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On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations

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A method for finding general solutions of second-order nonlinear ordinary differential equations by extending the Prelle–Singer (PS) method is briefly discussed. We explore integrating factors, integrals of motion and the general solution associated with several dynamical systems discussed in the current literature by employing our modifications and extensions of the PS method. We also introduce a novel way of deriving linearizing transformations from the first integrals to linearize the second-order nonlinear ordinary differential equations to free particle equations. We illustrate the theory with several potentially important examples and show that our procedure is widely applicable.

Keywords: integrability; integrating factor; linearization; equivalence problem

1. Introduction

Solving nonlinear ordinary differential equations (ODEs) is one of the classic but potentially important areas of research in the theory of dynamical systems (Arnold 1978; José & Saletan 2002; Wiggins 2003). Indeed, a considerable amount of research activity in this field was witnessed in the last century. Progress has been made through geometrical analysis and analytical studies. The modern geometrical theory originated with Poincaré and vigorously developed by Arnold, Moser, Birkhoff and others (Percival & Richards 1982; Guckenheimer & Holmes 1983; Litchenberg & Lieberman 1983; Wiggins 2003). Various analytical methods have been concurrently devised to tackle nonlinear ODEs. The ideas developed by Kovalevskaya, Painlevé and his co-workers have been used to integrate a class of nonlinear ODEs and obtain their underlying solutions (Ince 1956). As a consequence of these studies, nonlinear dynamical systems are broadly classified into two categories, namely, (i) integrable and (ii) nonintegrable systems. Indeed, one of the important current problems in nonlinear dynamics is to identify integrable dynamical systems (Ablowitz & Clarkson 1992; Lakshmanan & Rajasekar 2003). Of course, these methods have a close connection with the group theoretical approach introduced by Sophus Lie in the nineteenth century and subsequently extended by Cartan and Tresse to integrate ordinary and partial differential equations (e.g. see Olver (1995) and Bluman & Anco (2002)).

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Different techniques have been proposed for identifying such integrable dynamical systems, including Painlevé analysis (Conte 1999), Lie symmetry analysis (Bluman & Anco 2002) and direct methods of finding involutive integrals of motion (Hietarinta 1987). Each method has its advantages and disadvantages. For a detailed discussion about the underlying theory of each method and its limitations and applications we refer to Lakshmanan & Rajasekar (2003). Also, certain nonlinear ODEs can be solved through transformation to linear ODEs whose solutions are known. In fact, linearization of given nonlinear ODEs is one of the classic problems in ODE theory whose origin dates back to Cartan. For information on recent progress in this direction we refer readers to Olver (1995).

Prelle & Singer (1983) have proposed a procedure for solving first-order ODEs that presents the solution, if such a solution exists, in terms of elementary functions. The attractiveness of the Prelle–Singer (PS) method is that the method guarantees that a solution will be found if the given system of first-order ODEs has a solution in terms of elementary functions. Duarte *et al.* (2001) modified the technique developed by Prelle & Singer and applied it to second-order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second-order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. For a class of systems these authors (Duarte *et al.* 2001) have deduced first integrals through their procedure, in some cases for the first time.

In this paper we show that the theory of Duarte *et al.* (2001) can be extended in different directions to isolate two independent integrals of motion and obtain solutions. In the earlier study it was shown that the theory can be used to derive only one integral. In this work we extend their theory and deduce a general solution from the first integral. Our examples include those considered in Duarte *et al.*'s and certain important equations discussed in the recent literature whose solutions are not known. There are two objectives central to our study. First, it is to show that one can deduce general solutions in a straightforward and simple manner, as well as through finding first integrals. The method we propose is not confined to the PS method alone but can be treated as a general one. If one has a first integral for a given second-order ODE then our method provides the general solution in an algorithmic way for at least a class of equations. The reason for merging our procedure with the PS method, rather than any other method, is owing to the following facts.

(i) It has been conjectured that the PS method is guaranteed to provide first integrals for a given problem if a solution exists.

(ii) The PS method not only gives the first integrals but also the underlying integrating factors, that is, by multiplying the equation with these functions we can rewrite the equation as a perfect differentiable function, which gives the first integrals in a separate way upon integration.

(iii) The PS method can be used to solve nonlinear as well as linear second-order ODEs. As the PS method is based on the equations of motion rather than Lagrangian or Hamiltonian description, the analysis is applicable to both Hamiltonian and non-Hamiltonian systems.

Our second reason is to introduce and demonstrate a novel and straightforward technique for constructing and exploring linearizing transformations. The given second-order nonlinear ODEs can be transformed to linear equations, in particular, to free particle equations by exploring the transformations. As we illustrate below, these transformations can be deduced from the first integral, which is an entirely new technique in the current literature. In a nutshell, once a first integral is known then our procedure provides, at least for a class of problems, the general solutions as well as the linearizing transformations. The ideas proposed here can be applied to a coupled system of second-order ODEs as well as higher order ODEs, which will be presented separately.

The paper is organized as follows. In §2, we briefly describe the PS method applicable for second-order ODEs and indicate new features in finding the integrals of motion. In §3, we have extended the theory in three different directions, which indicates the novelty of the approach. The first significant application is that the second integral can be deduced from the method itself in many cases. The second application is that the general solution can be deduced from the first integral. Finally, we propose a method of identifying linearizing transformations. We emphasize the validity of the theory, with several illustrative examples arising in different areas of physics, in §4. In §5, we demonstrate the method for identifying linearizing transformations with three examples, including one studied in recent literature. We present our conclusions in §6.

2. Prelle–Singer method for second-order ODEs

In this section, we briefly discuss the theory introduced by Duarte *et al.* (2001) for second-order ODEs and extend it so that general solutions can be deduced from the modifications. Let us consider second-order ODEs of the form

$$\ddot{x} = \frac{P}{Q}, \quad P, Q \in \mathbb{C}[t, x, \dot{x}], \tag{2.1}$$

where $\dot{x}$ denotes differentiation with respect to time and P and Q are polynomials in t, x and $\dot{x}$ with coefficients in the field of complex numbers. Let us assume that the ODE (2.1) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential gives

$$\mathrm{d}I = I_t\,\mathrm{d}t + I_x\,\mathrm{d}x + I_{\dot{x}}\,\mathrm{d}\dot{x} = 0, \tag{2.2}$$

where the subscript denotes partial differentiation with respect to that variable. Rewriting equation (2.1) in the form $(P/Q)\mathrm{d}t - \mathrm{d}\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}\,\mathrm{d}t - S(t, x, \dot{x})\,\mathrm{d}x$ to the latter, we obtain the 1-form

$$\left(\frac{P}{Q} + S\dot{x}\right)\mathrm{d}t - S\,\mathrm{d}x - \mathrm{d}\dot{x} = 0. \tag{2.3}$$

Hence, on the solutions, the 1-forms given by (2.2) and (2.3) must be proportional. Multiplying (2.3) by the factor $R(t, x, \dot{x})$, which acts as the integrating factor for equation (2.3), we obtain

$$\mathrm{d}I = R(\phi + S\dot{x})\mathrm{d}t - RS\,\mathrm{d}x - R\,\mathrm{d}\dot{x} = 0, \tag{2.4}$$

where $\phi \equiv P/Q$. By comparing equation (2.2) with equation (2.4) we find the relations

$$\left.\begin{aligned} I_t &= R(\phi + \dot{x}S), \\ I_x &= -RS, \\ I_{\dot{x}} &= -R. \end{aligned}\right\} \tag{2.5}$$

Then, the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the equations (2.5) require that

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{2.6}$$

$$D[R] = -R(S + \phi_{\dot{x}}), \tag{2.7}$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \tag{2.8}$$

where

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}.$$

Equations (2.6)–(2.8) can be solved in the following way. One can obtain an expression for S by substituting the given expression of ϕ into equation (2.6) and solving it. Equation (2.7) becomes the determining equation for the function R once S is known. One can get an explicit form for R by solving equation (2.7). Now the functions R and S have to satisfy an extra constraint, that is, equation (2.8). Once a compatible solution satisfying all three equations has been found, then functions R and S fix the integral of motion $I(t, x, \dot{x})$ with the relation

$$\begin{aligned} I(t,x,\dot{x}) = &\int R(\phi + \dot{x}S)\mathrm{d}t - \int\left(RS + \frac{\mathrm{d}}{\mathrm{d}x}\int R(\phi + \dot{x}S)\mathrm{d}t\right)\mathrm{d}x \\ &- \int\left\{R + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}\left[\int R(\phi + \dot{x}S)\mathrm{d}t - \int\left(RS + \frac{\mathrm{d}}{\mathrm{d}x}\int R(\phi + \dot{x}S)\mathrm{d}t\right)\mathrm{d}x\right]\right\}\mathrm{d}\dot{x}. \end{aligned} \tag{2.9}$$

Equation (2.9) can be derived straightforwardly by integrating the equations (2.5). Note that for every independent set (S, R), equation (2.9) defines an integral.

Thus, two independent sets, (S_i, R_i), $i = 1, 2$, provide us with two independent integrals of motion through the relation (2.9), which guarantees the integrability of equation (2.1). Since we first solved equations (2.6) and (2.7) and then checked the compatibility of this solution with equation (2.8), we often found that the solutions that satisfied equations (2.6) and (2.7) did not need to satisfy equation (2.8) as equations (2.6)–(2.8) constitute an overdetermined system for the unknowns R and S. In fact, for a class of problems one often gets a set (S_1, R_1) which satisfies equations (2.6)–(2.8) and another set (S_2, R_2) that satisfies only equations (2.6) and (2.7), not equation (2.8). In this situation, we find that one can use the first integral, derived from the set (S_1, R_1), to deduce the second compatible solution $(S_2, \hat{R}_2)$. For example, let the set (S_2, R_2) be a solution of equations (2.6) and (2.7) and not of the constraint equation (2.8). After examining several examples we find that one can make the set (S_2, R_2) compatible by modifying the form of R_2 as

$$\hat{R}_2 = F(t, x, \dot{x}) R_2, \tag{2.10}$$

where $\hat{R}_2$ satisfies (2.7), so that we have

$$(F_t + \dot{x} F_x + \phi F_{\dot{x}}) R_2 + F D[R_2] = -F R_2 (S_2 + \phi_{\dot{x}}). \tag{2.11}$$

Furthermore, if F is a constant of motion (or a function of it), then the first term on the left-hand side vanishes and one gets the same equation (2.7) for R_2, provided F is non-zero. In other words, whenever F is a constant of motion or a function of it, then the solution of equation (2.7) may provide only a factor of the complete solution $\hat{R}_2$ without the factor F in equation (2.10). This general form of $\hat{R}_2$ with S_2 can now form a complete solution to the equations (2.6)–(2.8). In a nutshell, we describe the procedure as follows. First, we determine S and R from equations (2.6) and (2.7). If the set (S,R) satisfies equation (2.8) then we take it as a compatible solution and proceed to construct the associated integral of motion. On the other hand, if it does not satisfy equation (2.8) we then assume the modified form $\hat{R}_2 = F(I_1) R_2$, where I_1 is the first integral which has already been derived through a compatible solution, and find the explicit form of $F(I_1)$ from equation (2.8), which in turn fixes the compatible solution $(S_2, \hat{R}_2)$. This set $(S_2, \hat{R}_2)$ can be utilized to derive the second integral.

3. Generalization

(*a*) *Identifying a second integral of motion*

Duarte *et al.* (2001) have considered certain physically important systems and constructed first integrals. Furthermore, they mentioned that one can deduce the general solution by applying the original PS algorithm to these first integrals (by treating them as first-order ODEs). An interesting observation we make here is that there is no need to invoke the original PS procedure to deduce the general solution. In fact, as we show below, the general solution can be derived in a self-contained way. As the motivation of Duarte *et al.* (2001) was to construct only the first integral, they reported only one set of solutions (S, R) for the equations (2.6)–(2.8). However, we have observed that an additional independent set of solutions, namely, (S_2, R_2), of equations (2.6)–(2.8), may lead to another integral of motion, I_2, and if the latter is an independent function of I_1 then one can write down the general solution for the given problem from these two integrals alone. Now the question is whether one will be able to find a second pair of solutions for the system (2.6)–(2.8) and construct I_2 through the relation (2.9). After investigating several examples we observed the following.

(i) For a class of equations, including harmonic oscillator, equation coming from general relativity and generalized modified Emden equations with constant external forcing, one can easily construct a second pair of solutions (S_2, R_2) and deduce I_2 through the relation (2.9). We call this class Type I.

(ii) For another class of equations we can find (S_2, R_2) explicitly from equations (2.6)–(2.8) but are unable to integrate equation (2.9) exactly and unambiguously obtain the second integral I_2. We call this class Type II. The examples included in this category are Helmhotz oscillator and Duffing oscillator equations. For this class of equations we identify an alternative way to derive the second integration constant.

(iii) There exists another category in which the systems do not even admit a second pair (S_2, R_2) of solutions in simple rational forms for the equations (2.6)–(2.8) and we call this category Type III. An example is the Duffing–van der Pol oscillator, which is one of the prototype examples for the study of nonlinear dynamics in many branches of science. For this class of equations we identify an alternative way to obtain the second integral.

(*b*) *Method of deriving general solution*

To overcome the difficulties in constructing the second constant in Types II and III we propose the following procedure. As our aim is to derive the general solution for the given problem, we split the functional form of the first integral I into two terms so that one involves all the variables $(t, x, \dot{x})$ while the other excludes $\dot{x}$, that is

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \tag{3.1}$$

Now, let us split the function F_1 further in terms of two functions so that F_1 is a function of the product of the two functions, say, a perfect differentiable function $(\mathrm{d}/\mathrm{d}t)G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x)\right) + F_2(G_1(t, x)). \tag{3.2}$$

We note that while rewriting equation (3.1) in the form of equation (3.2), we require that the function $F_2(t, x)$ in equation (3.1) is automatically a function of $G_1(t, x)$. The reason for making such a specific decomposition is that in this case equation (3.2) can be rewritten as a simple first-order ODE for the variable G_1 (see equation (3.4) below). Actually, we originally realized this possibility for the integrable force-free Duffing–van der Pol oscillator equation (Chandrasekar *et al.* 2004), which has been generalized in the present case. Identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is

$$w = G_1(t, x), \quad z = \int_o^t G_2(t', x, \dot{x})\mathrm{d}t', \tag{3.3}$$

one obtains an explicit transformation to remove the time-dependent part in the first integral (2.9). We note here that the integration leading to z on the right-hand side of equation (3.3) can be performed provided the function G_2 is an exact derivative of t, that is, $G_2 = \mathrm{d}z(t, x)/\mathrm{d}t = \dot{x}z_x + z_t$, so that z turns out to be a function of t and x alone. In terms of the new variables, equation (3.2) can be modified to the form

$$I = F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) + F_2(w). \tag{3.4}$$

In other words

$$F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) = I - F_2(w). \tag{3.5}$$

By rewriting equation (3.4) one obtains a separable equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(w), \tag{3.6}$$

which can lead to the solution after integration. By rewriting the solution in terms of the original variables one obtains a general solution for equation (2.1).

(*c*) *Method of deriving linearizing transformations*

Finally, the following interesting point can be noted in the above analysis. Assuming $F_2(w)$ is zero in equation (3.4) obtains the simple equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \hat{I}, \tag{3.7}$$

where $\hat{I}$ is a constant. In other words, we have

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 0, \tag{3.8}$$

which is nothing but the free particle equation. In this case, the new variables z and w helps us to transform the given second-order nonlinear ODE into a second-order linear ODE, which in turn leads to the solution by trivial integration. The new variables z and w turn out to be the linearizing transformations. We discuss this possibility in detail in §5.

4. Applications

In this section, we demonstrate the theory discussed in the previous section with suitable examples. In particular, we consider several interesting examples, including those considered in Duarte *et al.* (2001), derive general solutions and establish complete integrability of these dynamical systems. We split our analysis into three categories. In the first category, we consider examples in which the I_i, $i=1,2$, can be easily derived from the relation (2.9). In the second and third categories, we follow our own procedure detailed in §3*b* and *c*, and deduce the second constant. We note that our procedure can be applied to a wide range of systems with second-order equations similar to equation (2.1) but we consider only a few examples for illustrative purposes.

(*a*) *Type-I systems*

As mentioned earlier, one can obtain the second pair of solutions (S_2, R_2) in an algorithmic way for certain equations from the determining equations (2.6)–(2.8) and construct I_2 through the relation (2.9). We observe that examples 1 and 2 discussed in Duarte *et al.* (2001) can be solved in this way and so we consider these two examples first and then a non-trivial example.

(i) *Example 1: an exact solution in general relativity*

Duarte *et al.* (2001) considered the following equation, which was originally derived by Buchdahl (1964) in the theory of general relativity,

$$x\ddot{x} = 3\dot{x}^2 + \frac{x\dot{x}}{t}, \tag{4.1}$$

and deduced the first integral I through their procedure. In the following, we briefly discuss their results and then illustrate our ideas. Substituting $\phi = (3\dot{x}^2/x) + (\dot{x}/t)$ into equations (2.6)–(2.8) we get

$$S_t + \dot{x}S_x + \frac{\dot{x}(3t\dot{x}+x)}{tx}S_{\dot{x}} = \frac{3\dot{x}^2}{x^2} + \frac{6t\dot{x}+x}{tx}S + S^2, \tag{4.2}$$

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x}+x)}{tx}R_{\dot{x}} = -RS - \frac{6\dot{x}t+x}{tx}R, \tag{4.3}$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \tag{4.4}$$

As mentioned in §2, let us first solve equation (4.2) and obtain an explicit form for the function S. To do so, Duarte *et al.* (2001) considered an ansatz for S of the form

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}, \tag{4.5}$$

where a, b, c and d are arbitrary functions of t and x. A rational form for S can be justified, since from equation (4.5) it may be noted that $S = (I_x/I_{\dot{x}})$. We consider only rational forms for S in $\dot{x}$ for all the examples which we consider in this paper. It may be noted that in certain examples, including the present one and examples 3 and 5 (below), this form degenerates into a polynomial form in $\dot{x}$. However, for other examples such as examples 2 and 4 (below), a rational form like equation (4.5) is required. To be general, we carry out an analysis with the form of equation (4.5).

By substituting equation (4.5) into equation (4.2) and equating the coefficients of different powers of $\dot{x}$ to zero, we get a set of partial differential equations for the variables a, b, c and d. By solving them we find that

$$S_1 = -\frac{3\dot{x}}{x}, \quad S_2 = -\frac{\dot{x}}{x}. \tag{4.6}$$

We note that Duarte *et al.* (2001) have reported the expression S_1 as the only solution for equation (4.2). However, we find S_2 also forms a solution for equation (4.2) and helps to deduce the general solution. Substituting forms S_1 and S_2 into equation (4.3) and solving the latter one can lead to an explicit form for the function R. Let us first consider S_1. By substituting S_1 into equation (4.3) we get the following equation for R:

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x}+x)}{tx}R_{\dot{x}} = \frac{3\dot{x}}{x}R - \frac{6\dot{x}t+x}{tx}R. \tag{4.7}$$

In order to solve equation (4.7) one has to make an ansatz. We assume the following form for R:

$$R = A(t,x) + B(t,x)\dot{x}, \tag{4.8}$$

where A and B are arbitrary functions of (t, x). Since $R = -I_{\dot{x}}$ (*vide* equation (2.5)) the form of R may be a polynomial or rational in $\dot{x}$. Depending upon the problem, one has to choose an appropriate ansatz. To begin with one can consider a simple polynomial (in $\dot{x}$) for R; if that fails one can go for rational forms. Let us start with equation (4.8). By substituting equation (4.8) into equation (4.7) and equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, $R_1 = (1/tx^3)$ can be obtained. The solution $S_1 = -(3\dot{x}/x)$ and $R_1 = (1/tx^3)$ has to satisfy the equation (4.4) in order to be a compatible solution, which it does. Once R and S have been found the first integral I can be fixed easily using the expression (2.9) as

$$I_1 = \frac{\dot{x}}{tx^3}. \tag{4.9}$$

One can easily check that I_1 is constant on the solutions, that is, $(\mathrm{d}I_1/\mathrm{d}t) = 0$. This integral has been deduced in Duarte *et al.* (2001). However, the second expression, S_2 has been ignored by the authors since the corresponding R_2 coming out of equation (4.3) does not form a compatible solution, that is, it does not satisfy equation (4.4). In the following we show how it can be made compatible and use it effectively to deduce the second integration constant.

By substituting the expression $S_2 = -(\dot{x}/x)$ into equation (4.3) and solving it in the same way as outlined in the previous paragraph we obtain the following form for R:

$$R_2 = \frac{1}{x^5 t}. \tag{4.10}$$

However, this set (S_2, R_2) does not satisfy the extra constraint in equation (4.4). In fact, not all forms of R from equation (2.7) satisfy equation (2.8). As we explained in §3, the form of R_2 given in equation (4.10) may not be the 'complete form' but might be a factor of the complete form. To recover the complete form of R it may be assumed that

$$\hat{R} = F(I_1)R, \tag{4.11}$$

where $F(I_1)$ is a function of the first integral I_1, and determine the form of $F(I_1)$ explicitly. For this purpose we proceed as follows. Substituting

$$\hat{R}_2 = F(I_1)R_2 = \frac{1}{tx^5}F(I_1) \tag{4.12}$$

into equation (4.4), we obtain the following equation for F:

$$I_1 F' + 2F = 0, \tag{4.13}$$

where the prime denotes differentiation with respect to I_1. Upon integrating equation (4.13) (after putting the constant of integration to zero) we get

$$F = \frac{1}{I_1^2} = \frac{t^2 x^6}{\dot{x}^2}, \tag{4.14}$$

which fixes the form of $\hat{R}_2$ as

$$\hat{R}_2 = \frac{1}{I_1^2} \frac{1}{x^5 t} = \frac{tx}{\dot{x}^2}. \tag{4.15}$$

It can easily be checked that this set $S_2 = -(\dot{x}/x)$ and $\hat{R}_2 = (tx/\dot{x}^2)$ is a compatible solution for equations (4.2)–(4.4). By substituting S_2 and $\hat{R}_2$ into equation (2.9) we get an explicit form for I_2, namely,

$$I_2 = t\left(t + \frac{x}{\dot{x}}\right). \tag{4.16}$$

From the integrals I_1 and I_2 one can deduce the general solution directly (without performing any further integration) for the problem in the form

$$x = \sqrt{\frac{1}{I_1(I_2 - t^2)}}. \tag{4.17}$$

Of course, the same result can be obtained solving equation (4.9) from the first integral. However, the point we want to emphasize here is that an independent second integral of motion can be deduced to find the solution without any further integration, which can be used profitably when the expression for I_1 cannot be easily solved.

(ii) *Example 2: simple harmonic oscillator*

To illustrate the above procedure also works for linear ODEs, we consider the simple harmonic oscillator and derive the general solution. As the procedure of deriving the first integral has been discussed in detail in Duarte *et al.* (2001), we omit the details and provide only the essential expressions in the following.

The equation of motion for the simple harmonic oscillator is

$$\ddot{x} = -x \tag{4.18}$$

so that equations (2.6)–(2.8) become

$$S_t + \dot{x}S_x - xS_{\dot{x}} = 1 + S^2, \tag{4.19}$$

$$R_t + \dot{x}R_x - xR_{\dot{x}} = -RS, \tag{4.20}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{4.21}$$

As shown in Duarte *et al.* (2001) a simple solution for equations (4.19)–(4.21) can be constructed with the form

$$S_1 = \frac{x}{\dot{x}}, \quad R_1 = \dot{x}, \tag{4.22}$$

which in turn gives the first integral

$$I_1 = \dot{x}^2 + x^2 \tag{4.23}$$

through relation (2.9). However, one can easily check that

$$S_2 = -\frac{\dot{x}}{x}, \quad R_2 = x \tag{4.24}$$

is also a solution for the set equations (4.17) and (4.20) (which has not been reported earlier) but does not satisfy the extra constraint of equation (4.21). Thus as before, let us seek an $\hat{R}_2$ of the form

$$\hat{R}_2 = F(I_1)R_2 = F(I_1)x, \tag{4.25}$$

where $F(I_1)$ is a function of I_1. Substituting equation (4.25) into equation (4.21) and integrating the resultant equation, we get $F=(1/I_1)$. Thus, $\hat{R}_2$ becomes

$$\hat{R}_2 = \frac{x}{I_1} = \frac{x}{x^2 + \dot{x}^2}. \tag{4.26}$$

Now, it can be checked that $(S_2, \hat{R}_2)$ satisfies equations (4.19)–(4.21) and furnishes the second integral through relation (2.9) of the form

$$\begin{aligned} I_2 &= -t - \int \frac{\dot{x}}{\dot{x}^2 + x^2}\,\mathrm{d}x - \int \left(\frac{x}{\dot{x}^2 + x^2} - \frac{\mathrm{d}}{\mathrm{d}\dot{x}} \int \frac{\dot{x}}{\dot{x}^2 + x^2}\,\mathrm{d}x \right) \mathrm{d}\dot{x}, \\ &= -t - \tan^{-1}\frac{\dot{x}}{x}. \end{aligned} \tag{4.27}$$

Using equations (4.23) and (4.27), we can write down the general solution for the simple harmonic oscillator directly in the form

$$x = \sqrt{I_1}\cos(t + I_2). \tag{4.28}$$

In a similar way, general solutions for a class of physically important systems can be deduced.

It may be noted that I_2 can also be obtained trivially in the above two examples by simply integrating the expressions (4.9) and (4.23) without using the extended procedure. We stress that for certain equations it is not possible to integrate and obtain the general solution in this simple way and the above said procedure has to be followed to obtain the second integral. In the following we discuss one such example for which, to our knowledge, an explicit solution was not previously known.

(iii) *Example 3: modified Emden-type equation with linear term*

It is known that the generalized Emden-type equation with linear and constant external forcing is also linearizable since it admits an eight point Lie symmetry group (Mahomed & Leach 1989*a*; Pandey *et al.* submitted). In the following we explore its general solution through the extended PS algorithm. Let us first consider the equation of the form

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x=0, \tag{4.29}$$

where k and λ_1 are arbitrary parameters. To explore the general solution for the equation (4.29) we again use the PS method. In this case, we have the following determining equations for functions R and S,

$$S_t+\dot{x}S_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)S_{\dot{x}}=k\dot{x}+\frac{k^2}{3}x^2+\lambda_1-Skx+S^2, \tag{4.30}$$

$$R_t+\dot{x}R_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)R_{\dot{x}}=-R(S-kx), \tag{4.31}$$

$$R_x-SR_{\dot{x}}-RS_{\dot{x}}=0. \tag{4.32}$$

As before, let us seek an ansatz for S of the form (4.5) to the first equation in (4.30)–(4.32). By substituting the ansatz (4.5) into equation (4.30) and equating the coefficients of different powers of $\dot{x}$ to zero we get

$$\left.\begin{aligned} db_x-bd_x-kd^2=0,\\ db_t-bd_t+cb_x-bc_x+a_xd-ad_x-2kcd-\left(\frac{k^2}{3}x^2+\lambda_1\right)d^2+kbdx-b^2=0,\\ cb_t-bc_t+da_t-ad_t+ca_x-ac_x-kc^2-2\left(\frac{k^2}{3}x^2+\lambda_1\right)cd+2kadx-2ab=0,\\ ca_t-ac_t-\left(\frac{k^2}{9}x^3+\lambda_1 x\right)(bc-ad)-\left(\frac{k^2}{3}x^2+\lambda_1\right)c^2+kacx-a^2=0,\end{aligned}\right\} \tag{4.33}$$

where subscripts denote partial derivative with respect to that variable. Solving equation (4.33) we can obtain two specific solutions

$$S_1=\frac{-\dot{x}+\frac{k}{3}x^2}{x},\quad S_2=\frac{kx+3\sqrt{-\lambda_1}}{3}-\frac{k\dot{x}}{kx+3\sqrt{-\lambda_1}}. \tag{4.34}$$

By putting the forms of S_1 and S_2 into equation (4.31) and solving it the respective forms of R can be obtained . To do so let us first consider S_1. By substituting the latter into equation (4.31) we get the following equation for R:

$$R_t+\dot{x}R_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)R_{\dot{x}}=\left(\frac{\dot{x}-\frac{k}{3}x^2}{x}+kx\right)R. \tag{4.35}$$

To solve equation (4.35) we make an ansatz of the form

$$R=\frac{A(t,x)+B(t,x)\dot{x}}{C(t,x)+D(t,x)\dot{x}+E(t,x)\dot{x}^2}. \tag{4.36}$$

By substituting equation (4.36) into equation (4.35), equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, we arrive at

$$R_1 = e^{-2\sqrt{-\lambda_1}t}\left(\frac{C_0 x}{(3\dot{x}+kx^2-3\sqrt{\lambda_1}x)^2}\right), \tag{4.37}$$

where $C_0 = 18\sqrt{-\lambda_1}$. It can easily be checked that S_1 and R_1 satisfy equation (4.32) and, as a consequence, obtain the first integral

$$I_1 = e^{-2\sqrt{-\lambda_1}t}\left(\frac{3\dot{x}+kx^2+3\sqrt{-\lambda_1}x}{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}\right). \tag{4.38}$$

We note that, unlike the other two examples, equation (4.38) cannot be easily integrated to provide the second integral (although one can, in fact, explicitly solve the resultant Riccati equation after some effort). We follow the procedure adopted in the previous two examples and construct I_2. By substituting the expression S_2 into equation (4.31) and solving it in the same way as outlined above, we obtain the following form for R:

$$R_2 = C_0\frac{kx+3\sqrt{-\lambda_1}}{k(3\dot{x}+kx^2-3\sqrt{-\lambda_1}x)^2}e^{-3\sqrt{\lambda_1}t}. \tag{4.39}$$

However, this set (S_2, R_2) does not satisfy the extra constraint (4.32) and so to deduce the correct form of R_2 we assume that

$$\hat{R}_2 = F(I_1)R_2 = C_0\frac{F(I_1)(kx+3\sqrt{-\lambda_1})e^{-3\sqrt{-\lambda_1}t}}{k(3\dot{x}+kx^2-3\sqrt{-\lambda_1}x)^2}. \tag{4.40}$$

By substituting equation (4.40) into equation (4.32) we obtain $F=(1/I_1^2)$, which fixes the form of $\hat{R}$ as

$$\hat{R}_2 = C_0\frac{kx+3\sqrt{-\lambda_1}}{k(3\dot{x}+kx^2+3\sqrt{-\lambda_1}x)^2}e^{\sqrt{-\lambda_1}t}. \tag{4.41}$$

Now, one can easily check that this set $(S_2, \hat{R}_2)$ is a compatible solution for the set (4.30)–(4.32), which in turn provides I_2 through the relation (2.9),

$$I_2 = -\frac{2}{k}e^{\sqrt{-\lambda_1}t}\left(\frac{9\lambda_1+3k\dot{x}+k^2x^2}{3\dot{x}+kx^2+3\sqrt{-\lambda_1}x}\right). \tag{4.42}$$

Using the explicit form of the first integrals I_1 and I_2, the solution can be deduced directly as

$$x = \left(\frac{3\sqrt{-\lambda_1}\left(I_1 e^{2\sqrt{-\lambda_1}t}-1\right)}{kI_1I_2e^{\sqrt{-\lambda_1}t}+k(1+I_1e^{2\sqrt{-\lambda_1}t})}\right). \tag{4.43}$$

To our knowledge, this is the first time equation (4.43), which explicitly solves the equation (4.29), has been given. It has several interesting consequences for nonlinear dynamics, which will be discussed separately.

(b) *Type-II systems*

In the previous category we considered examples which unambiguously give the integrals I_1 and I_2 through relation (2.9). In the present category we show that there are situations in which an explicit form of I_2 is difficult to obtain through relation (2.9), even though there is a compatible solution (2.6)–(2.8). An alternative method is necessary to obtain the general solution for the given problem. For this purpose, we make use of the method proposed in §3*b*. In the following, we give examples of where such a possibility occurs and how to overcome this situation.

(i) *Example 4: Helmholtz oscillator*

Recently, Almendral & Sanjuán (2003) studied the invariance and integrability properties of the Helmholtz oscillator with friction

$$\ddot{x} + c_1\dot{x} + c_2 x - \beta x^2 = 0, \tag{4.44}$$

where c_1, c_2 and β are arbitrary parameters, which is a simple nonlinear oscillator with quadratic nonlinearity. Using the Lie theory for differential equations, Almendral & Sanjuán (2003) found a parameteric choice, $c_2=(6c_1^2/25)$, for which the system is integrable and derived the general solution for this parametric value. In the following, we solve this problem through the extended PS method.

Substituting $\phi=-(c_1\dot{x}+c_2x-\beta x^2)$ into equations (2.6)–(2.8) we obtain

$$S_t + \dot{x}S_x - (c_1\dot{x} + c_2 x - \beta x^2)S_{\dot{x}} = c_2 - 2\beta x - c_1 S + S^2, \tag{4.45}$$

$$R_t + \dot{x}R_x - (c_1\dot{x} + c_2 x - \beta x^2)R_{\dot{x}} = -R(S - c_1), \tag{4.46}$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \tag{4.47}$$

Making the same form of an ansatz, *vide* equations (4.5) and (4.8), we find non-trivial solutions only exist for equations (4.45) and (4.46) for the parametric restrictions $c_2=\pm(6c_1^2/25)$. However, the case $c_2=-(6c_1^2/25)$ follows from the case $c_2=+(6c_1^2/25)$ in equation (4.44) through the simple translation $x=X+(6c_1^2/(25\beta))$. So we consider only the case $c_2=+(6c_1^2/25)$ in the following

$$S_1 = \frac{\left(\frac{2c_1\dot{x}}{5} + \frac{4c_1^2x}{25} - \beta x^2\right)}{\dot{x} + \frac{2c_1}{5}x}, \quad R_1 = -\left(\dot{x} + \frac{2c_1}{5}x\right)\mathrm{e}^{(6c_1t/5)}, \tag{4.48}$$

$$S_2 = \frac{\left(c_1\dot{x} + \frac{6c_1^2x}{25} - \beta x^2\right)}{\dot{x}}, \quad R_2 = -\dot{x}\mathrm{e}^{c_1t}. \tag{4.49}$$

Now, it can be easily checked that (S_1, R_1) satisfies the third equation (4.47) and, as a consequence, leads to the first integral of the form

$$I_1 = \mathrm{e}^{(6c_1t/5)}\left(\frac{\dot{x}^2}{2} + \frac{2c_1x\dot{x}}{5} + \frac{2c_1^2x^2}{25} - \frac{\beta x^3}{3}\right). \tag{4.50}$$

However, the second set (S_2, R_2) does not satisfy the extra constraint (4.47) and so we take

$$\hat{R}_2 = F(I)R_2 = -F(I)\dot{x}\,\mathrm{e}^{c_1 t}, \tag{4.51}$$

which in turn gives $F = C_0 I^{-(5/6)}$, where C_0 is an integration constant, so that

$$\hat{R}_2 = -\left(\frac{C_0}{I_1^{(5/6)}}\right)\dot{x}\,\mathrm{e}^{c_1 t} = -\frac{C_0\dot{x}}{\left(\frac{\dot{x}^2}{2} + \frac{2c_1 x\dot{x}}{5} + \frac{2c_1^2 x^2}{25} - \beta\frac{x^3}{3}\right)^{(5/6)}}. \tag{4.52}$$

It can be checked that $(S_2, \hat{R}_2)$ satisfy equations (4.45)–(4.47) and so one can proceed to deduce the second integration constant through relation (2.9). However, upon substituting $(S_2, \hat{R}_2)$ into (2.9) we arrive at

$$I_2 = \int \frac{c_1\dot{x} + \frac{6c_1^2 x}{25} - \beta x^2}{\left(\frac{\dot{x}^2}{2} + \frac{2c_1 x\dot{x}}{5} + \frac{2c_1^2 x^2}{25} - \beta\frac{x^3}{3}\right)^{(5/6)}}\,\mathrm{d}x. \tag{4.53}$$

It is very difficult to evaluate the integral and so an explicit form of I_2 for this problem cannot be obtained. A similar form of I_2 has been also derived by Jones *et al.* (1993) and Bluman & Anco (2002) for the Duffing oscillator problem (that is, the cubic nonlinearity in equation (4.44)).

Unlike the other examples discussed in Type I, the present example presents difficulties in evaluating the second integration constant, in fact, for a class of equations complicated integrals are faced. To overcome this, one has to look for an alternative way that allows the second constant to be deduced in a straightforward and simple manner. We tackled this situation in the following way. As we have seen, in most of the problems, we are able to deduce the first integral, that is, I_1, straightforwardly, and the first integral often admits explicit time-dependent terms. A useful way of overcoming this is to remove the explicit time-dependent terms by transforming the resultant differential equation into an autonomous form and integrate the latter and thus obtain the solution. In order to do this, one needs a transformation, and the latter can often be constructed through ad hoc methods. However, as we have shown in the theory in §3*b*, the required transformation coordinates can be deduced in a simple way from the first integral itself and the problem can be solved in a systematic manner.

Rewriting the first integral I_1 given by equation (4.50) in the form (3.1), we get

$$I_1 = \frac{1}{2}\left(\dot{x} + \frac{2c_1 x}{5}\right)^2 \mathrm{e}^{(6c_1 t/5)} - \frac{\beta x^3}{3}\mathrm{e}^{(6c_1 t/5)}. \tag{4.54}$$

Now, splitting the first term in equation (4.54) further in the form (3.2),

$$I_1 = \mathrm{e}^{(2c_1 t/5)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{2}}x\,\mathrm{e}^{(2c_1 t/5)}\right)\right)^2 - \frac{\beta}{3}(x\,\mathrm{e}^{(2c_1 t/5)})^3, \tag{4.55}$$

and identifying the dependent and independent variables from (4.55) and the relations (3.3), we obtain the transformation

$$w = \frac{1}{\sqrt{2}} x \, e^{(2c_1 t/5)}, \quad z = -\frac{5}{c_1} e^{-(c_1 t/5)}. \tag{4.56}$$

It is easy to check that equation (4.44) can be transformed to an autonomous form with the help of the transformation (4.56). We note that the transformation (4.56) exactly coincides with the earlier one constructed via Lie symmetry analysis in Almendral & Sanjuán (2003).

Using transformation (4.56), the first integral (4.54) can be rewritten in the form

$$\hat{I} = w'^2 - \frac{\hat{\beta}}{3} w^3, \tag{4.57}$$

which in turn leads to the solution by an integration. On the other hand, the transformation changes the equation of motion (4.44) to

$$w'' = \hat{\beta} w^2, \tag{4.58}$$

where $\hat{\beta} = 2\sqrt{2}\beta$, which upon integration gives (4.57). From equation (4.57), we obtain

$$w'^2 = 4w^3 - g_3, \tag{4.59}$$

where $z = 2\sqrt{(3/\hat{\beta})}\hat{z}$ and $g_3 = -(12I_1/\hat{\beta})$. The solution of this differential equation can be represented in terms of Weierstrass function $\wp(\hat{z}; 0, g_3)$ (Gradshteyn & Ryzhik 1980; Almendral & Sanjuán 2003).

(c) *Type-III systems*

In the previous two categories, we met the situation in which we are able to construct a pair of solutions (S_1, S_2) for the equations (2.6), from which R_1 and R_2 have been deduced. However, there are situations where only one set of solutions (R_1, S_1) can be constructed and its corresponding first integral and the second pair of solutions (R_2, S_2) cannot be obtained by a simple rational form of ansatz. In this situation, one can utilize our procedure and deduce the general solution for the given problem. In the following, we illustrate this with a couple of examples.

(i) *Example 5: force-free Duffing–van der Pol oscillator*

One of the well-studied but still challenging equations in nonlinear dynamics is the Duffing–van der Pol oscillator equation. Its autonomous version (force-free) is

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{4.60}$$

where an over-dot denotes differentiation with respect to time and α, β and γ are arbitrary parameters. Equation (4.60) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A vast amount of literature exists on this equation; for details see, for example, Lakshmanan & Rajasekar (2003) and references therein. In this case we have

$$S_t + \dot{x}S_x - ((\alpha+\beta x^2)\dot{x} - \gamma x + x^3)S_{\dot{x}} = (2\beta x\dot{x} - \gamma + 3x^2) - (\alpha+\beta x^2)S + S^2, \quad (4.61)$$

$$R_t + \dot{x}R_x - ((\alpha+\beta x^2)\dot{x} - \gamma x + x^3)R_{\dot{x}} = (\alpha+\beta x^2 - S)R, \quad (4.62)$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \quad (4.63)$$

To solve equations (4.61)–(4.63) we seek an ansatz for S and R of the form

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}, \quad R = A(t,x) + B(t,x)\dot{x}. \quad (4.64)$$

Upon solving equations (4.61)–(4.63) with the above ansatz, we find that a non-trivial solution exists only for the choice $\alpha = (4/\beta)$, $\gamma = -(3/\beta^2)$, and the corresponding forms of S and R reads

$$S = \frac{1}{\beta} + \beta x^2, \quad R = e^{(3t/\beta)}. \quad (4.65)$$

For this set, one can construct an invariant through the expression (2.9), which turns out to be (Senthilvelan & Lakshmanan 1995)

$$\dot{x} + \frac{1}{\beta}x + \frac{\beta}{3}x^3 = I e^{-(3t/\beta)}. \quad (4.66)$$

To obtain a second pair of solutions for the equations (4.61)–(4.63), one may seek a more general rational form of S and R by including higher polynomials in $\dot{x}$. However, they all lead to only functionally dependent integrals. As it is not possible to seek the second pair of solutions by a simple ansatz, an alternative way, as indicated in §3*b*, has to be sought. We can deduce the required transformation coordinates from the first integral and transform the latter to an autonomous equation and integrate it.

Using our algorithm given in §3*b*, one can deduce the transformation coordinates from the first integral itself, which turns out to be (Chandrasekar *et al.* 2004)

$$w = -x\, e^{(1/\beta)t}, \quad z = e^{-(2/\beta)t}, \quad (4.67)$$

where w and z are new dependent and independent variables, respectively. Substituting (4.67) into (4.60) with the parametric restriction $\alpha = (4/\beta)$, $\gamma = -(3/\beta^2)$, we get

$$w'' - \frac{\beta^2}{2}w^2 w' = 0, \quad (4.68)$$

where prime denotes differentiation with respect to z. Equation (4.68) can be integrated trivially to yield

$$w' - \frac{\beta^2}{6}w^3 = I, \quad (4.69)$$

where I is the integration constant. Equivalently, the transformation (4.67) reduces (4.60) to this form. Solving (4.69), we obtain (Gradshteyn & Ryzhik 1980)

$$z - z_0 = \frac{a}{3I}\left[\frac{1}{2}\log\left(\frac{(w+a)^2}{w^2 - aw + a^2}\right) + \sqrt{3}\arctan\left(\frac{w\sqrt{3}}{2a - w}\right)\right], \tag{4.70}$$

where $a = \sqrt[3]{6I/\beta^2}$ and z_0 is the second integration constant. Rewriting w and z in terms of old variables gives the explicit solution for equation (4.60).

We have shown that the systems (4.44) and (4.60) are integrable for certain specific parametric restrictions only. One may also assume that the functions S and R involve higher degree rational functions in $\dot{x}$ and then repeat the analysis. However, such an analysis does not provide any new integrable choice. In fact, the present results coincide exactly with the results obtained through other methods, namely, Painlevé analysis, Lie symmetry analysis and direct methods (Senthilvelan & Lakshmanan 1995; Almendral & Sanjuán 2003; Lakshmanan & Rajasekar 2003).

5. Linearizable equations

In §4 we discussed the complete integrability of nonlinear dynamical systems by constructing a sufficient number of integrals of motion and obtaining the general solutions explicitly. Another way of solving nonlinear ODEs is to transform them to linear ODEs, in particular, to a free particle equation and explore their underlying solutions. Even though this is one of the classic problems in the theory of ODEs, recently, considerable progress has been made (Mahomed & Leach 1989*b*; Steeb 1993; Olver 1995; Harrison 2002). In this direction it has been shown that a necessary condition for a second-order ODE to be linearizable is that it should be of the form (Mahomed & Leach 1989*b*)

$$\ddot{q} = D(t,q) + C(t,q)\dot{q} + B(t,q)\dot{q}^2 + A(t,q)\dot{q}^3, \tag{5.1}$$

where the functions A, B, C and D are analytic. Sufficient condition for the above second-order equation to be linearizable is (Mahomed & Leach 1989*b*)

$$\left.\begin{aligned} 3A_{tt} + 3CA_t - 3DA_q + 3AC_t + C_{qq} - 6AD_q + BC_q - 2BB_t - 2B_{tq} &= 0, \\ B_{tt} + 6DA_t - 3DB_q + 3AD_t - 2C_{tq} - 3BD_q + 3D_{qq} + 2CC_q - CB_t &= 0, \end{aligned}\right\} \tag{5.2}$$

where the suffices refer to partial derivatives.

For a given second-order nonlinear ODE, one can easily check whether it can be linearizable or not by using the above necessary and sufficient conditions. However, the non-trivial problem is how to deduce systematically the linearizing transformations if the given equation is linearizable. As far as our knowledge goes, Lie symmetries are often used to extract the linearizing transformations (Mahomed & Leach 1985). As we pointed out in §3, the linearizing transformations can also be deduced from the first integral itself, whenever the system is linearizable, in a simple and straightforward way, and we stress that our procedure is new to the literature. In fact, we use the same procedure discussed in §3*c* and deduce the linearizing transformations. The only difference is, that in the case of linearizing transformations, the function F_2 turns out to be zero in equation (3.2) and as a consequence, the latter becomes $(\mathrm{d}w/\mathrm{d}z)=I$ and the transformation coordinates become the linearizing transformations. We illustrate the theory with certain new examples in the following.

(*a*) *Example 1: general relativity*

To illustrate the underlying ideas let us begin with a simple and physically interesting example, namely, the general relativity equation which we discussed as example 1 in §4. We derived the solution (4.17) using the PS method. In this section we linearize the system and derive its solution. Rewriting the first integral (4.9) in the form (3.1),

$$I=-\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x^2}\right), \tag{5.3}$$

and identifying (5.3) with (3.2), we get

$$G_1=\frac{1}{x^2}, \quad G_2=-2t, \quad F_2=0. \tag{5.4}$$

With the above choices, equation (3.3) furnishes the transformed variables

$$w=\frac{1}{x^2}, \quad z=-t^2. \tag{5.5}$$

Substituting (5.5) into (4.1), the latter becomes the free particle equation, namely, $(\mathrm{d}^2w/\mathrm{d}z^2)=0$, whose general solution is $w=I_1z+I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of x and t one gets exactly (4.17), which has been derived in a different way.

(*b*) *Example 2: modified Emden-type equations*

Recently, several papers have been devoted to exploring the invariance and integrability properties of the modified Emden-type equations (Mahomed & Leach 1985; Duarte *et al.* 1987),

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3=0. \tag{5.6}$$

In fact, it is one of the rare second-order nonlinear ODEs which admit eight Lie point symmetries and, as a consequence, is a linearizable one. Recently, Pandey *et al.* (submitted) have obtained the explicit forms of the Lie point symmetries associated with the more general equation

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1x+\lambda_2=0, \tag{5.7}$$

where k, λ_1 and λ_2 are arbitrary parameters. They found that not only the Emden equation (5.6), but also its general form, that is, equation (5.7), admits eight Lie point symmetries. The authors have also reported the explicit forms of the symmetry generators. However, due to the complicated forms of the symmetry generators it is difficult to derive the first integrals and linearizing transformations from the symmetries straightforwardly (although in principle this is always possible). Nevertheless, we discussed the integrability of the case $\lambda_2=0$, $\lambda_1\neq 0$ of equation (5.7) as example 3 in §4 and deduced its general solution. In this section, we transform the equation into a free particle equation and deduce the general solution in an independent manner. We divide our analysis into two cases, namely, (i) $\lambda_1\neq 0$, $\lambda_2=0$ and (ii) $\lambda_1\neq 0$, $\lambda_2\neq 0$, and construct linearizing transformations and general solutions for both cases. As the procedure is the same as given in the previous examples we give only the results.

Case (i) $\lambda_2=0$, $\lambda_1\neq 0$: modified Emden-type equation with linear term

Restricting $\lambda_2=0$ in (5.7), we have

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x=0. \tag{5.8}$$

Since the first integral is already derived, *vide* equation (4.8), we utilize it here to deduce the linearizing transformations. Rewriting the first integral (4.38) in the form

$$I_1=-\frac{\mathrm{e}^{-\sqrt{-\lambda_1}t}kx^2}{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t}\right)\right] \tag{5.9}$$

and identifying (5.9) with (3.2), we get

$$G_1=\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t},\quad G_2=-\frac{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}{kx^2}\mathrm{e}^{\sqrt{-\lambda_1}t}. \tag{5.10}$$

With the above functions (3.3) furnishes

$$w=\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t},\quad z=\left(\frac{3}{kx}-\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{\sqrt{-\lambda_1}t}, \tag{5.11}$$

which is nothing but the linearizing transformation. Note that in this case, while rewriting the first integral I (equation (4.38)) in the form (3.1), the function F_2 disappears, and as a consequence we arrive at (*vide* equation (3.4))

$$\frac{\mathrm{d}w}{\mathrm{d}z}=I, \tag{5.12}$$

which, in turn, gives the free particle equation by differentiation or leads to the solution (4.43) by an integration. On the other hand, vanishing of the function F_2 in this analysis is precisely the condition for the system to be transformed into the free particle equation.

Case (ii) $\lambda_1\neq 0$, $\lambda_2\neq 0$: modified Emden-type equation with linear term and constant external forcing

Finally, we consider the general case, that is

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x+\lambda_2=0. \tag{5.13}$$

To explore the first integrals associated with the system (5.13), let us again seek the PS algorithm. The determining equations for the functions R and S move to be

$$S_t + \dot{x}S_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)S_{\dot{x}} = k\dot{x} + \frac{k^2}{3}x^2 + \lambda_1 - Skx + S^2, \quad (5.14)$$

$$R_t + \dot{x}R_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)R_{\dot{x}} = (kx - S)R, \quad (5.15)$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (5.16)$$

As before, let us seek an ansatz for S to solve the equation (5.14), namely,

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}. \quad (5.17)$$

Substituting (5.17) into (5.14) and equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, we arrive at

$$S_1 = \frac{kx + 3\alpha}{3} - \frac{k\dot{x}}{kx + 3\alpha}, \quad S_2 = \frac{kx + 3\beta}{3} - \frac{k\dot{x}}{kx + 3\beta}, \quad (5.18)$$

where $\alpha^3 + \alpha\lambda_1 - (k\lambda_2/3) = 0$ and $\beta = \left(-\alpha \pm \sqrt{-3\alpha^2 - 4\lambda_1}\right)/2$. Putting the forms of S_1 into (5.15) we get

$$R_t + \dot{x}R_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)R_{\dot{x}} = \left(\frac{k\dot{x}}{kx + 3\alpha} - \frac{kx + 3\alpha}{3} + kx\right)R. \quad (5.19)$$

Again, to solve this equation we make an ansatz

$$R = \frac{A(t,x) + B(t,x)\dot{x}}{C(t,x) + D(t,x)\dot{x} + E(t,x)\dot{x}^2}. \quad (5.20)$$

Substituting (5.20) into (5.19) and solving it we obtain the following form of R:

$$R_1 = \frac{C_0(kx + 3\alpha)\mathrm{e}^{\mp\hat{\alpha}t}}{\left(3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2\right)^2}, \quad (5.21)$$

where C_0 is constant and $\hat{\alpha} = \sqrt{-3\alpha^2 - 4\lambda_1}$. We find that the solution (S_1, R_1) satisfies (5.16). Equations (5.18) and (5.21) fix the first integral of the form

$$I_1 = \mathrm{e}^{\mp\hat{\alpha}t}\left(\frac{3k\dot{x} - 3\frac{(3\alpha \mp \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}\right), \quad (5.22)$$

where $C_0 = 9k\hat{\alpha}$. Rewriting the first integral (5.22) in the form (3.1),

$$I_1 = -\frac{\mathrm{e}^{(-3\alpha \mp \hat{\alpha}/2)t}(k_1 x + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2} \times \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \pm \hat{\alpha}}{2(3\alpha^2 + \lambda_1)}\right)\mathrm{e}^{(3\alpha \mp \hat{\alpha}/2)t}\right)\right] \quad (5.23)$$

and identifying (5.23) with (3.2), we get

$$\left.\begin{aligned} G_1 &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\pm\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\mp\hat{\alpha}/2)t},\\ G_2 &= -\frac{3k\dot{x}-3\dfrac{(3\alpha\pm\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2}{(kx+3\alpha)^2}\mathrm{e}^{(3\alpha\pm\hat{\alpha}/2)t}, \end{aligned}\right\} \tag{5.24}$$

so that (3.3) gives

$$\left.\begin{aligned} w &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\pm\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\mp\hat{\alpha}/2)t},\\ z &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\mp\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\pm\hat{\alpha}/2)t}, \end{aligned}\right\} \tag{5.25}$$

which is nothing but the linearizing transformation. Substituting (5.25) into (5.13) we get the free particle equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=0, \tag{5.26}$$

whose general can be written as $w=I_1z+I_2$. Rewriting w and z in terms of the original variable x and t one obtains

$$x=-\frac{3\alpha}{k}+\frac{6}{k}\left(\frac{(3\alpha^2+\lambda_1)(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})}{3\alpha(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})-2(3\alpha^2+\lambda_1)I_2\mathrm{e}^{(-3\alpha\pm\hat{\alpha}/2)t}\pm\hat{\alpha}(1+I_1\mathrm{e}^{\pm\hat{\alpha}t})}\right). \tag{5.27}$$

On the other hand, the general solution can also be derived by extending the PS method itself. To do so, one has to consider the functionS_2. Thus, substituting the expression S_2 into (5.15) and solving it in the same way as outlined in the previous paragraphs, we obtain the following form for R, that is,

$$R_2=\frac{C_0(kx+3\beta)\mathrm{e}^{3(\alpha\mp\hat{\alpha})t/2}}{\left(3k\dot{x}-3\frac{(3\alpha\pm\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)^2}. \tag{5.28}$$

However, this set, (S_2,R_2), does not satisfy the extra constraint (5.16), and to recover the full form of the integrating factor we assume that

$$\hat{R}_2=F(I_1)R_2. \tag{5.29}$$

Substituting (5.29) into equation (5.16) we obtain an equation for F, that is, $I_1F'+2F=0$, where prime denotes differentiation with respect to I_1. Upon integrating this equation we obtain $F=1/I_1^2$, which fixes the form of $\hat{R}$ as

$$\hat{R}_2=\frac{C_0(kx+3\beta)\mathrm{e}^{3\alpha\pm\hat{\alpha}t/2}}{\left(3k\dot{x}-3\frac{(3\alpha\mp\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)^2}. \tag{5.30}$$

Table 1. *Integral factors, integrals of motion, linearizing transformations and the general solution of equation (5.33)*

null forms and integrating factors

$$S_1=\frac{k_1x+3\alpha}{3}-\frac{k_1\dot{x}}{k_1x+3\alpha},\quad R_1=\frac{C_0(k_1x+3\alpha)\mathrm{e}^{\mp\hat{\alpha}t}}{\left(3k_1\dot{x}-\frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2\right)^2},$$

$$S_2=\frac{k_1x+3\beta}{3}-\frac{k_1\dot{x}}{k_1x+3\beta},\quad R_2=\frac{C_0(k_1x+3\beta)\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2}}{\left(3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2\right)^2},$$

$$\alpha^3-k_2\alpha^2+\alpha\lambda_1-\frac{k_1\lambda_2}{3}=0,\quad \hat{\alpha}=\sqrt{-3\alpha^2+2\alpha^2k_2+k_2^2-4\lambda_1},\quad \hat{\beta}=3\alpha-k_2,$$

$$\beta=\frac{-\alpha+k_2\pm\hat{\alpha}}{2}$$

first integrals

$$I_1=\mathrm{e}^{\mp\hat{\alpha}t}\left(\frac{3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}{3k_1\dot{x}-\frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}\right),\quad C_0=9k_1\hat{\alpha},$$

$$I_2=\frac{-2\hat{\alpha}\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2}}{\hat{\beta}\pm\hat{\alpha}}\left(\frac{3k_1\dot{x}-3k_1x(\alpha-k_2)+k_1^2x^2+9\alpha^2-9\alpha k_2+9\lambda_1}{3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}\right)$$

linearizing transformations

$$w=\left(\frac{-3}{k_1x+3\alpha}+\frac{\hat{\beta}\pm\hat{\alpha}}{2(3\alpha^2-2\alpha k_2+\lambda_1)}\right)\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2},\quad z=\left(\frac{-3}{k_1x+3\alpha}+\frac{\hat{\beta}\mp\hat{\alpha}}{2(3\alpha^2-2\alpha k_2+\lambda_1)}\right)\mathrm{e}^{\hat{\beta}\mp\hat{\alpha}t/2}$$

solution

$$x=-\frac{3\alpha}{k_1}+\frac{1}{k_1}\left(\frac{6(3\alpha^2-2\alpha k_2+\lambda_1)(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})}{\hat{\beta}(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})\pm(\hat{\beta}\pm\hat{\alpha})I_1I_2\mathrm{e}^{-\hat{\beta}\pm\hat{\alpha}t/2}\pm\hat{\alpha}(1+I_1\mathrm{e}^{\pm\hat{\alpha}t})}\right)$$

Now, one can easily check that this set S_2 and $\hat{R}_2$ is a compatible solution for the equations (5.14)–(5.16). Substituting S_2 and $\hat{R}_2$ into (2.9), we can obtain an explicit form for the second integral I_2, that is,

$$I_2=-\left(\frac{2\hat{\alpha}(3k\dot{x}-3\alpha kx+k^2x^2+9\alpha^2+9\lambda_1)\mathrm{e}^{3\alpha\pm\hat{\alpha}t/2}}{(3\alpha\pm\hat{\alpha})\left(3k\dot{x}-3\frac{(3\alpha\mp\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)}\right).\tag{5.31}$$

Rewriting equation (5.22) for $\dot{x}$ and substituting it into (5.31) we get the same expression (5.27) as the general solution.

(*c*) *Example 3: generalized modified Emden-type equation*

Recently, Pandey *et al.* (submitted) have considered the following Liénard equation:

$$\ddot{x}+f(x)\dot{x}+g(x)=0,\tag{5.32}$$

where f and g are arbitrary functions of their arguments, and classified systematically, all polynomial forms of f and g which admit eight Lie point symmetry generators with their explicit forms. They found that the most general nonlinear ODE which is linear in $\dot{x}$ whose coefficients are functions of the dependent variable alone should be of the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x + \lambda_2 = 0, \tag{5.33}$$

where k_i and λ_i, $i=1,2$, are arbitrary parameters, which is consistent with the criteria (5.1) and (5.2) given by Mahomed & Leach (1989*b*). Interestingly, equation (5.33) and all its sub-cases possess $sl(3,R)$ symmetry algebra. For example, we discussed the integrability and linearization of equation (5.33) with $k_2=0$ in the previous example. As the linearizing transformations and the general solution of equation (5.33) are yet to be reported, we include this equation as an example in the present work. As in the previous case we divide our analysis into three cases.

(i) $\lambda_1=0$, $\lambda_2=0$: modified Emden-type equation with quadratic and cubic nonlinearity

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 = 0. \tag{5.34}$$

(ii) $\lambda_1\neq 0$, $\lambda_2=0$: modified Emden-type equation with quadratic and linear terms

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0. \tag{5.35}$$

(iii) $\lambda_1\neq 0$, $\lambda_2\neq 0$: the full generalized modified Emden-type equation (5.33).

We have derived the integrating factors, integrals of motion, linearizing transformations and the general solutions for all the cases. As the calculations are similar to the ones discussed in the previous case, we present the results in tabular form (table 1), where the results for the most general case (5.33) have been given, from which the results for the limiting cases (5.34) and (5.34) can be deduced.

6. Conclusion

In this paper we have discussed the method of finding general solutions associated with second-order nonlinear ODEs through a modified PS method. The method can be considered as a direct one, complementing the well-known method of Lie symmetries. In particular, we have extended the theory of Duarte *et al.* (2001), such that new integrating factors and their associated integrals of motion can be recovered. These integrals of motion can be utilized to construct the general solution. In the situation where the second integral of motion cannot be recovered, we introduced another approach to derive the second integration constant. Interestingly, we showed that, in this case, it can be derived from the first integral itself, in a simple and elegant way. Apart from the above, we introduced a technique which can be utilized to derive linearizing transformation from the first integral. We illustrated the theory with several new examples and explored their underlying solutions.

In this paper we concentrated our studies only on single second-order ODEs. In principle, the method can also be extended to third-order ODEs and systems of second-order ODEs. The results will be published elsewhere.

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NOTICE OF CORRECTION

The sentence preceeding equation (4.48) is now present in its correct form.

Equation (4.49) is now present in its correct form.

A detailed erratum will appear at the end of volume 464.

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Errata

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On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations

By V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan

The sentence preceeding equation (4.48) is incorrect and should read as follows.

Making the same form of an ansatz, *vide* equations (4.5) and (4.8), we find non-trivial solutions only exist for equations (4.45) and (4.46) for the parametric restrictions $c_2 = \pm(6c_1^2/25)$. However, the case $c_2 = -(6c_1^2/25)$ follows from the case $c_2 = +(6c_1^2/25)$ in equation (4.44) through the simple translation $x = X + (6c_1^2/(25\beta))$. So we consider only the case $c_2 = +(6c_1^2/25)$ in the following

Equation (4.49) contains typographical errors that have no consequence for any other equations or results in the above paper.

$$S_2 = \frac{\left(c_1\dot{x} + \frac{6c_1^2 x}{25} - \beta x^2\right)}{\dot{x}}, \quad R_2 = -\dot{x}e^{c_1 t}. \tag{4.49}$$

The authors note that in addition to the first integral for the choice of the parameters $\alpha = (4/\beta)$, $\gamma = -(3/\beta^2)$ for equation (4.60) noted in the paper, there exists the more general case $\alpha = (3/\beta) - (\gamma\beta/3)$, for which time dependent integral exists. This has been shown later on in equation (104) of Chandrasekar *et al.* (2006).

Additional reference

Chandrasekar, V. K., Pandey, S. N., Senthilvelan, M. & Lakshmanan, M. 2006 A simple and unified approach to identify integrable nonlinear oscillators and systems. *J. Math. Phys.* **47**, 023 508. (doi:10.1063/1.2171520)