15}$$

Note that whilst each function $\hat{\phi}_j$ in this expansion is a function of the two spatial coordinates defining a general point in D, we choose to make explicit the dependence upon the slow scales by writing $\hat{\phi}_j = \hat{\phi}_j(\hat{s}, \hat{n}; S, N)$. Using (3.14), we can easily eliminate S and N to recover the dependence upon the two standard coordinates $\hat{s}$ and $\hat{n}$. At leading order, we obtain the homogeneous problem

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + 1\right)\hat{\phi}_0 = 0, \quad \hat{n} > 0, \tag{3.16}$$

$$\frac{\partial\hat{\phi}_0}{\partial\hat{n}} - \mathrm{i}\sin\alpha\hat{\phi}_0 = 0, \quad \hat{n} = 0, \tag{3.17}$$

which has an outgoing solution

$$\hat{\phi}_0(\hat{s}, \hat{n}; S, N) = \hat{A}_0(S, N)\mathrm{e}^{\mathrm{i}(\hat{s}\cos\alpha + \hat{n}\sin\alpha)}. \tag{3.18}$$

This leading-order boundary value problem for $\hat{\phi}_0$ is effectively that of a flat boundary, though we shall shortly see that boundary curvature will play a role in the determination of $\hat{A}_0(S, N)$.

In fact, we can determine some information about the amplitude $\hat{A}_0(S,\ N)$ by avoidance of secularity on the second-order boundary value problem

$$\left(\frac{\partial^2}{\partial\hat{s}^2}+\frac{\partial^2}{\partial\hat{n}^2}+1\right)\hat{\phi}_1=\frac{4N}{\sin\alpha}\frac{\partial^2\hat{\phi}_0}{\partial\hat{s}^2}-2\frac{\partial^2\hat{\phi}_0}{\partial\hat{s}\partial S}-2\frac{\partial^2\hat{\phi}_0}{\partial\hat{n}\partial N},\quad \hat{n},\quad N>0, \tag{3.19}$$

$$\frac{\partial\hat{\phi}_1}{\partial\hat{n}}-\mathrm{i}\sin\alpha\hat{\phi}_1=2\mathrm{i}\sin\alpha\mathrm{e}^{\mathrm{i}kx(s_0)+\mathrm{i}\hat{s}\cos\alpha+\mathrm{i}S^2}-\frac{\partial\hat{\phi}_0}{\partial N},\quad \hat{n}=N=0. \tag{3.20}$$

Notice that the non-constant coefficients first appear at this order and that it is appropriate for us to write the $\hat{n}$-dependent term in (3.11) in terms of N.

Once we insert (3.18) into (3.19) and (3.20), we suppress secularity provided $\hat{A}_0(S,N)$ satisfies

$$\cos\alpha\frac{\partial\hat{A}_0}{\partial S}+\sin\alpha\frac{\partial\hat{A}_0}{\partial N}=\frac{2\mathrm{i}\cos^2\alpha}{\sin\alpha}N\hat{A}_0,\quad N>0, \tag{3.21}$$

$$\frac{\partial\hat{A}_0}{\partial N}=2\mathrm{i}\sin\alpha\mathrm{e}^{\mathrm{i}kx(s_0)+\mathrm{i}S^2},\quad N=0. \tag{3.22}$$

Equation (3.21) has the general solution

$$\hat{A}_0(S,N)=F(S-N\cot\alpha)\mathrm{e}^{\mathrm{i}N^2\cot^2\alpha}, \tag{3.23}$$

whilst the boundary condition (3.22) yields that F satisfies the ordinary differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}S}=-\frac{2\mathrm{i}\sin^2\alpha}{\cos\alpha}\mathrm{e}^{\mathrm{i}kx(s_0)+\mathrm{i}S^2}. \tag{3.24}$$

Invoking the radiation condition, we find that the only acceptable solution for F is

$$F(S)=-\frac{2\mathrm{i}\sin^2\alpha}{\cos\alpha}\mathrm{e}^{\mathrm{i}kx(s_0)}\int_{-\infty}^{S}\mathrm{e}^{\mathrm{i}v^2}\,\mathrm{d}v \tag{3.25}$$

from which $\hat{A}_0(S,N)$ follows, using (3.23).

Combining all these results yields the leading-order wave structure local to the critical point in the form

$$\hat{\phi}_0\sim-\sqrt{8kR_0}\frac{\mathrm{i}\sin^{3/2}\alpha}{\cos\alpha}\mathrm{e}^{\mathrm{i}kx(s_0)+\mathrm{i}\hat{s}\cos\alpha+\mathrm{i}\hat{n}\sin\alpha+\mathrm{i}N^2\cot^2\alpha}\int_{-\infty}^{(S-N\cot\alpha)}\mathrm{e}^{\mathrm{i}v^2}\,\mathrm{d}v. \tag{3.26}$$

In order to understand the implications that this has for the outer field structure, we must consider (3.26) for large S and N and try to match this outer limit of the inner solution with the inner limit of the outer solution. Doing so yields a contribution

$$\phi^{\mathrm{spec}}\sim-\sqrt{2kR_0}\frac{\sin^{3/2}\alpha\mathrm{e}^{\mathrm{i}kx(s_0)+\mathrm{i}\hat{s}\cos\alpha+\mathrm{i}\hat{n}\sin\alpha+\mathrm{i}N^2\cot^2\alpha+\mathrm{i}(S-N\cot\alpha)^2}}{\cos\alpha(S-N\cot\alpha)} \tag{3.27}$$

valid on either side of, but well removed from, the line $S-N\cot\alpha=0$.

A detailed calculation shows that (3.27) precisely matches the specularly reflected field, with phase and amplitude given by (3.6) and (3.8), respectively. Though we omit the details of this calculation, a crucial feature is that the 'inner' version of the ray coordinates, $s_0+k^{-1}\hat{\rho}$ and $k^{-1}\hat{\tau}$, are related to $\hat{s}$ and $\hat{n}$ through the equations

$$\hat{\rho}\sim\hat{s}-\hat{n}\cot\alpha-\frac{1}{kR_0}\left(\frac{\hat{n}\hat{s}}{\sin^2\alpha}-\hat{n}^2\cot\alpha(2+\tfrac{3}{2}\cot^2\alpha)\right) \tag{3.28}$$

$$\hat{\tau}\sim\frac{\hat{n}}{\sin\alpha}-\frac{1}{kR_0}\left(\frac{-\hat{n}\hat{s}\cot\alpha}{\sin\alpha}+\frac{3\hat{n}^2\cot^2\alpha}{2\sin\alpha}\right). \tag{3.29}$$

There is, of course, another contribution to the far-field of (3.26), this being

$$\phi^{(SW)} \sim -\sqrt{8\pi k R_0}\frac{\mathrm{i}\sin^{3/2}\alpha}{\cos\alpha}\mathrm{e}^{\mathrm{i}(\pi/4)+\mathrm{i}kx(s_0)+\mathrm{i}\hat{s}\cos\alpha+\mathrm{i}\hat{n}\sin\alpha+\mathrm{i}N^2\cot^2\alpha} \tag{3.30}$$

and is valid in $S - N\cot\alpha \gg 1$. On the boundary, where $\hat{n} = N = 0$, this has plane wave structure proportional to $\mathrm{e}^{\mathrm{i}\hat{s}\cos\alpha}$ on this inner scale. This form is maintained off the boundary through the $\mathrm{e}^{\mathrm{i}\hat{n}\sin\alpha}$ term, though there is also a slow-scale modulation because of the variation of the exponent with N. To understand this we must consider the ray limit of the free-mode of the system, defined to be a non-trivial solution of the homogeneous problem comprising the Helmholtz equation (2.7) and boundary condition (3.1).

(c) *Free-mode excitation and propagation*

If we apply the ray ansatz (2.8) to (3.1) directly, we get the ray data

$$\frac{\partial u}{\partial n} = \sin\alpha, \quad n = 0, \tag{3.31}$$

at leading order. Since the eikonal equation (2.9) reduces to

$$\left(\frac{\partial u}{\partial n}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2 = 1 \tag{3.32}$$

on $n = 0$, (3.31) and the radiation condition imply that

$$\frac{\partial u}{\partial s} = \cos\alpha, \quad n = 0. \tag{3.33}$$

Since

$$\frac{\partial u}{\partial s} = \boldsymbol{t}\cdot\boldsymbol{\nabla}u, \quad n = 0, \tag{3.34}$$

and both $\boldsymbol{\nabla}u$ and $\boldsymbol{t}$ are unit vectors, (3.33) and (3.34) show that the angle between the direction of these vectors takes the constant value α. Since $\boldsymbol{\nabla}u$ is in the direction of the rays shed by the surface ray data in (3.31) and (3.33), we see that these shed rays all leave the boundary at an angle α to the local tangent. Therefore, the phase of the free-mode as a function of ray coordinates (ρ, τ) is

$$u(\rho,\tau) = \rho\cos\alpha + \tau, \tag{3.35}$$

and we can substitute this into (2.14) to get the amplitude variation in the form

$$A_0(\rho,\tau) = f(\rho)\left[\frac{\sin\alpha}{\sin\alpha - \psi'(\rho)\tau}\right]^{1/2}. \tag{3.36}$$

Further information about the function $f(\rho)$ can be obtained from the higher-order terms in the ray expansion of the boundary condition (3.1), giving

$$\frac{\partial A_0}{\partial n} = 0, \quad n = 0. \tag{3.37}$$

This in turn becomes

$$\frac{\partial A_0}{\partial \tau} - \cos\alpha\frac{\partial A_0}{\partial \rho} = 0, \quad \tau = 0, \tag{3.38}$$

in ray coordinates. Therefore we see that $f(\rho)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}\rho} - \frac{f\psi'}{\sin 2\alpha} = 0, \tag{3.39}$$

with general solution

$$f(\rho) = \gamma \exp\left(\frac{\psi(\rho)}{\sin 2\alpha}\right) \qquad (3.40)$$

for some constant γ. Piecing together all of these facts gives the leading-order term from the ray expansion for the free-mode as

$$\phi^{(SW)} \sim \gamma \exp\left(\frac{\psi(\rho)}{\sin 2\alpha}\right) \left[\frac{\sin\alpha}{\sin\alpha - \psi'(\rho)\tau}\right]^{1/2} \mathrm{e}^{\mathrm{i}k(\rho\cos\alpha + \tau)}. \qquad (3.41)$$

No further information can be obtained from this ray analysis and the value of the constant γ must be obtained by matching this outer solution to the inner diffraction solution. This leads to the evaluation

$$\gamma = \sqrt{8\pi k R_0}\frac{\sin^{3/2}\alpha}{\cos\alpha} \exp\left(\mathrm{i}kx(s_0) - \mathrm{i}ks_0\cos\alpha - \frac{\alpha}{\sin 2\alpha} - \mathrm{i}\frac{1}{4}\pi\right) \qquad (3.42)$$

leading to the full solution for the surface wave

$$\phi^{(\mathrm{SW})} \sim \sqrt{\frac{8\pi k R_0}{\sin\alpha - \tau\psi'(\rho)}}\frac{\sin^2\alpha}{\cos\alpha} \exp\left(-\left(\frac{\alpha - \psi(\rho)}{\sin 2\alpha}\right) - \mathrm{i}\frac{1}{4}\pi\right) \times \exp(\mathrm{i}k(x(s_0) - s_0\cos\alpha + \rho\cos\alpha + \tau)). \qquad (3.43)$$

4. Discussion

The solution (3.43) yields a completely specified closed-form expression for the radiation from the free-mode associated with this data for an *arbitrary* convex boundary. As stated previously, a similar analysis on more complicated boundary conditions provides an equivalent description of 'leaky' supersonic surface modes such as those that can occur on lightly fluid-loaded membranes and elastic solids (Smith 1995; Rogoff 1996). In all of these cases, the convexity condition is required in order to avoid a caustic being formed by the rays radiated into the outlying medium from the free-mode propagating around the boundary. Accounting for a change of sign in the boundary curvature would be an interesting extension of this work.

An interesting feature of the result in (3.43) is the exponential factor $\mathrm{e}^{-(\alpha-\psi(\rho))}$. This can be viewed as representing an $\mathcal{O}(1/k)$ correction term to the rest of the phase, though of course it has an $\mathcal{O}(1)$ effect on the solution. It might have appeared tempting to have calculated the traditional k-dependent phase from ray methods and the non-exponential amplitude prefactor (to within an unknown constant) using (2.14). A naive matching into the inner solution (3.30) would then have determined this constant, apparently fixing the solution. Of course, this procedure would have completely missed the exponential factor under discussion and this highlights the importance of correctly accounting for leading-order amplitude variations through differential equations such as (3.38) and (3.39), or their analogues. This simplified model calculation exemplifies this point very clearly.

As well as generalizing the boundary conditions, another possible extension is to consider different types of incoming wavefields. For example, if we were considering an incoming Gaussian beam, then we would be able to decompose the incident radiation in terms of *complex* rays (Deschamps 1971). An analysis similar to that presented here would then have to be performed about a complex point on the boundary, this point being determined by a complex version of Snell's law (see Keller & Karal

(1960) for details of the latter statement). The same Fresnel-type solutions will result and the excitation process for the surface mode will then be governed by the Stokes lines associated with this inner solution. This is referred to in general terms in the conclusion to the paper by Tew & Ockendon (1992) and a more complete theory for complex ray theory in terms of the Stokes phenomenon is presented by Chapman *et al.* (1997).

One final extension of the two-dimensional case is to consider closed convex bodies. In this case, the excitation procedure is as presented here and the only extra thing to account for would be complete circuits of the surface field as it propagates around the scatterer, which can be achieved by straightforward summation.

Of course, this raises the general case of three-dimensional surface wave excitation problems. An interesting hierarchy of problems then arises depending upon which of the curvatures associated with the two-dimensional boundary is largest. If they are comparable, and slowly varying as in the problem considered here, then it is highly likely that a major review of the application of the multiple scales methods used here would be needed. This aspect of the problem is currently under investigation.

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