Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity

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The equations for convective fluid motion in a porous medium of Brinkman or Forchheimer type are analysed when the viscosity varies with either temperature or a salt concentration. Mundane situations such as salinization require models which incorporate strong viscosity variation. Therefore, we establish rigorous a priori bounds with coefficients which depend only on boundary data, initial data and the geometry of the problem and which demonstrate continuous dependence of the solution on changes in the viscosity. A convergence result is established for the Darcy equations when the variable viscosity is allowed to tend to a constant viscosity.

Keywords: Brinkman; Forchheimer; salinization; stability; variable viscosity; convergence

1. Introduction

To theoretically model the process of salinization, whereby salts are transported upwards in soils in dry regions, Gilman & Bear (1996) have recently developed a porous flow model which hinges on a viscosity–concentration dependence. The work of Gilman & Bear (1996) presents a very interesting model which is highly nonlinear. This paper is motivated by that work and we analyse the manner in which the solution (velocity, concentration) depends on changes in the viscosity. Gilman & Bear (1996) point out that the viscosity dependence on concentration is 1.5 to 3 times greater than that of pure water whereas the variation in density is of the order of 15 to 30%. Thus, this indicates that convective motion of salt in a porous medium ought to take into account viscosity dependence on concentration.

Another area where viscosity changes may be significant in porous convection involves thermal convection in a saturated porous layer. Richardson & Straughan (1993) used a Brinkman model with a viscosity which depends linearly on temperature, $T$, and they developed rigorous nonlinear energy stability bounds. Qin et al. (1995) and Qin & Chadam (1996) presented nonlinear stability analyses for a model which does not employ a Brinkman term but instead includes a Forchheimer term.
In this paper the analysis presented, of continuous dependence on changes in the viscosity, also encompasses the situation where the viscosity depends on temperature rather than concentration.

The Brinkman and Forchheimer models for convective flow in porous media have recently been studied extensively in connection with practical stability studies, continuous dependence on modelling and spatial decay (see, for example, Guo & Kaloni 1995; Kaloni & Guo 1996; Nield & Bejan 1992; Payne & Song 1997; Payne & Straughan 1996, 1998; Qin & Chadam 1996; Qin et al. 1995; Qin & Kaloni 1994, 1998; Richardson & Straughan 1993, and references therein). This work continues in this vein, and we investigate the problem of how the solution depends on viscosity variation. Due to the many practical applications of convective flow in porous media where the viscosity variation is important, we believe our analysis is of value.

We now introduce the models to be studied. The Forchheimer and Brinkman models and their range of applicability are discussed at length in Nield & Bejan (1992). Very recent work, both theoretical and experimental, has justified the use of these models (see Firdaouss et al. 1997; Gilver & Altobelli 1994; Giorgi 1997; Kladias & Prasad 1991; Whitaker 1996).

Let \( u_i, \omega \) and \( p \) denote the fields of velocity, concentration (or temperature) and pressure. The Forchheimer equations for flow in a porous medium are

\[
\begin{align*}
bu_i |u| + (1 + \gamma_1 \omega)u_i &= -p_i + g_i \omega, \\
\frac{\partial u_i}{\partial x_i} &= 0, \\
\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} &= \Delta \omega,
\end{align*}
\]

where \( \gamma_1 \) and \( b \) are positive constants, \( g_i(x) \) is a gravity field which without loss of generality we assume satisfies

\[
|g| \leq 1.
\]

\( \Delta \) is the Laplacian operator, and standard indicial notation is employed throughout. Equations (1.1) hold on the region \( D \times (0, T) \) for \( D \), a bounded domain in \( \mathbb{R}^3 \), and for \( T, 0 < T < \infty \), a given number. These equations are effectively non-dimensionalized with suitable non-dimensional variables. The viscosity variation in (1.1) is accounted for by the term \( 1 + \gamma_1 \omega \), i.e. we are considering a viscosity \( \mu \) like \( \mu = \mu_1 (1 + \gamma_1 \omega) \).

The \( g_i \omega \) term arises due to an assumption of Boussinesq type. In view of the fact that we are primarily interested in continuous dependence on the coefficient \( \gamma_1 \) and convergence as \( \gamma_1 \to 0 \), the non-dimensional form (1.1), (1.2) is adequate for our purpose. (If \( \omega \) represents temperature, then \( \gamma_1 \) would normally be negative. The viscosity would have form \( \mu = \mu_1 (1 - \gamma_1 T) \geq \mu^* > 0 \), for a positive constant \( \mu^* \). The analysis in this case would employ the restriction \( \mu_1 (1 - \gamma_1 T) \geq \mu^* > 0 \).) Since \( \omega \) is a concentration we assume throughout that \( \omega(x, t) \geq 0 \), although if we know \textit{a priori} that \( u_i \) is bounded then this would follow from the maximum principle.

Let \( \partial D \) denote the boundary of \( D \), let \((\cdot, \cdot)\), \( \| \cdot \| \) denote the inner product and norm on \( L^2(D) \) and let \( \| \cdot \|_p \) denote the norm on \( L^p(D) \), \( 1 < p < \infty \). On the boundary of \( D \) we assume

\[
u_i n_i = f(x, t), \quad \omega = h(x, t) \geq 0, \quad x \in \partial D,
\]

(1.3)
where \( n_i \) is the unit outward normal to \( \partial D \). The concentration is given at \( t = 0 \), i.e.
\[
\omega(x, 0) = \omega_0(x), \quad x \in D. \tag{1.4}
\]
The questions of existence and uniqueness of solutions to systems like (1.1)–(1.4) may be addressed by the methods of Ly & Titi (1999) or those of Rodrigues (1986, 1992). While the work of Rodrigues is for a two-phase system, his techniques may evidently be adapted to the equations for porous media flow.

The analogous Brinkman system consists of the partial differential equations
\[
-\Delta u_i + (1 + \gamma_1 \omega) u_i = -p_i + g_i \omega,
\]
\[
\frac{\partial u_i}{\partial x_i} = 0,
\]
\[
\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \Delta \omega, \tag{1.5}
\]
on \( D \times (0, T) \), together with the boundary and initial conditions
\[
u_i = f_i(x, t), \quad \omega = h(x, t), \quad x \in \partial D, \tag{1.6}
\]
\[
\omega(x, 0) = \omega_0(x), \quad x \in D. \tag{1.7}
\]

The plan of the paper is as follows. In the next section we derive a maximum principle for a solution to (1.1)–(1.4). This maximum principle is essential in what follows. In §3 we derive continuous dependence on the viscosity coefficient \( \gamma_1 \) for (1.1)–(1.4). Section 4 indicates the changes necessary to the proof in §2 when the Brinkman system is employed and then shows how one may derive a continuous dependence on \( \gamma_1 \) analysis in this case. Finally, in §5, a convergence result is proved for convective flow when \( \gamma_1 \to 0 \).

2. An a priori estimate for the concentration

To derive a maximum principle for \( \omega \) which satisfies the Forchheimer system (1.1)–(1.4) we follow the method in the appendix of Payne & Straughan (1998a).

Thus with \( H \) satisfying
\[
\Delta H(x, t) = 0 \text{ in } D \times (0, T),
\]
\[
H(x, t) = h^{2p-1}(x, t) \text{ on } \partial D \times (0, T),
\]

we form the combination
\[
\int_0^t \int_D (H - \omega^{2p-1}) \left[ \frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} - \Delta \omega \right] dx \, dt = 0,
\]
and integrating by parts as in Payne & Straughan (1998a) now leads to
\[
\int_D \omega^{2p} dx + \frac{2(2p - 1)}{p} \int_0^t \int_D \omega_i^p \omega''_i dx \, d\eta = \int_D \omega_0^{2p} dx + 2p(H, \omega)
\]
\[
- 2p(H_0, \omega_0) - 2p \int_0^t \int_D H'_i \omega dx \, d\eta + 2p \int_0^t \int_D \omega_i u_i dx \, d\eta
\]
\[
+ 2p \int_0^t \int_{\partial D} \frac{\partial H}{\partial n} h dA \, d\eta - \int_0^t \int_{\partial D} f \omega^{2p} dA \, d\eta. \tag{2.1}
\]
The only difference between (2.1) and (A3) of Payne & Straughan (1998a) is the last term in (2.1). However, (A4) of Payne & Straughan (1998a) does not hold. Here the argument is complicated by the fact that \( u_i n_i \neq 0 \) on \( \partial D \). We bound the fifth term on the right of (2.1) as

\[
2p \int_0^t \int_D H u_i \omega, i \, dx \, d\eta \leq 2p \mathcal{H}_m \left( \int_0^t \|u\|^2 \, d\eta \int_0^t \|\nabla \omega\|^2 \, d\eta \right)^{1/2}, \tag{2.2}
\]

where \( \mathcal{H}_m = \max_{D \times [0, T]} |\mathcal{H}| \). We must now obtain a bound for \( \|u\| \). To this end we multiply (1.1) by \( u_i \) to derive

\[
b \int_D |\omega|^3 \, dx + \int_D (1 + \gamma_1 \omega) |u|^2 \, dx \leq \|\omega\| \|u\| - \int_{\partial D} p f \, dA. \tag{2.3}
\]

Next, let \( \psi \) be the solution to the Neumann problem with data \( f \), i.e.

\[
\begin{align*}
\Delta \psi &= 0 \text{ in } D, \\
\frac{\partial \psi}{\partial n} &= f \text{ on } \partial D, \\
\int_{\partial D} \psi \, dA &= 0.
\end{align*} \tag{2.4}
\]

Then, to handle the pressure term in (2.3),

\[
- \int_{\partial D} p f \, dA = - \int_{\partial D} p \frac{\partial \psi}{\partial n} \, dA
= - \int_D p, i \psi, i \, dx
= \int_D \psi, i (-f_i \omega + (1 + \gamma_1 \omega) u_i + bu_i |u|) \, dx
\leq \|\nabla \psi\| \|\omega\| + \left( \int_D (1 + \gamma_1 \omega) \psi, i \psi, i \, dx \int_D (1 + \gamma_1 \omega) u_i u_i \, dx \right)^{1/2}
+ b \left( \int_D |\nabla \psi|^3 \, dx \right)^{1/3} \left( \int_D |u|^3 \, dx \right)^{2/3}, \tag{2.5}
\]

where the Cauchy–Schwarz and Hölder’s inequalities have been employed. From the Steklo inequality,

\[
\int_{\partial D} \psi^2 \, dA \leq \frac{1}{p_2} \|\nabla \psi\|^2 \tag{2.6}
\]

(see, for example, Sperb 1981), where \( p_2 \) is the first non-zero eigenvalue in the variational characterization

\[
p_2 = \min_{\xi \in F} \frac{\|\nabla \xi\|^2}{\int_{\partial D} \xi^2 \, dA},
\]

with \( F \) being the space of admissible functions. Since \( \Delta \psi = 0 \) in \( D \), it follows that

\[
\|\nabla \psi\|^2 = \int_{\partial D} \psi \frac{\partial \psi}{\partial n} \, dA.
\]
and then by use of the Cauchy–Schwarz inequality and (2.6) we have

$$\|\nabla \psi\|^2 \leq \frac{1}{p_2} \oint_{\partial D} \left( \frac{\partial \psi}{\partial n} \right)^2 \, dA.$$  (2.7)

Hence, employing (2.7) and (2.4) in (2.5), we find, with further use of the Cauchy–
Schwarz inequality,

$$-\oint_{\partial D} pf \, dA \leq \frac{1}{\sqrt{p_2}} \oint_{\partial D} f^2 \, dA \|\omega\| + bm^{1/6} \|\nabla \psi\|_6 \|u\|_3^2$$

$$+ \left( \int_{D} (1 + \gamma_1 \omega) u_i u_i \, dx \right)^{1/2} \left[ \left( \int_{D} \|\nabla \psi\|^2 + \gamma_1 \|\omega\| \left( \int_{D} \|\nabla \psi\|^4 \, dx \right)^{1/2} \right]^{1/2},$$

where $m$ is the volume of $D$. We next use the Sobolev inequalities

$$\|Q\|_6 \leq c_1 \|Q\|_{H^1(D)}, \quad \|Q\|_4^2 \leq \tilde{c}_1 \|Q\|_{H^2(D)}^2,$$

and (2.7) to see that

$$-\oint_{\partial D} pf \, dA \leq \frac{1}{\sqrt{p_2}} \oint_{\partial D} f^2 \, dA \|\omega\|$$

$$+ bm^{1/6} \|u\|_3^2 \left( \int_{D} \|\nabla \psi\|^2 \, dx + \int_{D} \psi_{,ij} \psi_{,ij} \, dx \right)^{1/2}$$

$$+ \left( \int_{D} (1 + \gamma_1 \omega) u_i u_i \, dx \right)^{1/2}$$

$$\times \left\{ \frac{1}{p_2} \oint_{\partial D} f^2 \, dA + \gamma_1 \tilde{c}_1 \|\omega\| \left( \int_{D} \|\nabla \psi\|^2 \, dx + \int_{D} \psi_{,ij} \psi_{,ij} \, dx \right) \right\}^{1/2}.$$  (2.8)

Let now $d_3$ be the data term

$$d_3(t) = \frac{1}{p_2} \oint_{\partial D} f^2 \, dA,$$

and, as we have noted,

$$\|\nabla \psi\|^2 \leq d_3.$$  (2.9)

We use the fact that $\Delta \psi = 0$ and introduce the coordinate system $\theta^i = (\lambda, s^1, s^2)$
as in Payne & Straughan (1996, p. 238) (cf. Sperb 1981) and then we use (2.57) of
that paper to write

$$\int_{D} \psi_{,ij} \psi_{,ij} \, dx = \oint_{\partial D} \psi_{,i} \psi_{,i} n_j \, dA$$

$$= -\oint_{\partial D} \left[ b_{a} f^2 + \int_{\partial D} \left( \frac{1}{2} g^{22} \frac{\partial g_{33}}{\partial u} - \frac{\partial g_{33}}{\partial u} \frac{\partial \psi}{\partial u} + \frac{1}{2} g^{33} \left( \frac{\partial g_{33}}{\partial u} - \frac{\partial g_{33}}{\partial v} \frac{\partial \psi}{\partial v} \right) \right. \right.$$

$$\left. \left. + b_{1} (g^{22})^2 \right) \left( \frac{\partial \psi}{\partial u} \right)^2 + b_{2} (g^{33})^2 \left( \frac{\partial \psi}{\partial v} \right)^2 \right] \, dA,$$  (2.10)
where \( b^\alpha \) is twice the mean curvature and \( g_{ij}, g^{ij} \) denote the components of the metric tensor as defined in Payne & Straughan (1996, p. 238). We now integrate by parts in the surface coordinates \( u (= s^1), v (= s^2) \), to obtain

\[
\int_D \psi_{,ij} \psi_{,ij} \, dx = -\int_{\partial D} \left[ b^\alpha f^2 - 2 \frac{\partial (fg^{22})}{\partial u} \frac{\partial \psi}{\partial u} - 2 \frac{\partial (fg^{33})}{\partial v} \frac{\partial \psi}{\partial v} \right. \\
+ \frac{1}{2} fg^{22} \left( \frac{\partial g_{33}}{\partial u} - \frac{\partial g_{22}}{\partial u} \right) \frac{\partial \psi}{\partial u} + \frac{1}{2} fg^{33} \left( \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{33}}{\partial v} \right) \frac{\partial \psi}{\partial v} \\
+ b^1 (g^{22})^2 \left( \frac{\partial \psi}{\partial u} \right)^2 + b^2 (g^{33})^2 \left( \frac{\partial \psi}{\partial v} \right)^2 \left] \, dA. \right.
\]

The idea is now to estimate the surface terms in \( \psi \) on the right-hand side of (2.11) in terms of the data functions \( f \) and its tangential derivatives. In order to do this we use the Rellich identity (Payne & Weinberger 1958)

\[
0 = \int_D x^i \psi_{,i} \Delta \psi \, dx,
\]

or

\[
\frac{1}{2} \| \nabla \psi \|^2 + \int_{\partial D} x^i \psi_{,i} \frac{\partial \psi}{\partial n} \, dA = \frac{1}{2} \int_{\partial D} x^i n_i g^{rs} \psi_{,r} \psi_{,s} \, dA.
\]

Suppose \( D \) is star-shaped with respect to an origin and then let \( x^i n_i \geq \lambda_0 > 0 \) on \( \partial D \). With the metric tensor \( g^{ij} \) as indicated above we may then derive from the Rellich identity

\[
\frac{1}{2} \lambda_0 \int_{\partial D} \left[ g^{11} \left( \frac{\partial \psi}{\partial n} \right)^2 + g^{22} \left( \frac{\partial \psi}{\partial u} \right)^2 + g^{33} \left( \frac{\partial \psi}{\partial v} \right)^2 \right] \, dA
\]

\[
\leq \frac{1}{2} \| \nabla \psi \|^2 + \int_{\partial D} \frac{\partial \psi}{\partial n} \left[ g^{11} x_1 + g^{22} x_2 + g^{33} x_3 \frac{\partial \psi}{\partial n} \right] \, dA.
\]

Hence, using the arithmetic–geometric mean inequality we are led to

\[
\frac{1}{2} (\lambda_0 - \alpha) \int_{\partial D} g^{22} \left( \frac{\partial \psi}{\partial u} \right)^2 \, dA + \frac{1}{2} (\lambda_0 - \beta) \int_{\partial D} g^{33} \left( \frac{\partial \psi}{\partial v} \right)^2 \, dA
\]

\[
\leq \frac{1}{2} \| \nabla \psi \|^2 + \int_{\partial D} \left[ g^{11} x_1 - \frac{1}{2} \lambda_0 \right] + \frac{x_2^2 g^{22}}{2\alpha} + \frac{x_3^2 g^{33}}{2\beta} \left( \frac{\partial \psi}{\partial n} \right)^2 \, dA.
\]

We then use the Stekloff inequality on the first term on the right-hand side and pick \( \alpha = \beta = \frac{1}{2} \lambda_0 \) to find with

\[
k_1 = \frac{4}{\lambda_0} \left[ \frac{1}{p_2} + g^{11} (x_1 - \frac{1}{2} \lambda_0) + \frac{g^{22} x_2^2}{\lambda_0} + \frac{g^{33} x_3^2}{\lambda_0} \right],
\]

\[
\int_{\partial D} \left[ g^{22} \left( \frac{\partial \psi}{\partial u} \right)^2 + g^{33} \left( \frac{\partial \psi}{\partial v} \right)^2 \right] \, dA \leq \int_{\partial D} k_1 \left( \frac{\partial \psi}{\partial n} \right)^2 \, dA. \quad (2.12)
\]
If we now use (2.12) in (2.11) and define the data terms \( d_1(t) \) and \( d_2(t) \) by

\[
d_1(t) = \int_{\partial D} \left\{ b_0^2 f^2 + \left[ \frac{\partial (fg_{22})}{\partial u} \right]^2 + \left[ \frac{\partial (fg_{33})}{\partial v} \right]^2 \right\} \, dA ,
\]

\[
d_2(t) = \left[ \max_{\partial D} \left\{ 2 + \frac{|b_1^2 (g_{22}^2)^2|}{g_{22}^2}, 2 + \frac{|b_2^2 (g_{33}^2)^2|}{g_{33}^2} \right\} \right] \int_{\partial D} k_1 f^2 \, dA ,
\]

then we may obtain the bound

\[
\int_{\partial D} \psi_{,ij} \psi_{,ij} \, dx \leq d_1 + d_2 .
\]

(2.13)

We now return to (2.8) and employ (2.9) and (2.13) to derive

\[
-\int_{\partial D} pf \, dA \leq d_3^{1/2} \| \omega \| + d_4^{1/2} \| u \|^2 + (d_3 + d_5 \| \omega \|)^{1/2} \left[ \int_{\partial D} (1 + \gamma_1 \omega) u_i u_i \, dx \right]^{1/2} ,
\]

(2.14)

where the data terms \( d_4 \) and \( d_5 \) are given by

\[
d_4 = b^2 c_1^2 m^{1/3} (d_1 + d_2 + d_3) ,
\]

\[
d_5 = \gamma_1 c_1 (d_1 + d_2 + d_3) .
\]

Hence, use (2.14) in (2.3) together with Young’s inequality and the arithmetic–geometric mean inequality to arrive at

\[
b \int_{\partial D} |u|^3 \, dx + \int_{\partial D} (1 + \gamma_1 \omega) |u|^2 \, dx
\]

\[
\leq \| \omega \|^2 + \frac{1}{4} \| u \|^2 + d_3^{1/2} \| \omega \| + \frac{64}{81b^2} d_4^{3/2}
\]

\[
+ \frac{1}{2} b \int_{\partial D} |u|^3 \, dx + \frac{1}{4} \int_{\partial D} (1 + \gamma_1 \omega) |u|^2 \, dx + d_3 + d_5 \| \omega \| .
\]

(2.15)

This leads to the \textit{a priori} bound on \( \| u \| \) and \( \| u \|_3 \):

\[
b \int_{\partial D} |u|^3 \, dx + \int_{\partial D} (1 + \gamma_1 \omega) |u|^2 \, dx \leq 4 \| \omega \|^2 + d_6 ,
\]

(2.16)

where the data term \( d_6(t) \) is given by

\[
d_6 = 3d_3 + \frac{128}{81b^2} d_4^{3/2} + d_5^2 .
\]

A combination of (2.2) and (2.16) produces the result

\[
2p \int_0^t \int_{\partial D} \mathcal{H} u_i \omega_i \, dx \, d\eta \leq 2p \mathcal{H}_m \left[ \int_0^t (4 \| \omega \|^2 + d_6) \, d\eta \int_0^t \| \nabla \omega \|^2 \, d\eta \right]^{1/2} ,
\]

(2.17)

We now follow the steps of Payne & Straughan (1998a) from (A8) to (A20), \textit{mutatis mutandis}, to derive the exact estimates (A18)–(A20) but with \( T \) replaced by \( \omega \), \( \zeta \) replaced by \( \zeta = 8(1 + h_m^2) \) and \( d_1 \) of that paper replaced by

\[
d_1 = d_1 (\text{of (A16)}) + d_6 (1 + h_m^2) .
\]
Then following Payne & Straughan (1998a), inequality (A24) of that paper is replaced by
\[
\int_{D} \omega^{2p} \, dx \leq \int_{D} \omega_{0}^{2p} \, dx + 2p(\sqrt{2d_1 e^{cT}} + ||\omega_{0}||) \psi_{1}^{1/2} \left( \int_{\partial D} h^{4p-2} \, dA \right)^{1/2} \\
+ 2p^{2} \left( e^{cT} - 1 \right) \left( \int_{0}^{t} \psi_{1} \int_{\partial D} h^{4p-4} h_{r}^{2} \, dA \, d\eta \right)^{1/2} \\
+ 2p^{2} \int_{0}^{t} \int_{\partial D} h^{2} \, dA \, d\eta \left( \int_{0}^{t} \int_{\partial D} h^{4p-4} |\nabla_{s} h_{i}|^{2} \, dA \, d\eta \right)^{1/2} \\
+ \int_{0}^{t} \int_{\partial D} |f| \, h^{2p} \, dA \, d\eta + 2ph_{m}^{2p-1} \left[ \int_{0}^{t} \int_{\partial D} (8d_1 e^{cT} + d_0) \, dA \, d\eta \right]^{1/2}.
\]
(2.18)

After taking the power \(1/2p\) of this inequality we obtain an inequality of the form
\[
||\omega||_{2p} \leq \left[ ||\omega_{0}||_{2p}^{2p} + \left( \sum_{i=1}^{5} r_{i} \right) h_{m}^{2p} \right]^{1/2p}
\]
(2.19)
as in (A26) of Payne & Straughan (1998a), where \(r_{i}\) may be obtained from (2.18) and
\[
h_{m} = \max_{\partial D \times [0, T]} |h|.
\]
Taking the limit \(2p \to \infty\) leads to the \textit{a priori} bound
\[
\sup_{D \times [0, T]} |\omega| \leq \sup \left\{ ||\omega_{0}||_{m}, \sup_{[0, T]} h_{m} \right\},
\]
(2.20)
where \(||\omega_{0}||_{m} = \max_{D} |\omega_{0}|\).

3. Continuous dependence on the viscosity in the Forchheimer model

To investigate continuous dependence on the viscosity coefficient \(\gamma_1\) in (1.1) we let \((u_{i}, \omega, p)\) and \((v_{i}, \chi, q)\) be solutions to (1.1)–(1.4) for the same data functions \(f, h\) and \(\omega_{0}\), but for different viscosity coefficients, \(\gamma_1\) and \(\gamma_2\), respectively. Define the difference solution \((w_{i}, \theta, \pi)\) by
\[
w_{i} = u_{i} - v_{i}, \quad \theta = \omega - \chi, \quad \pi = p - q, \quad \gamma = \gamma_1 - \gamma_2.
\]
(3.1)

Then from (1.1)–(1.4) this solution satisfies the boundary–initial-value problem
\[
b[u_{i}|u| - v_{i}|v|] + w_{i} + \gamma w_{i} u_{i} + \gamma_2 \theta u_{i} + \gamma_2 \chi w_{i} = -\pi_{i} + g_{i} \theta, \\
\frac{\partial w_{i}}{\partial x_{i}} = 0, \\
\frac{\partial \theta}{\partial t} + w_{i} \frac{\partial \omega}{\partial x_{i}} + v_{i} \frac{\partial \theta}{\partial x_{i}} = \Delta \theta, \\
w_{i} = \theta = 0 \text{ on } \partial D, \\
\theta(x, 0) = 0, \quad x \in D.
\]
(3.2)

We observe for later convenience that (3.2) may be rearranged as
\[
b[u_{i}|u| - v_{i}|v|] + w_{i} + \gamma_{1} \omega w_{i} + \gamma \omega v_{i} + \gamma_2 v_{i} \theta = -\pi_{i} + g_{i} \theta.
\]
(3.3)
Let us denote the maximum on the right-hand side of (2.20) by $\omega_m$.
We multiply (3.2)$_1$ by $w_i$, integrate over $D$ and use (3.2)$_2$ and (3.2)$_4$ to find
\[
\frac{1}{b} \int_D (u_i|u| - v_i|v|)w_i \, dx + \int_D (1 + \gamma_2 \chi) w_i w_i \, dx
= g_i(\theta, w_i) - \gamma \int_D B w_i \, dx - \gamma_2 \int_D \theta u_i \, dx.
\]
Now use (2.20), the Cauchy–Schwarz inequality and (2.9) of Payne & Straughan (1998c) to find
\[
\frac{1}{b} \int_D (|u| + |v|)w_i w_i \, dx + \int_D (1 + \gamma_2 \chi) w_i w_i \, dx
\leq \|\theta\| \|w\| + \gamma \omega_m \|u\| \|w\| + \gamma_2 \left( \int_D |u| w_i w_i \, dx \right)^{1/2} \left( \int_D |\theta|^2 \, dx \right)^{1/2}
\leq \frac{1}{2\alpha} \|\theta\|^2 + \frac{1}{2} (1 + \frac{1}{2}) \|w\|^2 + \frac{\gamma_2 \omega_m^2}{2\beta} \|u\|^2
+ b \int_D |u| w_i w_i \, dx + \frac{\gamma_2 \omega_m^2}{4b} \left( \int_D |u|^3 \, dx \right)^{1/3} \left( \int_D |\theta|^3 \, dx \right)^{2/3},
\] (3.4)
where H"older’s inequality has also been used and $\alpha, \beta > 0$ are constants to be chosen.
We next employ the Sobolev inequality,
\[
\int_D \theta^4 \, dx \leq C^2 \left( \int_D \|\theta\|^2 \, dx \right)^{1/2} \left( \int_D |\nabla\theta|^2 \, dx \right)^{3/2},
\]
together with the Cauchy–Schwarz inequality in (3.4) to deduce
\[
\frac{1}{b} \int_D (|u| + |v|)w_i w_i \, dx + \int_D (1 + \gamma_2 \chi) w_i w_i \, dx
\leq \frac{1}{2\alpha} \|\theta\|^2 + \frac{1}{2} (1 + \beta) \|w\|^2 + \frac{\gamma_2 \omega_m^2}{2\beta} \|u\|^2
\]
\[+ b \int_D |u| w_i w_i \, dx + \frac{\gamma_2 \omega_m^2}{4b} \|u\|_3 \|\theta\| \|\nabla\theta\|. \] (3.5)
We now carry out a similar procedure starting from (3.3) to obtain
\[
\frac{1}{b} \int_D (|u| + |v|)w_i w_i \, dx + \int_D (1 + \gamma_1 \omega) w_i w_i \, dx
\leq \frac{1}{2\alpha} \|\theta\|^2 + \frac{1}{2} (1 + \beta) \|w\|^2 + \frac{\gamma_2 \omega_m^2}{2\beta} \|v\|^2
\]
\[+ b \int_D |v| w_i w_i \, dx + \frac{\gamma_2 \omega_m^2}{4b} \|v\|_3 \|\theta\| \|\nabla\theta\|. \] (3.6)
We add (3.5) and (3.6) to find
\[
\int_D (2 + \gamma_1 \omega + \gamma_2 \chi) w_i w_i \, dx \leq \frac{1}{\alpha} \|\theta\|^2 + (1 + \beta) \|w\|^2
\]
\[+ \frac{\gamma_2 \omega_m^2}{2\beta} \||u||^2 + \||v||^2\| + \frac{\gamma^2 \omega_m^2}{4b} \||u||_3 + \||v||_3\| \||\theta|| \||\nabla\theta||. \] (3.7)
For positive $\epsilon$ we may then deduce
\[
[2 - (\alpha + \beta)] \|w\|^2 \leq \left[ \frac{1}{\alpha} + \frac{\gamma^2 \omega_m^{4/3}}{64 \beta^2 \epsilon} (\|u\|_3 + \|v\|_3)^2 \right] \|\theta\|^2
\]
\[
+ \gamma^2 \omega_m^{2/3} (\|u\|^2 + \|v\|^2) + \epsilon \|\nabla \theta\|^2.
\]

(3.8)

Note now from (2.16) that
\[
\|u\|^2 \leq 4 \omega^2 + d_6, \quad \|u\|_3 \leq b^{-1/3} (4 \|\omega\|^2 + d_6)^{1/3},
\]
with similar estimates involving $v$.

The idea is to insert (3.9) in (3.8) and then note $\gamma$ is bounded by (2.20). We do this but also multiply (3.2) by $3$ and integrate to find
\[
\frac{1}{2} \|\theta\|^2 + \int_0^t \|\nabla \theta\|^2 \, d\eta = \int_0^t \int_D w_i \omega \theta_i \, dx \, d\eta.
\]

Upon using the Cauchy–Schwarz inequality,
\[
\|\theta\|^2 + \int_0^t \|\nabla \theta\|^2 \, d\eta \leq \omega_m^2 \int_0^t \|w\|^2 \, d\eta.
\]

(3.10)

We next use (3.9) in (3.8) and then employ (3.10). The result is integrated and (3.10) used again to derive
\[
\int_0^t \|w\|^2 \, d\eta \leq k_1 \int_0^t (t - \eta) \|w\|^2 \, d\eta + k_2(t) \gamma^2,
\]

(3.11)

where we have picked $\alpha = \beta = \frac{1}{2}$, $\epsilon = 1/2 \omega_m^2$ and
\[
k_1 = 2 \omega_m^2 + \frac{\gamma^2 \omega_m^{4/3}}{8 \beta^{2/3}} (4 \|\omega\|_{\text{max}}^2 + d_{\text{max}})^{2/3},
\]
\[
k_2(t) = \omega_m^2 (8 \|\omega\|_{\text{max}}^2 + 2d_6).
\]

In the expression for $k_1$ the maximum is over $[0, T]$. If we denote by $k_3(t)$ the term
\[
k_3(t) = \int_0^t k_2(\eta) e^{k_1(t-\eta)} \, d\eta,
\]
then from (3.11) we see that
\[
\int_0^t (t - \eta) \|w\|^2 \, d\eta \leq k_3(t) \gamma^2,
\]

(3.12)

\[
\int_0^t \|w\|^2 \, d\eta \leq k_4 \gamma^2,
\]

(3.13)

where $k_4 = k_1 k_3 + k_2$. Hence, employing (3.10) we also find
\[
\|\theta(t)\|^2 + \int_0^t \|\nabla \theta\|^2 \, d\eta \leq k_4 \omega_m^2 \gamma^2.
\]

(3.14)

Inequalities (3.12)–(3.14) demonstrate continuous dependence on the viscosity coefficient $\gamma_1$ and are truly a priori in that the coefficients of $\gamma^2$ depend only on boundary and initial data, and the geometry of $D$.

Note. We have dealt with the classical Forchheimer system given by (1.1). However, Firdaouss et al. (1997) observe that in Forchheimer’s original work an alternative model is proposed which extends the nonlinearity to include a cubic term. The generalization of (1.1) would be

$$c |u|^2 u_i + b|u| u_i + (1 + \gamma_1 \omega) u_i = -p_i + g_i \omega.$$  

Our analysis of §§2 and 3 may be adapted to this system. One uses the following inequality in conjunction with the Sobolev inequality:

$$\int_D \psi_i u_i |u|^2 \, dx \leq \|u\|_3^3 \|
abla \psi\|_4.$$

In fact, since Forchheimer’s work extends the nonlinearity by employing a polynomial nonlinearity we should observe that our analysis continues to work for the analogous Forchheimer system with (1.1) replaced by

$$\left( \sum_{m=1}^4 b_{m+1} |u|^m \right) u_i + (1 + \gamma_1 \omega) u_i = -p_i + g_i \omega.$$  

In the analysis one uses the inequality

$$\int_D \psi_i u_i |u|^4 \, dx \leq \|u\|_6^5 \|\nabla \psi\|_6,$$

together with the Sobolev inequality.

4. Continuous dependence on the viscosity in the Brinkman model

In this section we indicate how to prove continuous dependence on the viscosity coefficient $\gamma_1$ for the Brinkman system (1.5)–(1.7). The first step is to derive the maximum principle analogous to that of §2. Equation (2.1) still holds although $f$ in the last term is now $f_i n_i$. Inequality (2.2) is still valid.

To derive a bound for $\|u\|$ we let $\alpha$ solve the Stokes flow problem in $D$, namely

$$\begin{aligned}
\Delta \alpha_i &= \rho_i, \\
\frac{\partial \alpha_i}{\partial x_i} &= 0 \text{ in } D, \\
\alpha_i &= f_i \text{ on } \partial D, \\
\end{aligned}$$

where $\rho$ is a pressure term. By the triangle inequality,

$$\|u\| \leq \|u - \alpha\| + \|\alpha\|.$$  

Next, we employ $(1.5)_1$ and (4.1) to derive

$$\|\nabla (u - \alpha)\|^2 + \int_D (1 + \gamma_1 \omega)(u_i - \alpha_i)(u_i - \alpha_i) \, dx$$

$$= - \int_D (1 + \gamma_1 \omega)(u_i - \alpha_i) \alpha_i \, dx + \int_D g_i \omega (u_i - \alpha_i) \, dx.$$  

Let us denote the right-hand sides of (4.5) and (4.6) by $D_1(t)$ and $2D_2(t)$, respectively. Observe that $D_1$ and $D_2$ are data terms. Then from (4.4)–(4.6) we derive

$$
\| \nabla (u - a) \|^2 + \frac{1}{2} \int_D (1 + \gamma_1 \omega)(u_i - a_i)(u_i - a_i) \, dx
\leq \int_D (1 + \gamma_1 \omega)a_i a_i \, dx + \| \omega \|^2
\leq \| \omega \|^2 + \| a \|^2 + \gamma_1 \| \omega \| \| a \|_2^2
\leq \| \omega \|^2 + \| a \|^2 + \bar{c}_1 \gamma_1 \| \omega \| (\| \nabla a \|^2 + \| a \|^2).
$$

(4.4)

Payne (1964), (3.28) and (3.31), show that

$$
\| a \|^2 \leq 6d \int_{\partial D} f_i f_i \, dA + 4d^2 \int_D (a_{ij} - a_{j,i})(a_{ij} - a_{j,i}) \, dx
\leq (6d + 4d^2 \bar{k}_1) \int_{\partial D} f_i f_i \, dA + 4d^2 \bar{k}_2 \int_{\partial D} |\nabla_s f|^2 \, dA,
$$

(4.5)

where $d$ is the radius of the smallest circumscribed ball for $D$, $\bar{k}_1, \bar{k}_2$ are a priori constants given in Payne (1964) and $\nabla_s$ denotes the tangential derivative. Furthermore, we have that

$$
\| \nabla a \|^2 = \frac{1}{2} \int_D (a_{ij} - a_{j,i})(a_{ij} - a_{j,i}) \, dx + \int_D a_{ij} a_{j,i} \, dx
= \frac{1}{2} \int_D (a_{ij} - a_{j,i})(a_{ij} - a_{j,i}) \, dx + \int_{\partial D} (a_{ij} a_j - a_{j,i} a_i) n_i \, dA
= \frac{1}{2} \int_D (a_{ij} - a_{j,i})(a_{ij} - a_{j,i}) \, dx + \int_{\partial D} n^i s^j (a_j \nabla_s a_i - a_i \nabla_s a_j) \, dA
\leq \frac{1}{2} \bar{k}_1 \int_{\partial D} f_i f_i \, dA + \frac{1}{2} \bar{k}_2 \int_{\partial D} |\nabla_s f|^2 \, dA + \int_{\partial D} n^i s^j (f_j \nabla_s f_i - f_i \nabla_s f_j) \, dA,
$$

(4.6)

where $s^j$ denotes a tangential vector.

Let us denote the right-hand sides of (4.5) and (4.6) by $D_1^2(t)$ and $2D_2^2(t)$, respectively. Observe that $D_1$ and $D_2$ are data terms. Then from (4.4)–(4.6) we derive

$$
\| \nabla (u - a) \|^2 + \frac{1}{2} \int_D (1 + \gamma_1 \omega)(u_i - a_i)(u_i - a_i) \, dx
\leq \| \omega \|^2 + D_1^2(t) + 2\bar{c}_1 \gamma_1 D_2^2(t) \| \omega \|
\leq (\| \omega \|^2 + D_3)^2,
$$

where $D_3 = D_1$ if $D_1 \geq \bar{c}_1 \gamma_1 D_2^2$, else $D_3 = \bar{c}_1 \gamma_1 D_2^2$. Thus,

$$
\| a - u \| \leq 2 \| \omega \| + 2D_3.
$$

(4.7)

Hence from (4.2),

$$
\| a \| \leq 2 \| \omega \| + 2D_3 + D_1,
$$

from which we may derive that with $d_0(t)$, the data term $d_0 = 5(D_1 + 2D_3)^2$,

$$
\| a \|^2 \leq 5 \| \omega \|^2 + d_0.
$$

(4.8)
Inequality (4.8) serves the role of inequality (2.16) in § 2. The remainder of the proof in § 2 follows as before to yield the bound (2.20) for $|\omega|$, but for a solution to the Brinkman system (1.5)-(1.7).

We are now ready to demonstrate continuous dependence on $\gamma_1$. Hence, let $(u_i, \omega, p)$ and $(v_i, \chi, q)$ solve (1.5)-(1.7) for the same data $f_i, h$ and $\omega_0$, but for viscosity coefficients $\gamma_1$ and $\gamma_2$, respectively. The difference variables $(w_i, \theta, \pi)$ are defined as in § 3 and we see they satisfy the system

$$
-\Delta w_i + (1 + \gamma_2 \chi) w_i + \gamma \omega u_i + \gamma_2 \theta u_i = -\pi_i + g_i, \\
\frac{\partial w_i}{\partial x_i} = 0, \\
\frac{\partial \theta}{\partial t} + w_i \frac{\partial \omega}{\partial x_i} + v_i \frac{\partial \theta}{\partial x_i} = \Delta \theta, \\
w_i = \theta = 0 \text{ on } \partial D, \\
\theta(x, 0) = 0, \quad x \in D.
$$

(4.9)

The procedure leading to (3.10) leads to

$$
\|\theta(t)\|^2 \leq \frac{1}{2} \omega_m^2 \int_0^t \|w\|^2 \, d\eta.
$$

(4.10)

From equations (4.9) we derive

$$
\|\nabla w\|^2 + \int_D (1 + \gamma_2 \chi) w_i w_i \, dx \\
= -\gamma \int_D \omega u_i w_i \, dx - \gamma_2 \int_D \theta u_i w_i \, dx + g_i(\theta, w_i) \\
\leq \|\theta\| \|w\| + \gamma \omega_m \|u\| \|w\| + \gamma_2 \|\theta\| \left( \int_D u_i w_i u_j w_j \, dx \right)^{1/2}. 
$$

(4.11)

The last term is bounded using (2.16) of Payne (1964), i.e.

$$
\int_D u_i w_i u_j w_j \, dx \leq \frac{2}{\pi} \left( \|\nabla w\|^2 \int_{\partial D} f_i f_i \, dA + \|\nabla u\|^2 \|w\| \|\nabla w\| \right) \\
\leq \frac{2}{\pi} \left\{ \|\nabla w\|^2 \left[ \int_{\partial D} f_i f_i \, dA + \lambda^{-1/2} \|\nabla u\|^2 \right] \right\},
$$

(4.12)

where $\lambda_1$ is the Poincaré constant for $D$.

To employ (4.11) and (4.12) we need data bounds for $\|u\|$ and $\|\nabla u\|$. Since $\omega$ satisfies (2.20) we see from (4.8) that

$$
\|u\|^2 \leq 5m \omega_m^2 + d_0 = D_5^2.
$$

(4.13)

Also, from the triangle inequality

$$
\|\nabla u\| \leq \|\nabla (u - a)\| + \|\nabla a\|,
$$

and then from the inequality before (4.7) and (4.6) we find

$$
\|\nabla u\|^2 \leq D_4 = (m \omega_m^2 + D_1^2 + 2c_1 D_2 m \omega_m)^{1/2} + 2D_2.
$$

(4.14)

Hence, from (4.11) we conclude that

$$
\|\nabla w\|^2 + \int_D (1 + \gamma_1 \omega) w_i w_i \, dx \leq \|\theta\| \|w\| + \gamma \omega_m D_5 \|w\| + D_6 \|\theta\| \|\nabla w\|.
$$

(4.15)
where
\[ D_6 = \gamma_2 \sqrt{\frac{2}{\pi}} \left( \int_{\partial D} f_i f_i \, dA + \frac{D_4}{\sqrt{\lambda_1}} \right). \]

Thus, from (4.15) we may derive
\[ \| \nabla w \|^2 + \| w \|^2 \leq q_1 \gamma^2 + q_2 \| \theta \|^2, \]
where
\[ q_1 = 2 \omega_m^2 D_5^2, \quad q_2 = 2 + D_6^2. \]

Upon employing (4.10),
\[ \| \nabla w \|^2 + \| w \|^2 \leq \frac{1}{2} q_2 \omega_m^2 \int_0^t \left( \| \nabla w \|^2 + \| w \|^2 \right) \, d\eta + q_1 \gamma^2. \]

This inequality may be integrated to produce
\[ \int_0^t \left( \| \nabla w \|^2 + \| w \|^2 \right) \, d\eta \leq q_3(t) \gamma^2, \]
where \( q_3 \) is the data term
\[ q_3 = \int_0^t q_1(\eta) \exp\left( \frac{1}{2} \omega_m^2 \int_\eta^t q_2(s) \, ds \right) \, d\eta. \]

Furthermore, by using (4.17) and (4.10) we also have
\[ \| \nabla w(t) \|^2 + \| w(t) \|^2 \leq (q_1 + \frac{1}{2} q_2 q_3 \omega_m^2) \gamma^2, \]
\[ \| \theta(t) \|^2 \leq \frac{1}{2} \omega_m^2 q_3 \gamma^2. \]

Inequalities (4.18)–(4.20) are a priori bounds demonstrating continuous dependence of the solution on the viscosity coefficient \( \gamma_1 \).

**Note.** We could have kept a piece of the Dirichlet integral of \( \theta \) in (4.10) and this would then lead to an a priori bound like (4.20) but for \( \| \theta(t) \|_{H^1(D)} \), yielding continuous dependence in this measure.

### 5. Convergence to the constant viscosity solution

Strictly speaking, we should have included a parameter \( \lambda \), in front of the Laplacian term in (1.5). This is essentially an effective viscosity coefficient. The gist of §4 is unchanged if we do this. We observe that from estimates (3.12)–(3.14), or (4.18)–(4.20), we may obtain convergence of the solution to the Forchheimer or Brinkman system to the equivalent system with \( \gamma_1 = 0 \) (in the indicated measures). In this section we demonstrate convergence as \( \gamma_1 \to 0 \) but for the Darcy equations. The boundary–initial-value problem for the Darcy equations is obtained from (1.1)–(1.4) by setting \( b = 0 \). The proof of continuous dependence on \( \gamma_1 \) does not appear to carry over for the Darcy system. The difficulty stems from the term \( \gamma_2 \gamma w_i \) in (3.2), the treatment of which appears to require the stronger nonlinearity or extra dissipation in the Forchheimer or Brinkman equations, respectively. However, we can demonstrate convergence for the Darcy system.

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Let now \((u_i, \omega, p)\) and \((v_i, \chi, q)\) satisfy the following boundary–initial-value problems:

\[
(1 + \gamma \omega)u_i = -p_i + g_i \omega, \quad \frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \Delta \omega, \quad \frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \Delta \omega, \quad \frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \Delta \omega,
\]

in \(D \times (0, T)\), and

\[
u_i n_i = f, \quad \omega = h, \quad \text{on } \partial D \times (0, T), \quad \omega(x, 0) = \omega_0(x), \quad x \in D,
\]

\[
u_i = -q_i + g_i \chi, \quad \frac{\partial v_i}{\partial x_i} = 0, \quad \frac{\partial \chi}{\partial t} + v_i \frac{\partial \chi}{\partial x_i} = \Delta \chi,
\]

in \(D \times (0, T)\), and

\[
u_i n_i = f, \quad \chi = h, \quad \text{on } \partial D \times (0, T), \quad \chi(x, 0) = \chi_0(x), \quad x \in D.
\]

The difference variables \(w_i, \theta\) and \(\pi\) defined as in § 3 satisfy the boundary–initial-value problem

\[
w_i + \gamma \omega u_i = -\pi_i + g_i \theta, \quad \frac{\partial w_i}{\partial x_i} = 0, \quad \frac{\partial \theta}{\partial t} + w_i \frac{\partial \omega}{\partial x_i} + v_i \frac{\partial \theta}{\partial x_i} = \Delta \theta,
\]

in \(D \times (0, T)\), and

\[
w_i n_i = 0, \quad \theta = 0, \quad \text{on } \partial D \times (0, T), \quad \theta(x, 0) = 0, \quad x \in D.
\]

The proof of the maximum principle (2.20) for \(\omega\) may be shown to hold here. We then find, by multiplying (5.5) by \(w_i\), integrating and using the Cauchy–Schwarz inequality,

\[
\|w\| \leq \|\theta\| + \gamma \omega_m \|u\|,
\]

From (5.1) we may derive

\[
\int_D (1 + \gamma \omega)u_i u_i \, dx \leq \|u\| \|\omega\| - \int_{\partial D} u_i p n_i \, dA.
\]
The boundary term may be handled by introducing the function \( \psi \) of (2.4) as follows:

\[
- \oint_{\partial D} u_i n_i p \, dA = - \oint_{\partial D} \frac{\partial \psi}{\partial n} \, dA
\]

\[
= - \int_D p, \psi, \, d\mathbf{x}
\]

\[
= \int_D \psi, (1 + \gamma \omega)u_i - g_i \omega \, d\mathbf{x}
\]

\[
\leq \| \nabla \psi \| \left[ \| \omega \| + \left\{ \int_D (1 + \gamma \omega)^2 u_i, u_i \, d\mathbf{x} \right\}^{1/2} \right]
\]

\[
\leq \frac{1}{\sqrt{2}} \left( \oint_{\partial D} f^2 \, dA \right)^{1/2} \left\{ \| \omega \| + (1 + \gamma \omega_m) \| \mathbf{u} \| \right\}.
\]

(5.9)

We now use (5.9) in (5.8) and employ the arithmetic–geometric-mean inequality to derive

\[
\| \mathbf{u} \|^2 \leq \| \omega \|^2 + \frac{1}{\sqrt{2} p_2} \left( \oint_{\partial D} f^2 \, dA \right)^{1/2} \| \omega \| + \frac{1}{p_2} \left( \oint_{\partial D} f^2 \, dA \right) (1 + \gamma \omega)^2
\]

\[
\leq \zeta^2,
\]

(5.10)

where \( \zeta^2 \) is the data term

\[
\zeta^2 = m \omega_m^2 + \frac{m^{1/2} \omega_m^{1/2}}{p_2^{1/2}} \left( \oint_{\partial D} f^2 \, dA \right)^{1/2} + \frac{(1 + \gamma \omega_m)^2}{p_2} \oint_{\partial D} f^2 \, dA.
\]

Upon using (5.10) in (5.7) we may show that

\[
\| \mathbf{w} \|^2 \leq 2 \gamma^2 \omega_m^2 \zeta^2 + 2 \| \theta \|^2.
\]

(5.11)

Next, from (5.5) we derive

\[
\| \theta \|^2 + \int_0^t \| \nabla \theta \|^2 \, d\eta = \int_0^t \int_D w_i \omega \theta_i \, d\mathbf{x} \, d\eta.
\]

From this we may see that

\[
\| \theta \|^2 + \frac{1}{2} \int_0^t \| \nabla \theta \|^2 \, d\eta \leq \frac{1}{2} \omega_m^2 \int_0^t \| \mathbf{w} \|^2 \, d\eta.
\]

(5.12)

Thus, employing (5.12) in inequality (5.11),

\[
\| \mathbf{w} \|^2 \leq 2 \omega_m^2 \zeta^2 \gamma^2 + \omega_m^2 \int_0^t \| \mathbf{w} \|^2 \, d\eta.
\]

(5.13)

Upon integration (5.13) yields

\[
\int_0^t \| \mathbf{w} \|^2 \, d\eta \leq 2 \zeta^2 e^{\omega_m^2 \gamma^2}.
\]

(5.14)

Inequality (5.14) demonstrates convergence of \( u_i \) to \( v_i \) as \( \gamma \to 0 \), in the indicated measure. By combining (5.14) and (5.13), we also obtain convergence of \( w_i \) in \( L^2(D) \) norm, and from (5.12) we may obtain convergence of \( \theta \) in the \( L^2(D) \) and \( H^1(D) \) norms.

Note. We can evidently generalize the analysis of this paper to the case where the diffusivity in the ω equation is anisotropic, i.e. we replace the Δω term by $(D_{ij}ω_{,j})_i$ for $D_{ij}$ a bounded positive-definite symmetric matrix with coefficients dependent on $x$. This generalization has practical application since the diffusivity is often anisotropic (cf. Gilman & Bear 1996).

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