Distillation of secret key and entanglement from quantum states

By Igor Devetak\(^1\) and Andreas Winter\(^2\)

\(^1\)IBM T. J. Watson Research Center, PO Box 218, Yorktown Heights, NY 10598, USA (devetak@us.ibm.com)
\(^2\)Department of Computer Science, University of Bristol, Merchant Venturers Building, Woodland Road, Bristol BS8 1UB, UK (a.j.winter@bris.ac.uk)

We study and solve the problem of distilling a secret key from quantum states representing correlation between two parties (Alice and Bob) and an eavesdropper (Eve) via one-way public discussion: we prove a coding theorem to achieve the ‘wire-tapper’ bound, the difference of the mutual information Alice–Bob and that of Alice–Eve, for so-called classical–quantum–quantum-correlations, via one-way public communication. This result yields information-theoretic formulae for the distillable secret key, giving ‘ultimate’ key rate bounds if Eve is assumed to possess a purification of Alice and Bob’s joint state.

Specializing our protocol somewhat and making it coherent leads us to a protocol of entanglement distillation via one-way LOCC (local operations and classical communication) which is asymptotically optimal: in fact we prove the so-called ‘hashing inequality’, which says that the coherent information (i.e. the negative conditional von Neumann entropy) is an achievable Einstein–Podolsky–Rosen rate. This result is known to imply a whole set of distillation and capacity formulae, which we briefly review.

Keywords: entanglement; cryptography; quantum wiretap channel; Hashing inequality

1. Introduction

Entanglement and secret correlation share an ‘exclusiveness’—in the one case towards the total outside world, in the other towards an entity ‘Eve’—that has led quantum information scientists to speculate on a systematic relation between their theories: the works in this direction range from building analogies (Collins & Popescu 2002) to using entanglement to prove information theoretic security of quantum key distribution (Shor & Preskill 2000), to attempts to prove the equivalence of the distillability of secret key and of entanglement (Acín et al. 2003; Bruß et al. 2003).

Of course there are also conceptual differences: while the task of distilling secret perfect correlation derives from potential cryptographic applications (and requires a third, malicious, party to formulate the operational problem), entanglement is useful for simple transmission tasks between two perfectly cooperating parties, as
exemplified by dense coding (Bennett & Wiesner 1992) and teleportation (Bennett et al. 1993).

The present paper falls into the third of the above categories, for we address the two questions, of distilling secret key from many copies of a quantum state (itself a generalization of classical information theoretic work begun by Maurer (1993) and Ahlswede & Csiszár (1993)) by public discussion and of distilling Einstein–Podolsky–Rosen (EPR) pairs by local operations and classical communication (LOCC), in a unified way. To be more precise, after describing a protocol for secret-key distillation from a state by one-way public discussion, we show how secrecy codes of a particular structure can be converted into one-way LOCC entanglement distillation protocols achieving the coherent information, as was conjectured for some time under the name of the ‘hashing inequality’ (after the hashing protocol in Bennett et al. (1996b), which attains the bound for Bell-diagonal two-qubit states). It is well known from Horodecki et al. (2000) that this inequality yields information theoretic characterizations of distillable entanglement under general LOCC, as well as the quantum transmission capacity, without, with forward and with bidirectional classical side channel (the first of these capacity theorems proved recently by Shor (2004), following a heuristic argument of Lloyd (1997), and subsequently in Devetak (2005)). Our approach is very close to that of Devetak (2005), and, as far as secret-key distillation is concerned, (Cai et al. 2005): while here our resource is a three-party quantum state (‘static’ model), these papers deal with the ‘dynamic’ analogue, where the resource is a quantum/wiretap channel.

As for the structure of the paper: the main result of the cryptographic part is theorem 2.1 in §2; the form of the optimal rates is then not hard to obtain, as we shall show in the detailed discussion. It is theorem 2.1 which we return to in the entanglement distillation part: a very general modification of the coding procedure will give us theorem 3.1, the hashing inequality; and as before, the form of the optimal rates is not hard to get from there. A reader only interested in entanglement distillation can thus skip the second part of §2: there the general form of optimal one-way distillable secret key is derived. In §3 we turn to one-way entanglement distillation, proving the hashing inequality and exhibiting the general form of optimal one-way distillation; then in §4 the consequences of the hashing inequality are detailed. Appendixes collect the necessary facts about typical subspaces (A), some miscellaneous lemmas (B) and miscellaneous proofs (C).

2. One-way secret-key distillation

We will first study and solve the case of classical–quantum–quantum-correlations (cqq-correlations), which are those where the initial state $\rho_{ABE}$ has the form

$$\rho_{ABE} = \sum_{x \in X} P(x)|x\rangle\langle x| \otimes \rho_{BE}^x.$$  \hspace{1cm} (2.1)

Then $n$ copies of that state can be written

$$(\rho_{ABE}^{\otimes n}) = \sum_{x^n \in X^n} P^n(x^n)|x^n\rangle\langle x^n| \otimes \rho_{BE}^{x^n},$$

with $x^n = x_1 \cdots x_n$ and

$$|x^n\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle, \quad \rho_{x^n}^{BE} = \rho_{x_1}^{BE} \otimes \cdots \otimes \rho_{x_n}^{BE}.$$
Distillation of secret key and entanglement from quantum states

Let $X$ be a random variable with distribution $P$, and corresponding to the $n$ copies of $\rho$ consider independent identically distributed (i.i.d.) realizations $X_1, \ldots, X_n$ of $X$.

A one-way key distillation protocol consists of the following:

(i) a channel $T : x^n \rightarrow (\ell, m)$, with range $\ell \in \{1, \ldots, L\}$ and $m \in \{1, \ldots, M\};$

(ii) a positive-operator-valued measure (POVM) $D^{(\ell)} = (D^{(\ell)} m)_{m=1}^M$ on $B^n$ for every $\ell$.

The idea is that Alice generates $T(X^n) = (\Lambda, K)$; her version of the key is $K = m$, while she sends $\Lambda = \ell$ to Bob. He obtains his $K'$ by measuring his system, $B$, using $D^{(\ell)}$:

$$\Pr\{K' = m \mid \Lambda = \ell, X^n = x^n\} = \text{Tr}(D^{(\ell)} m \rho^B_{x^n}).$$

For technical reasons we assume that the communication has a rate $L \leq 2^{nF}$, for some constant $F$.

We call this an $(n, \epsilon)$-protocol if

1. $\Pr\{K \neq K'\} \leq \epsilon$;

2. $\left\| \text{Dist}(K) - \frac{1}{M} 1_{\{1,\ldots,M\}} \right\|_1 \leq \epsilon$;

3. there is a state $\sigma_0$ such that, for all $m$,

$$\left\| \sum_{x^n,\ell} \Pr\{X^n = x^n, \Lambda = \ell \mid K = m\} |\ell\rangle\langle \ell | \otimes \rho^E_{x^n} - \sigma_0 \right\|_1 \leq \epsilon.$$

We call $R$ an achievable rate if for all $n$ there exist $(n, \epsilon)$-protocols with $\epsilon \to 0$ and $(1/n) \log M \to R$ as $n \to \infty$. (The convention in this paper is that log and exp are understood to be to basis 2.) Finally, we define

$$K \rightarrow (\rho) := \text{sup}\{R : R \text{ achievable}\},$$

the one-way (or forward) secret-key capacity of $\rho$.

Before we can formulate our first main result, we have to introduce some information notation: for a quantum state $\rho$ we denote the von Neumann entropy $H(\rho) = -\text{Tr} \rho \log \rho$, and the Shannon entropy of a probability distribution $P$,

$$H(P) = -\sum_x P(x) \log P(x).$$

If the state is the reduced state of a multi-party state, like the $\rho^{ABE}$ above, we write $H(A) = H(\rho^A)$, etc. In the particular case of equation (2.1), obviously $H(\rho^A) = H(P)$. For a general bipartite state $\rho^{AB}$ we define the (quantum) mutual information

$$I(A : B) = H(A) + H(B) - H(AB),$$

which for the cqq-state of equation (2.1) is easily checked to be equal to

$$H(\rho^B) - \sum_x P(x) H(\rho^B_x).$$
I. Devetak and A. Winter

a quantity known as the *Holevo bound* (Holevo 1973) and which we denote $I(P; \rho^B)$, reflecting in the notation the distribution $P$ and the *cq-channel* (using terminology introduced by Holevo (1977)) $x \mapsto \rho^B_x$. We shall often use the abbreviation $I(P; \rho^B_B)$ for this latter, if the states and distribution of the random variable $X$ are implicitly clear: this latter notation has the advantage that for any $U$ jointly distributed with $X$, $I(U; B)$ makes sense immediately, without our having to write down a composite state.

Finally, for a tripartite state $\rho^{ABC}$, define the (quantum) conditional mutual information

$$I(A : C | B) := H(AB) + H(BC) - H(ABC) - H(B),$$

which is non-negative by strong subadditivity (Lieb & Ruskai 1973). Usually the state these notations refer to will be clear from the context; where not we add it in subscript. Observe that for a classically correlated system $B$, the conditional mutual information takes the form of a probability average over mutual informations: e.g. for the state of equation (2.1),

$$I(B : E | A) = \sum_x P(x) I(B : E)_{\rho^B_x}.$$  

Also, for conditional mutual information we make use of the hybrid notation involving random variables: for example, for random variables $T$ and $U$, jointly distributed with $X$, $I(U; B | T)$ is the average over $T$ of Holevo quantities as above.

**Theorem 2.1.** For every cqq-state $\rho$,

$$K \rightarrow (\rho) \geq I(X; B) - I(X; E).$$

**Proof.** The idea is as follows: the state

$$\rho^{AB} = \sum_x P(x) |x\rangle^A \otimes \rho^B_x$$

contains the description of a cq-channel with channel states $\rho_x$. We will cover ‘evenly’ all typical type classes of block length $n$ by channel codes $C_\ell$ to transmit $\approx n I(X; B)$ bits, most of which are ‘good’ in the sense that they have small error probability. All of them are of the kind such that the state of $E$, when taking the average over the last $\approx n I(X; E)$ bits of the input, is almost a constant operator, $\sigma_\ell$, independent of the leading bits.

The key distillation scheme works then as follows: on observing $x^n$, which is typical with high probability, Alice announces its type and a random $\ell$, such that $x^n$ is a codeword of the code $C_\ell$, to Bob. He is able to decode it with high probability (because the code will be good with high probability), and they take the leading $\approx n (I(X; B) - I(X; E))$ bits of the message as the key. This is uniformly distributed because the code is entirely within one type class. Eve knows almost nothing about the key since she only has a state very close to $\sigma_\ell$, independent of the key (cf. figure 1).

In precise detail: let $Q$ be an $n$ type. (Ultimately we will only be interested in typical $Q$, i.e. $\|P - Q\|_1 \leq \delta$.) Consider random variables $U^{(\ell)}$, i.i.d. according

\[I.\]
Distillation of secret key and entanglement from quantum states

Figure 1. A schematic of the anatomy of the code: the typical sequences are covered by sets \( C_\ell \), which are good transmission codes for \( B \). A magnified view of one \( C_\ell \) (to the lower right) reveals its inner structure: it is composed of \( S_{\ell m} \), which are good privacy amplification codes against \( E \).

to the uniform distribution on the type class \( T_Q^n \) (see Appendix A), \( \ell = 1, \ldots, L \), \( m = 1, \ldots, M \), \( s = 1, \ldots, S \). Let

\[
\sigma(Q) := \frac{1}{|T_Q^n|} \sum_{x_n \in T_Q^n} \rho^E_{x_n} = \mathbb{E}\rho^E_{U^{(\ell ms)}},
\]

We are interested in the probability of the following various random events (for \( 0 < \epsilon < \frac{1}{2} \)).

\( \epsilon \)-evenness. For all \( x^n \in T_Q^n \),

\[
(1 - \epsilon) \frac{LMS}{|T_Q^n|} \leq \sum_{\ell m s} 1_{U^{(\ell ms)}}(x^n) \leq (1 + \epsilon) \frac{LMS}{|T_Q^n|},
\]

with the indicator functions \( 1_{U^{(\ell ms)}} \) on \( T_Q^n \).

\( \epsilon \)-secrecy. For all \( \ell, m \), the average of \( \rho^E_{x^n} \) over \( S_{\ell m} = \{ U^{(\ell ms)} : s = 1, \ldots, S \} \) is close to \( \sigma(Q) \):

\[
\left\| \frac{1}{S} \sum_{s} \rho^E_{U^{(\ell ms)}} - \sigma(Q) \right\|_1 \leq \epsilon.
\]

Codes \( C_\ell \) are \( \epsilon \)-good. Define the code \( C_\ell \) as the collection of codewords \( \{ U^{(\ell ms)} : s = 1, \ldots, S \} \). We call it \( \epsilon \)-good if there exists a POVM \( (D_{ms}^{(\ell)})_{m,s} \) such that

\[
\frac{1}{MS} \sum_{ms} \text{Tr}(\rho^E_{U^{(\ell ms)}} D_{ms}^{(\ell)}) \geq 1 - \epsilon.
\]
Using the Chernoff bound for the indicator functions $1_{U^\ell_{ms}}$ evaluated at all points in $T_Q^n$ (lemma 2.3 and following remarks), we obtain
\[
\Pr\{\epsilon\text{-evenness}\} \geq 1 - |X|^n \exp\left(-L M S \frac{\epsilon^2}{2 \ln 2 |T_Q^n|}\right).
\] (2.2)

Proposition 2.4 gives us (observing $MS \leq |X|^n$), for every $\delta > 0$ and sufficiently large $n$,
\[
\Pr\{\epsilon\text{-secrecy}\} \geq 1 - 2d^n |X|^n \exp\left(-S \epsilon^2 \frac{288}{228 \ln 2}\right),
\] (2.3)

with $\log \iota = -I(Q; \rho^E) - \delta$.

Finally, proposition 2.5 yields, for every $\delta > 0$ and if $MS \leq \exp(n(I(Q; \rho^E) - I(Q; \rho^B) - 3\delta))$ ($n$ sufficiently large),
\[
\forall \ell \Pr\{C_\ell \text{ $\epsilon$-good}\} \geq 1 - \epsilon.
\] (2.4)

Since the individual events in this equation are independent, another application of the Chernoff bound (to the indicator function of '\(\epsilon\)-goodness') gives
\[
\Pr\{A \text{ fraction } 1 - 2\epsilon \text{ of the } C_\ell \text{ is } \epsilon\text{-good}\} \geq 1 - \exp\left(-L \epsilon^2 \frac{4}{2 \ln 2}\right).
\] (2.5)

Thus, if we pick
\[
S = \exp[n(I(Q; \rho^E) + 2\delta)],
M = \exp[n(I(Q; \rho^B) - I(Q; \rho^E) - 3\delta)],
L = \exp[n(H(Q) - I(Q; \rho^B) + 2\delta)],
\]
and observing $|T_Q^n| \leq \exp(n H(Q))$, the right-hand sides of equations (2.2), (2.3) and (2.5) converge to 1 as $n \to \infty$, and hence by the union bound also the conjunction of these three events approaches unit probability asymptotically.

Thus, for sufficiently large $n$, there exist codewords $u_{(ms)}^{(\ell)} \in T_Q^n$ which together have the property of $\epsilon$-evenness, $\epsilon$-secrecy and that a fraction of at least $1 - 2\epsilon$ of the $C_\ell = (u_{(ms)}^{(\ell)})_{m,s}$ is $\epsilon$-good. Clearly, we can construct such code sets for all types $Q$, of which there are at most $(n + 1)|X|$ many.

Now, the protocol works as follows: on observing $x^n$ from the source, Alice determines its type $Q$ and sends it to Bob. If $x^n$ is not typical, i.e. if $\|P - Q\|_1 > \delta$, the protocol aborts here. Otherwise she selects a random $\ell$ such that $x^n$ is a codeword of $C_\ell$, as well as random $m, s$ such that $u_{(ms)}^{(\ell)} = x^n$. (The latter choice of course is unique most of the time: if $C_\ell$ is a good code, only a fraction of at most $\epsilon$ of the codewords have a collision.) She also informs Bob of $\ell$; if $C_\ell$ is not $\epsilon$-good, the protocol aborts.

Note that by the $\epsilon$-evenness of the codewords, the state of $ABE$ conditional on $Q$ and $\ell$ is
\[
\frac{1}{MS} \sum_{m,s} (1 \pm \epsilon) |ms\rangle \langle ms|^A \otimes \rho_{u_{(ms)}^{(\ell)}}^B.
\] (2.6)

(By way of notation, ‘$1 \pm \epsilon$’ stands for any number in the interval $[1 - \epsilon; 1 + \epsilon]$.) Now, since $C_\ell$ is a good code, Bob can apply the decoding POVM $D^{(\ell)}$ to his part of the
Distillation of secret key and entanglement from quantum states

system, and transform the state in equation (2.6) into a state $\theta$ with the property

$$\frac{1}{2} \left\| \theta - \frac{1}{MS} \sum_{m,s} |ms\rangle\langle ms|^{A} \otimes |ms\rangle\langle ms|^{B} \otimes \rho^{E}_{u(lms)} \right\|_{1} \leq 2\epsilon.$$ 

Both Alice and Bob measure $m$ and end up with a perfectly uniformly distributed key of length $n$

$$n(I(Q; \rho^{B}) - I(Q; \rho^{E}) - 3\delta) \geq n(I(P; \rho^{B}) - I(P; \rho^{E}) - 3\delta - \delta'),$$

with probability $1-3\epsilon$, where

$$\delta' = 2\delta \log(d_{A}d_{B}d_{E}) + 2\tau(\delta),$$

with the dimensions $d_{B}$ and $d_{E}$ of Bob’s and Eve’s local systems, respectively. (Recall that $Q$ is typical, and use the Fannes inequality, stated in Appendix B as lemma B 1.) By the above property of $\theta$, Alice and Bob disagree with probability $\leq \epsilon$.

Finally, thanks to the $\epsilon$-secrecy, for all $\ell$ and $m$,

$$\left\| \frac{1}{S} \sum_{s} \rho^{E}_{u(lms)} - \sigma(Q) \right\|_{1} \leq \epsilon,$$

so Eve’s state after the protocol (including her knowledge of $Q$ and $\ell$) is almost constant, whatever the value of $m$. $\blacksquare$

Remark 2.2. The communication cost of the protocol described in the above proof is asymptotically

$$H(X) - I(X; B) = H(A \mid B)$$

bits of forward communication (per copy of the state): the information which code $C_{\ell}$ to apply from Alice to Bob.

Here we give the facts we use in the proof.

Lemma 2.3 (‘operator Chernoff bound’ (Ahlswede & Winter 2002)). Let $X_{1}, \ldots, X_{M}$ be i.i.d. random variables taking values in the operators $B(H)$ on the $D$-dimensional Hilbert space $H$, $0 \leq X_{j} \leq 1$, with $A = \mathbb{E}X_{j} \geq \alpha 1$, and let $0 < \eta < \frac{1}{2}$. Then

$$\Pr \left\{ \frac{1}{M} \sum_{j=1}^{M} X_{j} \notin [(1 - \eta)A; (1 + \eta)A] \right\} \leq 2D \exp \left( -\frac{M \alpha \eta^{2}}{2 \ln 2} \right),$$

where $[A; B] = \{X : A \leq X \leq B\}$ is an interval in the operator order.

Note that for the case $D = 1$ this reduces to the classical Chernoff bound for bounded real random variables (Chernoff 1952). Also the case of finite vectors of bounded real random variables is included by considering the matrices with vector entries on the diagonal and zero elsewhere. It is essential in the proof of the following result.

Proposition 2.4. For a classical–quantum-channel $W : X \to S(\mathcal{H})$ and a type $P$, let $U^{(i)}$ be i.i.d. according to the uniform distribution on the type class $\mathcal{T}^n_P$, $i = 1, \ldots, M$. Define the state
\[
\sigma(P) = \frac{1}{|\mathcal{T}^n_P|} \sum_{x^n \in \mathcal{T}^n_P} W^n_{x^n} = E W^n_{U^{(i)}}.
\]
Then for every $\epsilon, \delta > 0$, and sufficiently large $n$,
\[
\Pr\left\{ \left\| \frac{1}{M} \sum_{j=1}^M W^n_{U^{(j)}} - \sigma(P) \right\|_1 \geq \epsilon \right\} \leq 2d^n \exp\left( -M\epsilon^2 \frac{288 \ln 2}{n} \right),
\]
with $\log \epsilon = I(P; W) - \delta$.

Proof. The proof is very close to that of the compression theorem for POVMs (Winter 2004). We reproduce a version of the argument in Appendix C. \hfill \blacksquare

Proposition 2.5 (HSW theorem). Consider a cq-channel $W : X \to S(\mathcal{H})$ and a type $P$, and let $U^{(i)}$ be i.i.d. according to the uniform distribution on the type class $\mathcal{T}^n_P$, $i = 1, \ldots, N$. Then for every $\epsilon, \delta > 0$ and sufficiently large $n$, if $\log N \leq n(I(P; W) - \delta)$,
\[
\Pr\{ \mathcal{C} = (U^{(i)})^N_{i=1} \text{ is } \epsilon\text{-good} \} \geq 1 - \epsilon.
\]
Here we call a collection of codewords $\epsilon$-good if there exists a POVM $(D_i)_{i=1}^N$ on $\mathcal{H}^\otimes n$ such that
\[
\frac{1}{N} \sum_{i=1}^N \text{Tr}(W^n_{U^{(i)}} D_i) \geq 1 - \epsilon.
\]

Proof. This really is only a slight modification of the Holevo-Schumacher-Westmoreland argument (Holevo 1998; Schumacher & Westmoreland 1997); we give the proof in Appendix C. \hfill \blacksquare

This coding theorem puts us in a position in which to prove the following formula for the one-way secret-key distillation capacity of a cqq-state.

Theorem 2.6. For every cqq-state $\rho$,
\[
K_{-}\rho = \lim_{n \to \infty} \frac{1}{n} K^{(1)}(\rho^\otimes n),
\]
with
\[
K^{(1)}(\rho) = \max_{T[U]} [I(U; B \mid T) - I(U; E \mid T)],
\]
where the maximization runs over all random variables $U$ depending on $X$ and $T$ depending on $U$, i.e. there are channels $Q$ and $R$ such that $U = Q(X)$ and $T = R(U)$, and the above formula refers to the state
\[
\omega^{TUABE} = \sum_{t,u,x} R(t \mid u)Q(u \mid x)P(x)[t|x]^{T} \otimes |u\rangle\langle u|^{U} \otimes |x\rangle\langle x|^{A} \otimes \rho^{BE}.
\]
The ranges of $U$ and $T$ may be taken to have cardinalities $|T| \leq |X|$ and $|U| \leq |X|^2$, and furthermore $T$ can be taken a (deterministic) function of $U$. 

Distillation of secret key and entanglement from quantum states

Proof. Let us begin with the converse part, i.e. the inequality ‘\(\leq\)’: consider an \((n, \epsilon)\)-protocol with rate \(R\); then by its definition, and using the standard quantum data-processing inequality (in the second and third line below) and the Fannes inequality (lemma B.1) several times,

\[
nR \leq H(K) + n(\tau(\epsilon) + \epsilon R + \epsilon F)
\]

\[
\leq I(K; K') + n(2\tau(\epsilon) + \epsilon R + \epsilon F)
\]

\[
\leq I(K; B\Lambda) - I(K; E\Lambda) + n(3\tau(\epsilon) + \epsilon R + 2\epsilon F + \epsilon \log d_E)
\]

\[
= I(K; B\Lambda) - I(K; E\Lambda) + n\delta.
\]

Letting \(U = (K, \Lambda)\) and \(T = \Lambda\) we obtain

\[
R \leq \frac{1}{n} K^{(1)}(\rho^{\otimes n}) + \delta,
\]

with arbitrarily small \(\delta\) as \(n \to \infty\).

The proof of the properties of \(U\) and \(T\) is the same as that for the classical analogue in Ahlswede & Csiszár (1993): it is an application of Carathéodory’s theorem, lemma B.4; see also Csiszár & Körner (1981), where the technique is explained as ‘support lemma’. In Appendix C the analogous fact in theorem 2.8 is proved in full: the statement wanted here is obtained by replacing the convex set of density operators there by the probability simplex.

Now we come to the proof of the direct part, i.e. the inequality ‘\(\geq\)’: it is clearly sufficient to show that, for given \(U\) and \(T\), the rate \(R = I(U; B | T) - I(U; E | T)\) is achievable. To this end, consider a protocol where Alice generates \(U\) and \(T\) for each copy of the state i.i.d. and broadcasts \(T\): this leaves Alice, Bob and Eve in \(n\) copies of \(\tilde{\rho} = \sum_t R(t | u) Q(u | x) P(x | u) |u\rangle^A \otimes \rho_x^{BE} \otimes |t\rangle^B \otimes |t\rangle^E\).

Observing

\[
R = I(U; BB') - I(U; EE'),
\]

we can invoke theorem 2.1, and are done.

Remark 2.7. Comparing this with the classical analogue in Ahlswede & Csiszár (1993), it is a slight disappointment to see that here we do not get a single-letter formula. The reader may want to verify that the technique used in Ahlswede & Csiszár (1993) to single-letterize the upper bound does not work here, as it introduces conditioning on quantum registers, while our \(T\) has to be classical.

One can clearly also use a general three-party state \(\rho^{ABE}\) to generate secret key between Alice and Bob; a particular strategy certainly is for Alice to perform a quantum measurement described by the POVM \(Q = (Q_x)_{x \in X}\), which leads to the state

\[
\tilde{\rho}^{ABE} = \sum_x |x\rangle\langle x|^A \otimes \text{Tr}_A(\rho^{ABE}(Q_x \otimes 1^{BE})).
\]

Then, starting from many copies of the original state \(\rho\), we now have many copies of \(\tilde{\rho}\), and theorem 2.6 can be applied. Because we can absorb the channel \(U | X\) into the POVM, we obtain the direct part (‘\(\geq\)’) in the following statement.
Theorem 2.8. For every state $\rho^{AEB}$,

$$K_\to(\rho) = \lim_{n \to \infty} \frac{1}{n} K^{(1)}(\rho^{\otimes n}),$$

with

$$K^{(1)}(\rho) = \max_{Q,T} \left[ I(X;B \mid T) - I(X;E \mid T) \right],$$

where the maximization is over all POVMs $Q = (Q_x)_{x \in X}$ and channels $R$ such that $T = R(X)$, and the information quantities refer to the state $\omega^{TA_{\ell \ell'}BE} = \sum_{t,x} R(t \mid x) P(x) |t\rangle \langle t| \otimes |x\rangle \langle x|^{A_{\ell \ell'}} \otimes \Tr_A(\rho^{ABE}(Q_x \otimes 1^{BE})).$

The range of the measurement $Q$ and the random variable $T$ may be assumed to be bounded as follows: $|T| \leq d_A^2$ and $|X| \leq d_A^4$, and furthermore $T$ can be taken a (deterministic) function of $X$.

Proof. After our remarks preceding the statement of the theorem, we have only the converse to prove. This will look very similar to the converse of theorem 2.6. Even though we have not so far defined what a key distillation protocol is in the present context, we can easily do that now (and check that the procedure above is of this type): it consists of a measurement POVM $Q = (Q_{\ell m})_{L,M}^{\ell,m}$ for Alice and the POVMs $D^{(\ell)}$ for Bob, with the same conditions (1)–(3) as in the first paragraphs of this section, where as before we assume a rate bound on the public discussion: $L \leq 2^{nF}$. This obviously generalizes the definition we gave for cqq-states.

Consider an $(n,\epsilon)$-protocol with rate $R$; once more using the standard quantum data-processing inequality and the Fannes inequality, lemma B 1, we can estimate as follows:

$$nR \leq H(K) + n(\tau(\epsilon) + \epsilon R)$$

$$\leq I(K;K'|\Lambda) + n(2\tau(\epsilon) + \epsilon R + \epsilon F)$$

$$\leq I(K;B|A) + n(2\tau(\epsilon) + \epsilon R + \epsilon F)$$

$$\leq I(K;B|A) - I(K;E|A) + n(3\tau(\epsilon) + \epsilon R + 2\epsilon F + \epsilon \log d_E)$$

$$= I(K;B|A) - I(K;E|A) + n\delta.$$  

The measurement $Q$ and $T(\ell, m) = \ell$ are permissible in the definition of $K^{(1)}$, hence we obtain

$$R \leq \frac{1}{n} K^{(1)}(\rho^{\otimes n}) + \delta,$$

with arbitrarily small $\delta$ as $n \to \infty$.

It remains to prove the bounds on the range of $X$ and $T$ for which we imitate the proof of the corresponding statement in theorem 2.6: the full argument is given in Appendix C.

Remark 2.9. Clearly, the worst case for Alice and Bob is when Eve holds the system $E$ of a purification $|\psi^{ABE}\rangle$ of $\rho^{AB}$, because clearly every other extension $\rho^{ABE}$ of $\rho^{AB}$ can be obtained from the purification by a quantum operation acting on $E$.  

Distillation of secret key and entanglement from quantum states

Our result (at least in principle) characterizes those bipartite states $\rho^{AB}$ for which one-way key distillation is possible at a positive rate. We have to leave open the question of characterizing the states for which positive rates can be obtained by general two-way public discussion (cf. the classical case (Ahlswede & Csiszár 1993; Maurer 1993; Maurer & Wolf 1996; Renner & Wolf 2003)!).

Note that the classical analogue of the ‘worst case’ is Eve’s having total knowledge about both Alice’s and Bob’s random variables—which makes key distillation totally impossible. For quantum states thus, it must be some ‘non-classical’ correlation which makes positive rates possible; it is tempting to speculate that a manifestation of entanglement is behind this effect.

We do not fully resolve this issue in the present paper; nevertheless, in a similar vein, we show in the following section that if $\rho^{AB}$ allows one-way distillation of EPR pairs at positive rates, then our cryptographic techniques give a construction of an entanglement distillation protocol by a modification of key distillation protocols of a particular form.

3. One-way entanglement distillation

Consider an arbitrary state $\rho^{AB}$ between Alice and Bob. In Bennett et al. (1996a,b) the task of distilling EPR pairs at optimal rate from many copies of $\rho$, via local operations and classical communication (LOCC), was introduced (see also Rains 1999).

A one-way entanglement distillation protocol consists of

(i) a quantum instrument $T = (T_\ell)_{\ell=1}^L$ for Alice (an instrument (Davies & Lewis 1970) is a quantum operation with both classical and quantum outputs—it is modelled in general as a cp-map valued measure; for our purposes it is a finite collection of cp-maps which sum to a ctp map);

(ii) a quantum operation $R_\ell$ for Bob for each $\ell$.

We call it an $(n, \epsilon)$-protocol if it acts on $n$ copies of the state $\rho$ and produces a maximally entangled state

$$ |\Phi_M \rangle = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} |m\rangle^A \otimes |m\rangle^B $$

up to fidelity $1 - \epsilon$:

$$ F\left(\Phi_M, \sum_{\ell=1}^{L} (T_\ell \otimes R_\ell)(\rho^{\otimes n})\right) \geq 1 - \epsilon. $$

Note that we may assume without loss of generality that $T_\ell$ and $R_\ell$ output states supported on the reduced states of $\Phi_M$ on Alice’s and Bob’s system, respectively: otherwise we could improve the fidelity.

A number $R$ is an achievable rate if there exist, for every $n$, $(n, \epsilon)$-protocol, with $\epsilon \to 0$ and

$$ \frac{1}{n} \log M \to R \quad \text{as} \quad n \to \infty. $$

Finally, 

\[ D_{\to}(\rho) := \sup \{ R : R \text{ achievable} \} \]

is the one-way (or forward) entanglement capacity of \( \rho \).

In Bennett et al. (1996b) the case of Bell-diagonal two-qubit states,

\[ \rho = p_{00}\Phi^+ + p_{01}\Phi^- + p_{10}\Psi^+ + p_{11}\Psi^-, \]

was considered and proved that \( D_{\to}(\rho) \geq 1 - H(\{p\}) \), by a method called ‘hashing protocol’ (this was generalized recently to higher dimensions in Vollbrecht & Wolf (2003)). Concerning lower bounds, not much more is known, but there are numerous works dealing with upper bounds on the distillable entanglement: the entanglement of formation \( E_F(\rho) \) (Bennett et al. 1996b), the relative entropy of entanglement \( E_{re}(\rho) \) (Vedral et al. 1997), the Rains bound \( R(\rho) \) (Rains 2001) (which may well equal the regularized relative entropy of entanglement with respect to the states with positive partial transpose, see Audenaert et al. 2002)), and the recently introduced squashed entanglement \( E_{sq}(\rho) \) (Christandl & Winter 2004).

To connect to the cryptographic setting discussed so far, construct a purification \( |\psi\rangle_{ABE} \) of \( \rho \), of which we are particularly interested in its Schmidt form

\[ |\psi\rangle_{ABE} = \sum_x \sqrt{P(x)} |x\rangle_A \otimes |\psi_x\rangle_{BE}. \]

Consider the following special strategy for a one-way secret-key distillation protocol, on \( n \) copies of the state. Alice measures \( x^n \) (i.e. the above Schmidt basis), and applies the secret-key distillation protocol from theorem 2.1: it is easy to evaluate the key rate

\[ I(\rho; |\psi^B\rangle) - I(\rho; |\psi^E\rangle) = H(B) - H(E) = H(B) - H(AB). \]

By letting Alice and Bob execute this protocol ‘coherently’, we can prove the following theorem.

**Theorem 3.1 (Hashing inequality).** For any state \( \rho^{AB} \),

\[ D_{\to}(\rho) \geq H(\rho^B) - H(\rho^{AB}). \]

The right-hand side here equals the negative conditional von Neumann entropy, \(-H(A | B)\), a quantity known as coherent information (Barnum et al. 1998; Schumacher et al. 1996; Schumacher & Nielsen 1996), which we denote (acknowledging its directionality) \( I_c(A;B) \). If the state this is referring to is not apparent from the context, we add it in subscript: \( I_{c}(A;B)_{\rho} \).

**Proof.** Recall the structure of the protocol in the proof of theorem 2.1: for each typical type \( Q \) we have a collection of codewords \( u^{(\ell ms)} \), \( \ell = 1, \ldots, L, \ m = 1, \ldots, M \) and \( s = 1, \ldots, S \) from \( T_Q \) satisfying \( \epsilon \)-evenness, \( \epsilon \)-secrecy, and a fraction of at least \( 1 - 2\epsilon \) of the codes \( C_\ell = (u^{(\ell ms)})_{m,s} \) are \( \epsilon \)-good.

The first step of the protocol is that Alice measures the type \( Q \) non-destructively and informs Bob about the result. The protocol aborts if \( Q \) is not typical, i.e. if \( \|P - Q\|_1 > \delta \). This leaves the post-measurement state

\[ \sqrt{\frac{1}{|T_Q|}} \sum_{x^n \in T_Q} |x^n\rangle_A \otimes |\psi_{x^n}\rangle_{BE}. \]
Distillation of secret key and entanglement from quantum states

Define now a quantum operation for Alice, with Kraus elements

\[ C_\ell = \sqrt{\frac{1}{1 + \epsilon}} LMS \sum_{m,s} |ms\rangle\langle u(\ell ms)|, \]

\[ C_\emptyset = \sqrt{1 - \sum_Q C_\ell^\dagger C_\ell}, \]

which we interpret as an instrument with outcomes \( \ell \) and \( \emptyset \) (Davies & Lewis 1970): \( T_\ell(\sigma) = C_\ell \sigma C_\ell^\dagger \). This outcome is communicated to Bob. The fact that these are really permissible Kraus operators we obtain from the \( \epsilon \)-evenness condition.

The outcome \( \emptyset \) (resulting in abortion of the protocol) is observed with probability at most \( \epsilon \) (if \( n \) is large enough). The other outcomes \( \ell \) all occur with the probability

\[ \gamma(Q) = P^\otimes_n (T^\emptyset_Q) \frac{1}{1 + \epsilon} L, \]

in which case the output state of the instrument is

\[ \sqrt{\frac{1}{MS} \sum_{ms} |ms\rangle^A \otimes |\psi_{u(\ell ms)}^{(\ell ms)}\rangle_{BE}}. \]

(3.1)

(The absence of the \( 1 \pm \epsilon \) factors when compared with the analogous equation (2.6) in the proof of theorem 2.1 is due to our having introduced the error event \( \emptyset \).)

Now, just as in the proof of theorem 2.1, Bob decodes \( m \) and \( s \), at least if \( C_\ell \) is \( \epsilon \)-good (which fails to happen with probability only \( 2\epsilon \)). But he does it coherently, by introducing an ancilla system \( B' \) in a standard state \( |0\rangle \) and applying a unitary to extract \( ms \) into \( B \), leaving in \( B' \) whatever is necessary to make the map unitary. This transforms the state in equation (3.1) into a state

\[ |\emptyset\rangle_{ABB'E} = \sqrt{\frac{1}{MS} \sum_{ms} |ms\rangle^A \otimes (\sqrt{1 - e_{ms}} |ms\rangle^B |\varphi_{\ell ms}^{OK}\rangle_{B'E} + \sqrt{e_{ms}} |\varphi_{\ell ms}^{bad}\rangle_{BB'E})}, \]

where \( e_{ms} \) is the probability of the code incorrectly identifying \( ms \), and \( |\varphi_{\ell ms}^{bad}\rangle \) is orthogonal to \( |ms\rangle |\varphi_{\ell ms}^{OK}\rangle \). Now, because the code is \( \epsilon \)-good,

\[ F\left(|\emptyset\rangle, \sqrt{\frac{1}{MS} \sum_{ms} |ms\rangle^A \otimes |ms\rangle^B |\varphi_{\ell ms}^{OK}\rangle_{B'E} \right) \geq 1 - \sqrt{\epsilon} \geq 1 - 2\sqrt{\epsilon}, \]

where we have used the Markov inequality: at most a fraction of \( \sqrt{\epsilon} \) of the \( e_{ms} \) can be larger than \( \sqrt{\epsilon} \). Since the decoding only affects Bob’s registers, but certainly not \( E \), we have

\[ (1 - e_{ms}) |\varphi_{\ell ms}^{OK}\rangle_{E} + e_{ms} |\varphi_{\ell ms}^{bad}\rangle_{E} = |\psi_{u(\ell ms)}^{E}\rangle, \]

and hence we can assume that

\[ |\varphi_{\ell ms}^{OK}\rangle_{E} |\psi_{u(\ell ms)}^{E}\rangle \geq \sqrt{1 - e_{ms}}. \]

(3.2)

This implies

\[ F\left(|\emptyset\rangle, \sqrt{\frac{1}{MS} \sum_{ms} |ms\rangle^A \otimes |ms\rangle^B |\psi_{u(\ell ms)}^{E}\rangle_{B'E} \right) \geq 1 - 3\sqrt{\epsilon}. \]

(3.3)

At this point, Alice and Bob almost have their maximal entanglement of the $m$-variable. All that remains is to be done is to disentangle Eve.

To begin, Alice measures the $s$-component of her register in the Fourier-transformed basis:

$$|\hat{t}\rangle = \frac{1}{\sqrt{S}} \sum_{s=1}^{S} e^{2\pi i st/S} |s\rangle,$$

and tells Bob the result $t$. He then applies the phase shift

$$\sum_{s=1}^{S} e^{2\pi i st/S} |s\rangle\langle s|$$

to the $s$-component of his register $B$. This transforms $|\vartheta\rangle_{AB'BE}$ into a state $|\Theta\rangle_{AB'BE}$ with

$$F\left(|\Theta\rangle, \sqrt{\frac{1}{MS}} \sum_{m} |m\rangle^A \otimes |m\rangle^B |\psi_{t_{\text{rms}}}^E\rangle^B \right) \geq 1 - 3\epsilon,$$

invoking the non-decrease of the fidelity under quantum operations, applied to equation (3.3).

Absorbing $s$ into the register $B'$, the right hand state in the last equation can be rewritten as

$$\frac{1}{\sqrt{M}} \sum_{m} |m\rangle^A \otimes |m\rangle^B |\tilde{\psi}_{t_{\text{rms}}}^E\rangle^B,$$

with

$$|\tilde{\psi}_{t_{\text{rms}}}^E\rangle^B = \frac{1}{\sqrt{S}} \sum_{s} |s\rangle^B |\hat{\psi}_{u(t_{\text{rms}})}^E\rangle^B.$$

The reduced states of Eve of the $|\tilde{\psi}_{t_{\text{rms}}}^E\rangle^B$ is

$$\sigma_{t_{\text{rms}}} = \frac{1}{S} \sum_{s} |\tilde{\psi}_{u(t_{\text{rms}})}^E\rangle^B,$$

where we made use of equation (3.2), which is, by the $\epsilon$-secrecy, at trace distance at most $\epsilon$ from a state we denoted $\sigma(Q)$ in the proof of theorem 2.1. By lemma B 2 in Appendix B, $F(\sigma_{t_{\text{rms}}}, \sigma(Q)) \geq 1 - \epsilon$.

Choosing a purification $|\zeta\rangle^B$ of $\sigma(Q)$, this means that there are unitaries $U_{t_{\text{rms}}}$ on $B'$ such that

$$F\left((U_{t_{\text{rms}}} \otimes 1) |\tilde{\psi}_{t_{\text{rms}}}^E\rangle, |\zeta\rangle\right) \geq 1 - \epsilon,$$

because the mixed-state fidelity equals the maximum pure-state fidelity over all purifications of the states and all purifications are related by unitaries on the purifying system (Jozsa 1994; Uhlmann 1976). Hence, if Bob applies

$$U := \sum_{m} |m\rangle\langle m| \otimes U_{t_{\text{rms}}}$$

to his share of the state, then the state in equation (3.5) is transformed into a state $|\Xi\rangle^{AB'BE}$ with

$$F\left(|\Xi\rangle, \frac{1}{\sqrt{M}} \sum_{m} |m\rangle^A \otimes |m\rangle^B |\zeta\rangle^{B'} \right) \geq 1 - \epsilon.$$
Distillation of secret key and entanglement from quantum states

Of course, he actually works on $|\Theta\rangle$, so they end up with the state $(\mathbb{1} \otimes U \otimes \mathbb{1})|\Theta\rangle$, which has fidelity $1 - 3\sqrt{\epsilon}$ to $|\Xi\rangle$; hence, with equation (3.4) we conclude (by simple geometry) that it has fidelity $\geq 1 - 12\sqrt{\epsilon}$ to $|{\Phi}_M^{AB} \otimes |\zeta\rangle^E\rangle$.

Non-typical $Q$, the event $\emptyset$ or bad code $C_t$, happen with total probability at most $4\epsilon$. In the ‘good’ case, Alice and Bob distill, up to fidelity $1 - 2\sqrt{\epsilon}$, a maximally entangled state of log Schmidt rank

$$|I(Q; \psi^B) - I(Q; \psi^E) - 3\delta| \geq n(I(P; \psi^B) - I(P; \psi^E) - 3\delta - \delta') = n(H(B) - H(E) - 3\delta - \delta').$$

with $\delta'$ just as at the end of the proof of theorem 2.1.

\textbf{Remark 3.2.} The communication cost of the above protocol is asymptotically

$$H(A) - I(X; B) + I(X; E) = H(A) - H(B) + H(E)$$

$$= H(A) + H(E) - H(AE)$$

$$= I(A : E)$$

bits of forward classical communication per copy of the state: the information which code $C_t$ to use, plus the information from the measurement of the Fourier-transformed basis $|\tilde{t}\rangle_t$.

Even though at first sight there seems to be little reason to believe that our procedure is optimal for this resource (consider, for example, a separable initial state: Alice will have mutual information with a purification but clearly the best thing is to do nothing), it is amusing to see the quantum mutual information show up here.

It is in fact possible to show that subject to another optimization, the quantum mutual information between Alice and Eve, indeed gives the minimum forward communication cost (Devetak \textit{et al.} 2005).

\textbf{Example 3.3.} It is interesting to compare our method with the original hashing protocol of Bennett \textit{et al.} (1996b), for the case of mixtures of Bell states

$$\rho = \sum_{i,j=0}^{1} p_{ij} |\Phi_{ij}\rangle,$$

with the numbering of the Bell states introduced in Bennett \textit{et al.} (1996b):

$$|\Phi_{00}\rangle = |\Phi^+\rangle, \quad |\Phi_{01}\rangle = |\Phi^-\rangle, \quad |\Phi_{10}\rangle = |\Psi^+\rangle, \quad |\Phi_{11}\rangle = |\Psi^-\rangle.$$

The purification we use in the proof reads

$$|\psi\rangle^{ABE} = \sum_{i,j=0}^{1} \sqrt{p_{ij}} |\Phi_{ij}\rangle^{AB} \otimes |ij\rangle^E$$

$$= \frac{1}{\sqrt{1}}(|0\rangle^A |\psi_0\rangle^{BE} + |1\rangle^A |\psi_1\rangle^{BE}),$$

with

$$|\psi_0\rangle^{BE} = \sqrt{p_{00}} |0\rangle^E |0\rangle^E + \sqrt{p_{01}} |0\rangle^E |1\rangle^E + \sqrt{p_{10}} |1\rangle^E |0\rangle^E + \sqrt{p_{11}} |1\rangle^E |1\rangle^E,$$

$$|\psi_1\rangle^{BE} = \sqrt{p_{00}} |1\rangle^E |0\rangle^E - \sqrt{p_{01}} |1\rangle^E |1\rangle^E + \sqrt{p_{10}} |0\rangle^E |1\rangle^E - \sqrt{p_{11}} |0\rangle^E |1\rangle^E.$$
Note that this is indeed a Schmidt decomposition. First of all, the communication cost of our protocol evaluates (using the symmetry between $A$ and $B$)

\[
I(A : E) = I(B : E) = H(B) + H(E) - H(BE) = 1 + H(\{p\}) - 1 = H(\{p\}),
\]

which is the same as in Bennett et al. (1996b). But the way of the hashing protocol is to ‘hash’ information about the identity of the state in the Bell ensemble into approximately $nH(\{p\})$ of the states, which then are measured locally and the results communicated. Our protocol in contrast has two very distinct communication parts: there is the ‘code information’ (which amounts to error correction between Alice and Bob, with built-in privacy amplification for Eve’s information about the basis state), and there is the ‘phase information’ from the measurement in the Fourier transformed basis. The first amounts to

\[
H(X) - I(X; \psi^B) = H(p_{00} + p_{01}, p_{10} + p_{11}),
\]

while the second is

\[
I(X; E) = H(\{p\}) - H(p_{00} + p_{01}, p_{10} + p_{11}).
\]

Our result leads to the general formula for one-way distillable entanglement.

**Theorem 3.4.** For any bipartite state $\rho^{AB}$,

\[
D_{\rightarrow}(\rho) = \lim_{n \to \infty} \frac{1}{n} D^{(1)}(\rho^{\otimes n}),
\]

with

\[
D^{(1)}(\rho) := \max_T \sum_{\ell=1}^L \lambda_{\ell} I_c(A)B_{\rho_{\ell}},
\]

where the maximization is over quantum instruments $T = (T_1, \ldots, T_L)$ on Alice’s system,

\[
\lambda_{\ell} = \text{Tr} T_\ell(\rho^A) \quad \text{and} \quad \rho_{\ell} = \frac{1}{\lambda_{\ell}} (T_\ell \otimes \text{id})(\rho).
\]

The range of $\ell$ can be assumed to be bounded, $L \leq d_A^2$, and moreover each $T_\ell$ can be assumed to have only one Kraus operator: $T_\ell(\sigma) = A_\ell \sigma A_\ell^\dagger$.

**Proof.** First, for the direct part, it is sufficient to consider an instrument $T$ on one copy of the state: if Alice performs the instrument $T$ on each copy and communicates the result to Bob, they end up with the new state

\[
\tilde{\rho} = \sum_{\ell} \lambda_{\ell} \rho_{\ell}^{AB} \otimes |\ell\rangle\langle\ell|^{B'}.
\]

Observe that

\[
I_c(A)B_{\tilde{\rho}} = \sum_{\ell} \lambda_{\ell} I_c(A)B_{\rho_{\ell}};
\]

thus application of theorem 3.1 to $\tilde{\rho}$ gives achievability.
For the converse, consider any one-way distillation protocol with rate $R$, and denote Alice’s instrument by $T = (T_\ell)_{\ell}$ and Bob’s quantum operations by $R_\ell$. Write

$$\Omega = \sum_\ell (T_\ell \otimes R_\ell)(\rho^{\otimes n}) =: \sum_\ell \lambda_\ell \Omega_\ell.$$ 

Then, using Fannes’s inequality lemma B 1, the convexity of the coherent information in the state (Lieb & Ruskai 1973) and quantum data processing (Barnum et al. 1998; Schumacher et al. 1996; Schumacher & Nielsen 1996),

$$nR \leq H(\Omega^B) - H(\Omega^{AB}) + 2n(\tau(\epsilon) + \epsilon R)$$

$$= \sum_\ell \lambda_\ell I_c(A)B_{\Omega_\ell} + 2n(\tau(\epsilon) + \epsilon R)$$

$$\leq \sum_\ell \lambda_\ell I_c(A)B_{\omega_\ell} + 2n(\tau(\epsilon) + \epsilon R),$$

where

$$\omega_\ell = \frac{1}{\lambda_\ell} (T_\ell \otimes \text{id})(\rho^{\otimes n}).$$

Hence we get

$$R \leq \frac{1}{n} D^{(1)}(\rho^{\otimes n}) + \delta',$$

with arbitrarily small $\delta'$ as $n \to \infty$, and we are done.

As for the bound on $L$ and the structure of $T$, observe that if one $T_\ell$ has more than one Kraus element, one can decompose $T_\ell(\sigma)$ into a sum of terms $A_{\ell j} \sigma A_{\ell j}$; for the corresponding probabilities $\lambda_\ell = \sum_j \lambda_{\ell j}$ and for the post-measurement states $\lambda_\ell \rho_\ell = \sum_j \lambda_{\ell j} \rho_{\ell j}$. Then by the convexity of $I_c$ in the state (Lieb & Ruskai 1973),

$$\lambda_\ell I_c(A)B_{\rho_\ell} \leq \sum_j \lambda_{\ell j} I_c(A)B_{\rho_{\ell j}}.$$

By the polar decomposition and invariance of $I_c$ under local unitaries we may further assume that $A_\ell \geq 0$, i.e.

$$A_\ell = \sqrt{A_\ell^\dagger A_\ell};$$

in this form the whole instrument is actually described by the POVM $(A_\ell^2)_{\ell}$, and each POVM corresponds to an instrument by taking as the $A_\ell$ the square roots of the POVM operators.

Now, invoking a theorem of Davies (1978) (which actually is another application of Carathéodory’s theorem, lemma B 4), any POVM is a convex combination of extremal POVMs, which have at most $d_A^2$ non-zero elements each, and this convex decomposition clearly carries over to the instruments: $T = \sum_j \pi_j T_j$. Since then $\sum_\ell \lambda_\ell I_c(A)B_{\rho_\ell}$ is the same convex combination of similar such terms for the instruments $T_j$, at least one of these gives a higher yield $\sum_\ell \lambda_{\ell j} I_c(A)B_{\rho_{\ell j}}$. Note that the cp-maps of $T_j$ are scalar multiples of the $T_\ell$; hence, the output state of $T_j$ with classical result $\ell$ is $\rho_\ell$. ■
4. Quantum and entanglement capacities

Horodecki et al. (2000) have observed that the hashing inequality implies information theoretic formulae for a number of quantum capacities and the distillable entanglement.

In particular, the quantum capacity of a quantum channel, either unassisted or assisted by forward or two-way communication is given by a formula involving coherent information (where we indicate the assisting resource in the subscript):

**Theorem 4.1.** Let $T : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be any quantum channel. Then

$$Q_\emptyset(T) = Q_\rightarrow(T) = \lim_{n \to \infty} \frac{1}{n} \max_{|\psi\rangle} I_c(A^n|B^n)_{\omega},$$

with any pure state $\psi$ on $A^n$ and the state

$$\omega = (\text{id} \otimes T)^\otimes (\psi^{A^n}).$$

Furthermore,

$$Q_{\leftrightarrow}(T) = \lim_{n \to \infty} \frac{1}{n} \sup_{|\psi, V\rangle} I_c(A^n|B^n)_{\omega},$$

with any pure state $\psi$ on $A^n$, two-way LOCC operation $V$ and the state

$$\omega = V[(\text{id} \otimes T)^\otimes (\psi^{A^n})].$$

**Proof.** See Horodecki et al. (2000). The fact that forward communication does not help was proved in Barnum et al. (2000), and that the right-hand side is an upper bound to $Q_\emptyset$ was shown in Schumacher et al. (1996), Schumacher & Nielsen (1996) and Barnum et al. (1998) and Lloyd (1997).

The idea of achievability is to distill the state $\omega$ and then use teleportation—this involves forward communication but either it is free or the whole procedure including the distillation and teleportation uses only forward communication, which by Barnum et al. (2000) can be removed.

In Horodecki et al. (2000) a similar formula (involving a coherent information $I_c(B^n|A)_{\sigma}$) was proposed for the quantum capacity with classical feedback. However, the proof as indicated above does not work in this case. Indeed we may use the back-communication to help distillation, but teleportation needs a forward communication, so we end up with a quantum channel code using two-way classical side communication, which is not known to be reducible to just back-communication; in fact, the results of Bowen (2004) might be taken as indication that for the erasure channel the capacities with feedback and with two-way side communication are different.

We have already given a formula for the distillable entanglement using one-way LOCC in theorem 3.4.

**Theorem 4.2.** For any state $\rho^{AB}$,

$$D(\rho) = \lim_{n \to \infty} \frac{1}{n} \sup_{|\psi, V\rangle} I_c(A^n|B^n)_{\omega},$$

with any two-way LOCC operation $V$ and the coherent information refers to the state $\omega = V(\rho^{\otimes n})$. 

*Proc. R. Soc. A (2005)*
Proof. For the direct part (‘$\geq$’) it is obviously sufficient to consider any two-way LOCC operation $V$ on the bipartite system, which, applied to $\rho$, gives a state $\sigma$. Doing that for $n$ copies of $\rho$, application of theorem 3.1 shows that we can distill EPR pairs at rate $I_c(A'\rangle B')_\sigma$ from this.

Conversely, let $V_0$ be a two-way LOCC producing a state $\Omega$ with $\|\Omega - \Phi_M\|_1 \leq \epsilon$, $nR = \log M$. Without loss of generality we may assume that $\Omega$ is supported within the tensor product of the supports of the reduced states of $\Phi_M$. Thus,

$$nR \leq H(\Omega^B) - H(\Omega^{AB}) + 3n(\epsilon R + \tau(\epsilon))$$

$$= I_c(A\rangle B)_\Omega + 3n(\epsilon R + \tau(\epsilon))$$

$$\leq \sup\{I_c(A\rangle B)_{V(\rho^\otimes n)} : V \text{ two-way LOCC}\} + 3n(\epsilon R + \tau(\epsilon)),$$

and we are done. 

It was shown, furthermore, in Horodecki et al. (2000) that for an ensemble $\{p_i, \rho_i\}$ of bipartite pure states the hashing inequality implies

$$\Delta D := \sum_i p_i D(\rho_i) - D(\rho) \leq \Delta I := H(\rho) - \sum_i p_i H(\rho_i).$$

This inequality was first exhibited in Eisert et al. (2000) for a class of examples, and conjectured to be true in general. Note that the inequality is trivially true (using only concavity of the entropy) for the loss of coherent information on the left-hand side.

(a) History and relation to other work

The coherent information made its appearance in Schumacher et al. (1996), Schumacher et al. (1996), Schumacher & Nielsen (1996), Barnum et al. (1998), where its relation to quantum channel capacity was conjectured and many of its properties proved. Independently, Lloyd (1997) proposed this quantity and a heuristic argument which, however, fell short of a proof of the coding theorem. Only recently Hamada (2002) succeeded in giving a lower bound on quantum channel capacity in terms of coherent information—still with a crucial restriction to stabilizer codes. A full proof was not known until the work of Shor (2004)—but then quite quickly Devetak (2005) discovered a proof based on private information transmission, an idea inspired by the work of Schumacher & Westmoreland (1998).

Regarding entanglement distillation, the hashing inequality appears to have been a folk conjecture from the publication of Bennett et al. (1996b) on. This, however, has received much less attention than the quantum channel coding problem. It was codified as an important conjecture in Horodecki et al. (2000).

While completing the writing of the present paper we learned of the work by Horodecki & Horodecki (2005), in which it is shown that the proof by random coding of the channel capacity theorem can be used to obtain the hashing inequality. It may be interesting to compare the proofs (Horodecki & Horodecki 2005; Shor 2004) for the achievability of the coherent information to ours and that of Devetak (2005). While we, on the face of it, take a detour via secret-key distillation, the final procedure can be argued to be more direct: in particular, we do not require the ‘double blocking’ which in the other approaches seem necessary to reduce to a situation in which Alice’s end is in a maximally mixed state. Thus, presumably, our codes achieve rates approaching the coherent information more quickly, i.e. for smaller block length.
5. Conclusion

Our findings not only transport an existing classical theory of distilling secret key from prior correlation (Ahlswede & Csiszár 1993; Maurer 1993, and follow-up work) to the quantum case, but also link this subject to entanglement distillation in an operational way: a coherent implementation of the basic secret-key distillation protocol yielded an entanglement distillation protocol achieving the coherent information—this then implies information theoretic formulae for distillable entanglement and quantum capacities.

We want to draw the reader’s attention to several questions we have left open. First of all, operationally and from the formulae we have derived, it is clear that $D_\rightarrow \leq K_\rightarrow$. Are there states for which $D_\rightarrow(\rho) < K_\rightarrow(\rho)$? Are there maybe even bound entangled states with positive key rate? A first step might be to find states such that $D^{(1)}(\rho) < K^{(1)}(\rho)$. Note that the potential gap between $K_\rightarrow$ and $D_\rightarrow$ comes from the possibility of having more general measurements at Alice’s side than the complete von Neumann measurement in the Schmidt basis that was our starting point in the proof of theorem 3.1 (actually any complete von Neumann measurement would do). Namely, in key distillation, a viable option for Alice is to discard part of her state (corresponding to using higher-rank POVM elements), but keep that part secret from Eve all the same; while in entanglement distillation, ‘Eve’ is everything except Alice and Bob, so it is as if she would get the parts Alice decided to toss away.

A second group of open questions: in general, the optimizations in theorems 2.6, 2.8 and 3.4 are quite nasty, mostly so because they involve a limit of many copies of the state. In the classical theory of secret-key distillation, a single-letter formula for the optimal one-way key rate is proved in Ahlswede & Csiszár (1993), so there might be hope at least for theorem 2.6. In contrast, the optimal rate of two-way protocols or even a procedure to decide if it is non-zero is still to be found (see the very well-informed reviews by Horodecki (2001) and Horodecki & Horodecki (2001)), which is why we concentrate on one-way protocols for now. It is known that distillability of entanglement may be absent for a single copy of a state, but could appear for collective operations on several copies (see Horodecki & Horodecki 2001, §§6.3, 7.2 and references therein), so there are only limited possibilities for making theorem 3.4 into a single-letter formula. Note in particular that the results of Di Vincenzo et al. (1998); Shor & Smolin (1996)—see also the discussion of Schumacher et al. (1996), Schumacher & Nielsen (1996) and Barnum et al. (1998), where the failure of subadditivity for the coherent information is observed—imply that single-letter maximization of the coherent information will certainly not achieve the optimum distillability. It would therefore be good to have at least an a priori bound on the number $n$ of copies of the state which we have to consider to have $D^{(1)}(\rho^{\otimes n})$ within, say, $\epsilon$ of the optimal rate. In general, good single-letter lower and upper bounds (Christandl & Winter 2004) are still wanted!

Finally, we would like to know what the public/classical communication cost is of distilling secret key and entanglement, respectively, in particular in the one-way scenario (which at any rate seems to be the one open to analysis). More generally, if we limit the amount of communication, can we determine the optimal distillation rates (see Devetak & Winter (2005) for the communication cost of common randomness distillation)? In the entanglement case this should link up with initial efforts to understand the communication cost of various state transformation tasks (Ambainis
Distillation of secret key and entanglement from quantum states


We thank Patrick Hayden and Debbie Leung for a discussion at the conception of this work. After the present research had been done, we learned from Michał and Paweł Horodecki that they had a different approach to proving the hashing inequality, which they were able to carry through after hearing of our result. We thank them for showing us an early draft of their paper (Horodecki & Horodecki 2005). I.D. is supported in part by the NSA under US Army Research Office (ARO) grants DAAG55-98-C-0041 and DAAD19-01-1-06. A.W. is supported by the UK Engineering and Physical Sciences Research Council.

Appendix A. Types and typical subspaces

The following material can be found in most textbooks on information theory, e.g. Cover & Thomas (1991), Csiszár & Körner (1981), or in the original literature on quantum information theory (Jozsa & Schumacher 1994; Schumacher et al. 1995; Schumacher & Westmoreland 1997; Winter 1999).

For strings of length \( n \) from a finite alphabet \( \mathcal{X} \), which we generically denote \( x^n = x_1 \cdots x_n \in \mathcal{X}^n \), we define the type of \( x^n \) as the empirical distribution of letters in \( x^n \); i.e. \( P \) is the type of \( x^n \) if
\[
\forall x \in \mathcal{C} \quad P(x) = \frac{1}{n} |\{ k : x_k = x \}|.
\]

It is easy to see that the total number of types is upper bounded by \((n + 1)^{|\mathcal{X}|}\).

The type class of \( P \), denoted \( T^n_P \), is defined as all strings of length \( n \) of type \( P \). Obviously, the type class is obtained by taking all permutations of an arbitrary string of type \( P \).

The following is an elementary property of the type class:
\[
(n + 1)^{-|\mathcal{X}|} \exp(nH(P)) \leq |T^n_P| \leq \exp(nH(P)),
\]
with the (Shannon) entropy \( H(P) \).

For \( \delta > 0 \), and for an arbitrary probability distribution \( P \), we define the set of \( P \)-typical sequences as
\[
T^n_{P,\delta} := \left\{ x^n : \left| -\frac{1}{n} \log P^\otimes n(x^n) - H(P) \right| \leq \delta \right\}.
\]

By the law of large numbers, for every \( \epsilon > 0 \) and sufficiently large \( n \),
\[
P^\otimes n(T^n_{P,\delta}) \geq 1 - \epsilon.
\]

Furthermore,
\[
|T^n_{P,\delta}| \leq \exp(n(H(P) + \delta)),
\]
\[
|T^n_{P,\delta}| \geq (1 - \epsilon) \exp(n(H(P) - \delta)).
\]

For a (classical) channel \( W : \mathcal{X} \to \mathcal{Y} \) (i.e. a stochastic map, taking \( x \in \mathcal{X} \) to a probability distribution \( W_x \) on \( \mathcal{Y} \)) and a string \( x^n \in \mathcal{X}^n \) of type \( P \), we denote the output distribution of \( x^n \) in \( n \) independent uses of the channel by
\[
W^\otimes n_{x^n} = W_{x_1} \otimes \cdots \otimes W_{x_n}.
\]
Let $\delta > 0$, and define the set of conditional $W$-typical sequences as
\[
T_{W,\delta}^n(x^n) := \left\{ y^n : \left| \frac{1}{n} \log W^n(y^n) - H(W | P) \right| \leq \delta \right\},
\]
where $H(W | P) = \sum_x P(x) H(W_x)$ is the conditional entropy.

Once more by the law of large numbers, for every $\epsilon$ and sufficiently large $n$,
\[
W^n_x(T_{W,\delta}^n(x^n)) \geq 1 - \epsilon. \tag{A 5}
\]
Furthermore,
\[
|T_{W,\delta}^n(x^n)| \leq \exp(n(H(W | P) + \delta)), \tag{A 6}
\]
\[
|T_{W,\delta}^n(x^n)| \geq (1 - \epsilon) \exp(n(H(W | P) - \delta)). \tag{A 7}
\]

All of these concepts and formulae have analogues as ‘typical projectors’ $\Pi$ for quantum states: by virtue of the spectral decomposition, the eigenvalues of a density operator can be interpreted as a probability distribution over eigenstates. The subspaces spanned by the typical eigenstates are the ‘typical subspaces’. The trace of a density operator with one of its typical projectors is then the probability of the corresponding set of typical sequences.

Notations like $\Pi^n_{\rho,\delta}$, $\Pi^n_{W,\delta}(x^n)$, etc., should be clear from this.

There is only one such statement for density operators that we shall use, which is not of this form.

Lemma A 1 (operator law of large numbers). Let $x^n \in \mathcal{X}^n$ be of type $P$, and let $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$ be a cq-channel. Denote the average output state of $W$ under $P$ as
\[
\rho = \sum_x P(x) W_x.
\]
Then, for every $\epsilon > 0$ and sufficiently large $n$,
\[
\text{Tr}(W^n_x(\Pi^n_{\rho,\delta})) \geq 1 - \epsilon.
\]

Appendix B. Miscellaneous facts

This appendix collects some standard facts about various functionals we use: entropy, fidelity and trace norm.

Lemma B 1 (Fannes 1973). Let $\rho$ and $\sigma$ be states on a $d$-dimensional Hilbert space, with $||\rho - \sigma||_1 \leq \delta$. Then $|H(\rho) - H(\sigma)| \leq \delta \log d + \tau(\delta)$, with
\[
\tau(\delta) = \begin{cases} -\delta \log \delta & \text{if } \delta \leq \frac{1}{4}, \\ \frac{1}{2} & \text{otherwise}. \end{cases}
\]
Note that $\tau$ is a monotonic and concave function.

Lemma B 2 (Fuchs et al. 1999). Let $\rho$ and $\sigma$ be any two states on a Hilbert space. Then
\[
1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} ||\rho - \sigma||_1 \leq \sqrt{1 - F(\rho, \sigma)}.
\]
Lemma B 3 (gentle measurement (Winter 1999)). Let $\rho$ be a (subnormalized) density operator, i.e. $\rho \geq 0$ and $\text{Tr} \rho \leq 1$, and let $0 \leq X \leq 1$. Then, if $\text{Tr}(\rho X) \geq 1 - \lambda$, 
$$\|\sqrt{X} \rho \sqrt{X} - \rho\|_1 \leq \sqrt{8\lambda}.$$ 

Lemma B 4 (Carathéodory’s theorem (Ziegler 1995, theorem 1.6)). Let $v_1, \ldots, v_n$ be points in a $d$-dimensional $\mathbb{R}$-vector space, and let $p(1), \ldots, p(n)$ be probabilities (i.e. non-negative and summing to 1). Then the convex combination 
$$v = \sum_{i=1}^{n} p(i) v_i$$
can be expressed as a convex combination of (at most) $d + 1$ of the $v_i$. As a consequence, there exist probability distributions $p_j$ on $\{1, \ldots, n\}$ and probability weights $\lambda_j$ such that, for all $j$, 
$$v = \sum_{i=1}^{n} p_j(i) v_i, \quad |\text{supp} p_j| \leq d + 1.$$ 

Appendix C. Miscellaneous proofs

Proof of proposition 2.4. The proof follows closely the argument of Winter (2004) and of Ahlswede & Winter (2002). Begin by constructing the typical projectors $\Pi_{x^n}$ of the $W_{x^n}$, which, for $x^n$ of type $P$, is defined as the sum of the eigenstate projectors of $W_{x^n}$ with eigenvalues in the interval 
$$[\exp(-n(H(W | P) + \delta)); \exp(-n(H(W | P) - \delta))],$$
with the conditional entropy $H(W | P) = \sum_x P(x) H(W_x)$. For sufficiently large $n$, by the law of large numbers, $\text{Tr}(W_{x^n} \Pi_{x^n}) \geq 1 - \epsilon$. Now define 
$$\omega_{x^n} := \Pi W_{x^n} \Pi_{x^n} \Pi,$$ 
where $\Pi$ is the typical projector of $\rho = \sum_x P(x) W_x$, i.e. the sum of the eigenstate projectors with eigenvalues in the interval 
$$[\exp(-n(H(\rho) + \delta)); \exp(-n(H(\rho) - \delta))].$$
These concepts are taken from Schumacher et al. (1995), Jozsa & Schumacher (1994) and Schumacher & Westmoreland (1997), but see also Appendix A. By the law of large numbers, for sufficiently large $n$, 
$$\text{Tr}(W_{x^n} \Pi) \geq 1 - \frac{1}{8} \epsilon^2.$$ 
From this and the gentle measurement lemma B 3, we get 
$$\|\omega_{x^n} - W_{x^n}\|_1 \leq 2\epsilon. \quad (C1)$$ 
The strategy is now to apply the operator Chernoff bound to the $\omega_{x^n}$: they are supported on a subspace of dimension $\leq \exp(n(H(\rho) + \delta))$, and are all upper bounded by $\exp(-n(H(W | P) - \delta)) \Pi$. 

The only remaining obstacle is that we need a lower bound on 
\[ \bar{\omega} = \frac{1}{|T^n|} \sum_{x^n \in T^n} \omega_{x^n}. \]
To this end, let \( \hat{\Pi} \) be the projector onto the subspace spanned by eigenvectors of \( \bar{\omega} \) with eigenvalues \( \geq \exp(-n(H(\rho) - 2\delta)) \). In this way, for sufficiently large \( n \),
\[ \text{Tr}(\varpi \hat{\Pi}) \geq 1 - \epsilon. \]
Defining the operators
\[ \tilde{\omega}_{x^n} := \hat{\Pi} \omega_{x^n} \hat{\Pi}, \]
we can now apply lemma 2.3 to the (rescaled) \( \tilde{\omega}_{U(j)} \), and get
\[
\Pr\left\{ \frac{1}{M} \sum_{j=1}^{M} \tilde{\omega}_{U(j)} \notin [(1 \pm \epsilon) \hat{\Pi} \varpi \hat{\Pi}] \right\} \leq 2d^n \exp\left(-M \exp(-n(I(P;W) + 3\delta)) \frac{\epsilon^2}{2\ln 2}\right). \tag{C 2}
\]
But
\[ \Omega := \frac{1}{M} \sum_{j=1}^{M} \tilde{\omega}_{U(j)} \in [(1 \pm \epsilon) \hat{\Pi} \varpi \hat{\Pi}] \]
implies \( \| \hat{\Pi}(\Omega - \varpi) \hat{\Pi} \|_1 \leq \epsilon \), which in turn implies
\[ \| \hat{\Pi} \Omega \hat{\Pi} - \varpi \|_1 \leq 2\epsilon. \tag{C 3} \]
In particular we get, invoking equation (C 1),
\[ \text{Tr} \Omega \geq \text{Tr} \varpi - 2\epsilon \geq 1 - 4\epsilon. \]
Hence, with the gentle measurement lemma B 3, Appendix B, we obtain
\[ \| \hat{\Pi} \Omega \hat{\Pi} - \Omega \|_1 \leq \sqrt{32\epsilon}. \tag{C 4} \]
Combining equations (C 3) and (C 4) via the triangle inequality gives
\[ \| \Omega - \varpi \|_1 \leq 2\epsilon + \sqrt{32\epsilon}, \tag{C 5} \]
and, using equation (C 1) to replace \( \omega_{x^n} \) by \( W^n_{x^n} \) in both of the above operators, we finally get
\[ \left\| \frac{1}{M} \sum_{j=1}^{M} W^n_{U(j)} - \sigma(P) \right\|_1 \leq 6\epsilon + \sqrt{32\epsilon} \leq 12\sqrt{\epsilon}. \]
The complement of this event has probability smaller than equation (C 2), and since \( \delta \) was arbitrary we obtain our claim. \( \blacksquare \)

Proof of proposition 2.5. In Schumacher & Westmoreland (1997), it is proved that selecting
\[ N' = 2(n + 1)|X| \exp(n(I(P;W) - \delta)) \]
codewords \( u^{(i)} \) at random, i.i.d. according to \( P^{\otimes n} \), one can construct a canonical decoding POVM such that, for the expectation (over the code \( C_{\text{HSW}} \)) of the average error probability, \( p_E \) goes to zero:
\[ \langle p_E \rangle_{C_{\text{HSW}}} \leq 9\epsilon + N' \exp(-n(I(P;W) - \frac{1}{2}\delta)) \tag{C 6} \]

Distillation of secret key and entanglement from quantum states

(see Schumacher & Westmoreland 1997, eqn (34)). The first thing we note is that (for sufficiently large \( n \)) \( \epsilon = \exp(-\gamma n) \) for a constant \( \gamma > 0 \) depending on \( \delta \): this follows by inspection of §III of Schumacher & Westmoreland (1997), where \( \epsilon \) is introduced as the loss of probability mass by removing non-typical contributions. As non-typicality is defined as large deviation events for a sum of independent random variables, of the form

\[
\log \lambda_{x^n} = \sum_k \log \lambda_{x_k},
\]

the Chernoff bound allows us to put exponential bounds on the non-typical mass.

Hence equation (C6) can be rewritten, for sufficiently large \( n \),

\[
\langle p_E \rangle_{\text{HSW}} \leq \exp(-n\beta), \tag{C 7}
\]

with some \( \beta > 0 \).

We want to show that \( C_{\text{HSW}} \cap T^n_P \) is a good approximation to a random code from the type class \( T^n_P \). Of course, it is not quite that, if only because it has a variable number of codewords! There is an easy fix to this problem: define, with \( N = \exp(n(I(P; W) - \delta)) \),

\[
C := \text{first } N \text{ elements of } C_{\text{HSW}} \cap T^n_P,
\]

which makes sense because we can put the codewords in the order in which we select them. If the intersection is too small, we define \( C \) to be empty.

First of all, let us bound the error probability of \( C \):

\[
p_E(C) \leq \frac{1}{N} \sum_{w^{(i)} \in T^n_P} (1 - \text{Tr}(W^n_{u^{(i)}}) D_i) \leq \frac{1}{N} \sum_{i=1}^{N'} (1 - \text{Tr}(W^n_{u^{(i)}}) D_i) = \frac{N'}{N} p_E(C_{\text{HSW}}). \tag{C 8}
\]

Now, the event that \( |C_{\text{HSW}} \cap T^n_P| < N \) happens extremely rarely: because \( P^\otimes n (T^n_P) \geq (n + 1)^{-|X|} \), the expected cardinality of the intersection is larger than \( 2N \), for sufficiently large \( n \). But then, using the Chernoff bound,

\[
\Pr\{|C_{\text{HSW}} \cap T^n_P| < N\} \leq \exp\left(-N' \frac{1}{8 \ln 2(n + 1)^{|X|}}\right) \leq \exp(-\frac{1}{4}N).
\]

By symmetry, it is clear that conditional on \( C \neq \emptyset \), the code \( C \) is a uniformly random code of \( N \) words from \( T^n_P \), i.e. it can be described by i.i.d. and uniformly picking codewords.

Hence, denoting by \( \tilde{C} \) a truly random code of \( N \) words from \( T^n_P \), we have

\[
\frac{1}{2} \| \text{Dist}(C) - \text{Dist}(\tilde{C}) \|_1 \leq \exp(-\frac{1}{4}N).
\]

Observe that the left-hand side is the total variational distance of distributions. Thus, putting this together with equations (C8) and (C7), we obtain

\[
\langle p_E \rangle_{\tilde{C}} \leq \langle p_E \rangle_C + \exp(-\frac{1}{4}N) \leq 2(n + 1)^{|X|} \exp(-n\beta) + \exp(-\frac{1}{4}N) \leq \exp(-\frac{1}{2}n\beta),
\]

\[\text{Proc. R. Soc. A (2005)}\]
for sufficiently large \( n \). But this in turn implies that
\[
\Pr\{p_E(\tilde{C}) > \exp(-\frac{1}{4} n\beta)\} \leq \exp(-\frac{1}{4} n\beta),
\]
by the Markov inequality. ■

**Proof of range bounds in theorem 2.8.** Here we prove that we may assume that \( T \) is a deterministic function of \( X \), \( |T| \leq d_A^2 \) and \( |X| \leq d_A^4 \).

Denote the POVM elements of the measurement producing \( x \) and \( t \) by \( P_{xt} \), and introduce the coarse-grained operators \( P_t = \sum_x P_{xt} \). To decompose the POVM using convexity arguments, we rewrite the completeness conditions as
\[
\frac{1}{d_A^2} 1_A = \sum_t \frac{\text{Tr} P_t}{d_A^2} \frac{P_t}{\text{Tr} P_t} =: \sum_t q(t) \pi_t,
\]
\[
\pi_t = \sum_x \frac{\text{Tr} P_{xt} \quad P_{xt}}{\text{Tr} P_t \quad \text{Tr} P_{xt}} =: \sum_{xt'} q(xt' \mid t) \pi_{xt'}.
\]
Using Carathéodory’s theorem, lemma B 4, we can write
\[
q = \sum_k \mu_k q_k, \tag{C 9}
\]
with distributions \( q_k \) of support \( \leq d_A^2 \) and such that, for all \( k \),
\[
\sum_t q_k(t) \pi_t = \frac{1}{d_A^2} 1_A. \tag{C 10}
\]
Using Carathéodory’s theorem once more, we obtain, for each \( t \), a decomposition
\[
q(\cdot \mid t) = \sum_j \lambda_{jt} q_j(\cdot \mid t), \tag{C 11}
\]
with conditional distributions \( q_j(\cdot \mid t) \) of support \( \leq d_A^2 \) and such that, for all \( j \),
\[
\sum_{xt'} q_j(xt' \mid t) \pi_{xt'} = \pi_t. \tag{C 12}
\]
Now, let \( \tilde{X} := XT \), of which \( T \) clearly is a function. Then, equations (C 9) and (C 11) define random variables \( J \) and \( K \), respectively; by equations (C 10) and (C 12), for each value \( JK = jk \) we have a POVM \( P^{jk} \) (whose output variable we denote \( \tilde{X}_{jk} \)) the function \( T \) of which we denote \( T_{jk} \). Then (cf. previous proof)
\[
H(B \mid T) - H(E \mid T) = H(B \mid TJ) - H(E \mid TJ) = \sum_{jk} \Pr\{JK = jk\}[H(B \mid T_{jk}) - H(E \mid T_{jk})],
\]
\[
-H(B \mid XT) + H(E \mid XT) = -H(B \mid \tilde{X}JK) + H(E \mid \tilde{X}JK) = \sum_{jk} \Pr\{JK = jk\}[-H(B \mid \tilde{X}_{jk}) + H(E \mid \tilde{X}_{jk})].
\]
Hence there exist \( j \) and \( k \) such that
\[
I(X; B \mid T) - I(X; E \mid T) \leq I(\tilde{X}_{jk}; B \mid T_{jk}) - I(\tilde{X}_{jk}; E \mid T_{jk}),
\]
and \( \tilde{X}_{jk} \) and \( T_{jk} \) satisfy the range constraints. ■
Distillation of secret key and entanglement from quantum states

References


Ambainis, A. & Yang, K. 2002 Towards the classical communication complexity of entanglement distillation protocols with incomplete information. (e-print quant-ph/0207090.)


Hamada, M. 2002 Information rates achievable with algebraic codes on quantum discrete memoryless channels. (e-print quant-ph/0207113.)


Distillation of secret key and entanglement from quantum states

Shor, P. W. & Smolin, J. A. 1996 Quantum error-correcting codes need not completely reveal the error syndrome. (e-print quant-ph/9604006.)

As this paper exceeds the maximum length normally permitted, the authors have agreed to contribute to production costs.