Dynamics of a piecewise linear map with a gap

BY S. J. Hogan*, L. Higham and T. C. L. Griffin

Department of Engineering Mathematics, University of Bristol, Queen’s Building, University Walk, Bristol BS8 1TR, UK

In this paper, we consider periodic solutions of discontinuous non-smooth maps. We show how the fixed points of a general piecewise linear map with a discontinuity (‘a map with a gap’) behave under parameter variation. We show in detail all the possible behaviours of period 1 and period 2 solutions. For positive gaps, we find that period 2 solutions can exist independently of period 1 solutions. Conversely, for negative gaps, period 1 and period 2 solutions can coexist. Higher periodic orbits can also exist and be stable and we give several examples of how these solutions behave under parameter variation. Finally, we compare our results with those of Jain & Banerjee (Jain & Banerjee 2003 Int. J. Bifurcat. Chaos 13, 3341–3351) and Banerjee et al. (Banerjee et al. 2004 IEEE Trans. Circ. Syst. II 51, 649–654) and explain their numerical simulations.

Keywords: non-smooth maps; map with a gap; periodic solutions

1. Introduction

Non-smooth systems are found in many areas of engineering and science including mechanical systems involving friction, electrical circuits containing diodes, transistors and thyristors, earthquake engineering, suspension bridge dynamics and neurons in the brain firing above a certain threshold.

Often the analysis of such a system is simplified by reducing the dynamics to an equivalent non-smooth map. A non-smooth map is one where the region of definition of the map is divided into two or more parts, separated by boundaries (surfaces of discontinuity), with different expressions for the map in each part. A non-smooth map is generally smooth wherever it is defined except at the boundaries, where it can be either continuous (but not differentiable) or discontinuous (‘a map with a gap’).

There is an enormous body of research on the continuous case (see Zhusubaliyev & Mosekilde 2003; Di Bernardo et al. 2006). Less work has been done on discontinuous non-smooth systems. These systems arise naturally in systems with time delays or imperfections or spatial discontinuities such as backlash. The earliest work known to the authors is Keener (1980), where dynamics on the interval were studied and where the presence of chaos in such a system was shown. More recent work has included LoFaro (1996), where such systems were shown to possess period-adding bifurcations; Qu et al. (1997), where the presence of multiple devil’s staircases was demonstrated; and Qu et al. 

* Author for correspondence (s.j.hogan@bristol.ac.uk).
(1998), where type V intermittency was found. A quadratic map with a gap defined on the interval was studied by Avrutin & Schanz (2004, 2005). In Kollar et al. (2004), the dynamics of symmetric bilinear maps representing classes of controlled sampled systems with both backlash and delay was examined and in Belykh et al. (2000), a discontinuous map was used to help explain the emergence of bursting oscillations in cell models. Another work on area-preserving maps includes Christiansen et al. (1992), Guan et al. (1995), Wang et al. (2001), He et al. (2004) and Lai et al. (2005).

Discontinuous non-smooth maps that are not restricted to the unit interval are the subject of this paper. Less work has been done in this area. Jain & Banerjee (2003) made the key observation that important classes of switching systems (such as thyristor circuits) yield discontinuous maps under discrete time modelling. The authors presented the first attempt to classify non-smooth bifurcations of these maps and successfully applied their results to a practical example of a switching circuit with a delay in the feedback loop. Their analysis was largely based on the numerical simulation.

The paper by Banerjee et al. (2004) continues the work of Jain & Banerjee (2003) with emphasis on the experiments. The authors pointed out that past investigations on nonlinear phenomena in DC–DC converters had assumed ideal switching. They showed that the unavoidable imperfections in switching result in a discontinuity in the map. They demonstrated these effects experimentally and predicted the bifurcations in such systems with reasonable accuracy. They also derived the limiting conditions for reliable period 1 operation when these effects are considered. They made the remarkable experimental observation that when the inductor current is plotted on a bifurcation diagram as a function of input voltage, stable period 1 and period 2 solutions can exist together. This was further supported by the numerical simulations for which they give complete details of the parameter values used.

In §2 of this paper, we consider the discontinuous version of the piecewise map in Di Bernardo et al. (1999) in its general $n$-dimensional form. Then, we examine the specific case of the one-dimensional map, examining period 1 solutions in §3 and period 2 solutions in §4. Both sets of results are brought together in §5. Higher period orbits are examined in §6 and comparison with other work is made in §7. An unpublished report by Higham (2000), containing the general analysis of §2 as well as the analysis of the higher order periodic solutions of §6, forms the basis of this work.

2. General case

We consider maps of the form

$$x^{(k+1)} = \begin{cases} 
A_1 x^{(k)} + c\mu, & x_n^{(k)} > 0, \\
A_2 x^{(k)} + c(\mu + \gamma), & x_n^{(k)} < 0,
\end{cases}$$

where $\mu, \gamma \in \mathbb{R}$ are scalars, $A_1$ and $A_2$ real $n \times n$ matrices and $c$ an $n \times 1$ vector. We denote the $k$th iteration of the map by $x^{(k)}$, where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. In Di Bernardo et al. (1999), the case $\gamma = 0$ was examined in detail. It is important
to emphasize that, as shown in Di Bernardo et al. (2006), when $\gamma=0$, many important non-smooth flows do not have this bilinear normal form. Instead, they have a power law dependence on one side of their surface of discontinuity. However, as this was the case with the work of Di Bernardo et al. (1999), the study of this bilinear system (2.1) can be expected to shed considerable light on these cases.

A simple scaling argument shows that there are actually only two cases to be considered when $\gamma \neq 0$, namely $\gamma = \pm 1$. These latter two cases are the subject of this paper. Note that the map defined in equation (2.1) is discontinuous at $x(z_k) = 0$, $\mu = 0$ for all $k$. The map is smooth for $x(z_j) = 0$, $\mu = 0$, $j = 1, \ldots, n-1$ and so the matrices $A_1 = [a_{ij}^{(1)}]$ and $A_2 = [a_{ij}^{(2)}]$ are such that $a_{ij}^{(1)} = a_{ij}^{(2)}$, $\forall i, j \neq n$. We proceed to examine the behaviour of the fixed points as the parameter $\mu$ is varied. The form of the general analysis is almost identical to that of §3 of Di Bernardo et al. (1999) and hence the relevant details are included here (a slightly different notation has also been used).

\begin{equation}
(a) \text{ Existence of fixed points}
\end{equation}

Let $M^*$ and $M^{**}$ be the fixed points of (2.1). So

\begin{align}
M^* &= A_1 M^* + c\mu, \quad m^*_n > 0, \\
M^{**} &= A_2 M^{**} + c(\mu + \gamma), \quad m^{**}_n < 0,
\end{align}

where $m^*_k = [M^*]_k$, $m^{**}_k = [M^{**}]_k$. Thus, assuming $A_1 - I$ and $A_2 - I$ to be invertible, we find

\begin{equation}
M^* = -\frac{\text{adj}(A_1 - I)}{p^*(1)} c\mu,
\end{equation}

where $p^*(1)$ is the characteristic polynomial of $A_1$ evaluated at 1. Similarly,

\begin{equation}
M^{**} = -\frac{\text{adj}(A_2 - I)}{p^{**}(1)} c(\mu + \gamma),
\end{equation}

where $p^{**}(1)$ is the characteristic polynomial of $A_2$ evaluated at 1.

In scalar form, we have

\begin{equation}
m^*_k = \frac{b^*_k}{p^*(1)} \mu, \quad m^{**}_k = \frac{b^{**}_k}{p^{**}(1)} (\mu + \gamma),
\end{equation}

where $b^*_k = [-\text{adj}(A_1 - I)c]_k$ and $b^{**}_k = [-\text{adj}(A_2 - I)c]_k$. Using a proof identical to that in Di Bernardo et al. (1999), it can be shown that $b^*_n = b^{**}_n =: b_n$, hence

\begin{equation}
m^*_n = \frac{b_n}{p^*(1)} \mu, \quad m^{**}_n = \frac{b_n}{p^{**}(1)} (\mu + \gamma).
\end{equation}

We require $m^*_n > 0$ and $m^{**}_n < 0$ for each fixed point to exist. So in a region where both fixed points exist together, or in a region where neither of them exists, the product $m^*_n m^{**}_n < 0$. In a region where just one fixed point exists, this product will be positive.

Therefore, either both fixed points will coexist or none of them will exist if
\[
\frac{\mu(\mu + \gamma)}{p^*(1)p^{**}(1)} < 0,
\]  
and only one of the fixed points will exist if
\[
\frac{\mu(\mu + \gamma)}{p^*(1)p^{**}(1)} > 0.
\]

Note that when \( \gamma = 0 \), equations (2.8) and (2.9) reduce to eqns (17) and (18) of Di Bernardo et al. (1999), respectively, as expected.

In equations (2.8) and (2.9), we can see that there are three different regions of behaviour as \( \mu \) is varied, with the number of solutions changing at \( \mu = 0 \) and \( \mu = -\gamma \). If \( \gamma = 1 \), then the changes occur at \( \mu = -1, 0 \), whereas if \( \gamma = -1 \), the changes occur at \( \mu = 0, 1 \).

In Di Bernardo et al. (1999), it was shown that knowledge of the characteristic polynomials was enough to distinguish between the different types of behaviour. Here, the process is subtler with the gap (\( \gamma \neq 0 \)) in the map allowing the changes to occur more slowly as \( \mu \) varies.

In fact, if \( p^*(1)p^{**}(1) < 0 \), then as \( \mu \) varies (denoted by \( \leftrightarrow \)), we have the possibility that the ordering is either
\[
\text{no fixed point} \leftrightarrow \text{one fixed point} \leftrightarrow \text{two fixed points}, 
\]  
(2.10)

or
\[
\text{two fixed points} \leftrightarrow \text{one fixed point} \leftrightarrow \text{no fixed point}. 
\]  
(2.11)

Whereas if \( p^*(1)p^{**}(1) > 0 \), then we have either
\[
\text{one fixed point} \leftrightarrow \text{two fixed points} \leftrightarrow \text{other fixed point}, 
\]  
(2.12)

or
\[
\text{one fixed point} \leftrightarrow \text{no fixed point} \leftrightarrow \text{other fixed point}. 
\]  
(2.13)

In Di Bernardo et al. (1999), where \( \gamma = 0 \), equations (2.10) and (2.11) both reduce to no fixed point \( \leftrightarrow \) two fixed points, thus behaving like a saddle-node bifurcation, and equations (2.12) and (2.13) both reduce to one fixed point \( \leftrightarrow \) other fixed point, a transition between the two solutions.

(b) Classification through eigenvalues

In §2a, we have shown that, depending on the product \( p^*(1)p^{**}(1) \), the number of fixed points changes as \( \mu \) varies; but the roots of a characteristic polynomial are the eigenvalues of the map, so we can also discuss the number of fixed points in terms of eigenvalues. In particular, let \( \{\alpha_i\}_{i=1,2,...,n} \) and \( \{\beta_i\}_{i=1,2,...,n} \) be the eigenvalues of \( A_1 \) and \( A_2 \), respectively, then
\[
p^*(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_n), 
\]  
(2.14)
\[
p^{**}(\lambda) = (\lambda - \beta_1)(\lambda - \beta_2)\cdots(\lambda - \beta_n), 
\]  
(2.15)

so
\[
p^*(1) = (1 - \alpha_1)(1 - \alpha_2)\cdots(1 - \alpha_n), 
\]  
(2.16)
\[ p^{**}(1) = (1 - \beta_1)(1 - \beta_2) \ldots (1 - \beta_n). \] (2.17)

Hence the sign of \( p^*(1)p^{**}(1) \) will depend on the number of real eigenvalues greater than 1. Thus, if we define \( \sigma^+_\alpha \) to be the number of real eigenvalues \( (\alpha_i) \) of \( A_1 \) greater than 1 and \( \sigma^+_\beta \) to be the number of real eigenvalues \( (\beta_i) \) of \( A_2 \) greater than 1, we find that

(i) \( p^*(1)p^{**}(1) < 0 \Leftrightarrow \sigma^+_\alpha + \sigma^+_\beta \) is odd \( \Leftrightarrow \) two/no fixed points \( \Leftrightarrow \) one fixed point \( \Leftrightarrow \) no/two fixed points, and

(ii) \( p^*(1)p^{**}(1) > 0 \Leftrightarrow \sigma^+_\alpha + \sigma^+_\beta \) is even \( \Leftrightarrow \) one fixed point \( \Leftrightarrow \) no/two fixed points \( \Leftrightarrow \) other fixed point.

\section*{3. A One-Dimensional Piecewise Linear Map with a Discontinuity}

For the rest of the paper, we study in detail the following scalar version of equation (2.1)

\[ x_{n+1} = \begin{cases} \alpha x_n - \mu, & x_n > 0, \\ \beta x_n - (\mu + \gamma), & x_n < 0, \end{cases} \] (3.1)

where \( \alpha, \beta, \gamma \) and \( \mu \) are real parameters. For definiteness following Di Bernardo \textit{et al.} (1999), we take \( \alpha > 0, \beta < 0 \). As before \( \gamma = 0, \pm 1 \) only. Note that equation (3.1) is invariant separately under each of the transformations

\[ (\alpha, \beta, \mu) \rightarrow (-\beta, -\alpha, -(\mu + \gamma)), \] (3.2)

\[ (\alpha, \beta, \mu) \rightarrow (\beta, \alpha, -(\mu + \gamma)). \]

We need a notation that can distinguish between fixed points, higher order periodic solutions and provide information about stability. Thus, following Di Bernardo \textit{et al.} (1999), we define

(i) \( A, a \) denotes a \textit{stable, unstable} fixed point (period 1 solution) \( > 0 \),

(ii) \( B, b \) denotes a \textit{stable, unstable} fixed point (period 1 solution) \( < 0 \),

(iii) \( AB, ab \) denotes a \textit{stable, unstable} period 2 solution with one positive iterate and one negative iterate, and

(iv) \( A^{k-p}B^p, a^{k-p}b^p \) denotes a \textit{stable, unstable} period \( k \) solution with \( k-p \) positive iterates and \( p \) negative iterates.

In §3a, we examine the occurrence of simple (period 1) fixed points in equation (3.1).

(a) \textit{Simple fixed points}

For completeness, we recap results from Di Bernardo \textit{et al.} (1999) for the case \( \gamma = 0 \). In this case, the map defined by equation (3.1) has one fixed point \( M^* \equiv x^* \), given by

\[ x^* = \frac{\mu}{\alpha - 1}. \] (3.3)
By definition, $x^*/C_3/O_0$ and so exists for $m/O_0$ if $a/O_1$, and for $m/O!$ if $a/O!$. In addition, it has eigenvalue $a$ and hence stable ($A$) for $a/O!$ and unstable ($A$) for $a/O!$. Similarly, the map (3.1) has another fixed point $M--h/x^*/C_3/C_3$, given by

$$\frac{\mu}{\beta-1}. \quad (3.4)$$

Here, $x^{**}/C_3/O_0$ and so exists only for $\mu/O$ (since $\beta/O$). It has eigenvalue $\beta$ and hence stable ($B$) for $\beta/O$, and unstable ($B$) for $\beta/O$. Thus, for $-1/\beta/O$ and $0/O$, the simplest non-smooth bifurcation is the simple transition $A\leftrightarrow B$. On the other hand, if $-1/\beta/O$ and $\alpha/O$, then we have the non-smooth equivalent of the saddle-node bifurcation, namely $\emptyset\leftrightarrow a\leftrightarrow B$. In a similar way, if $\beta/O$, the simplest non-smooth bifurcations are $A\leftrightarrow b$ when $0/O$, and $\emptyset\leftrightarrow a\leftrightarrow b$ when $\alpha/O$. These results are summarized graphically in figure 1.

For $\gamma/O$, the map defined by equation (3.1) is shown in figure 2 for the case $\alpha/O$, $\beta/O$ for three ranges of $\mu$. For $\mu/O$, there is only one fixed point $x^*$ (in blue), given by equation (3.3). As $\mu/O$, the two arms of the map move upward until (when $\mu=0$) the left-hand end of the right-hand branch intersects the (dotted) line $x_{n+1}=x_n$ to give rise to a second fixed point $x^{**}$ (in red). These two fixed points coexist until $\mu/O$ when $x^*$ ceases to exist, leaving $x^{**}$ as the only fixed point in the range $\mu/O$. 

**Figure 1.** Behaviour of fixed points of equation (3.1) in parameter space for $\gamma/O$. 

**Figure 2.** Equation (3.1) for $\gamma/O$ with $\alpha/O$, $\beta/O$ as $\mu/O$. Note the two coexisting solutions when $\mu/O$. Horizontal axis $x_n$, vertical axis $x_{n+1}$, dotted line $x_{n+1}=x_n$.
The fixed point $x/C3$ exists for $m \neq 0$, if $a \neq 1$ when it is unstable (A), and exists for $m \neq 0$, if $a \neq 1$ when it is stable (B). The fixed point $x/C3/C3$ is given by

$$x/C3/C3 = \frac{m - 1}{\beta - 1}. \quad (3.5)$$

Since $x^{**} < 0$, it exists only for $m + 1 > 0$ because $\beta < 0$. It has eigenvalue $\beta$ and hence stable (B) for $\beta > -1$ and unstable (b) for $\beta < -1$. These results are summarized graphically in figure 3.

When $\gamma = -1$, the map is shown in figure 4 for the case $\alpha \in (0,1)$, $\beta \in (-1,0)$ for three ranges of $\mu$. For $\mu \in (1, \infty)$, there is only one fixed point $x^*$ (in blue), again given by equation (3.3). As $\mu$ decreases, the two arms of the map move upward until (when $\mu = 1$) the left-hand branch no longer intersects $x_{n+1} = x_n$. For $\mu \in (0,1)$, neither branch of the map intersects $x_{n+1} = x_n$. Only when $\mu = 0$ the right-hand branch gives rise to a second fixed point $x^{**}$ (in red) in the range $\mu \in (-\infty, 0)$ given by

$$x^{**} = \frac{\mu - 1}{\beta - 1}. \quad (3.6)$$

Since $x^{**} < 0$, it exists only for $\mu - 1 > 0$ because $\beta < 0$. It has eigenvalue $\beta$ and hence stable (B) for $\beta > -1$ and unstable (b) for $\beta < -1$. These results are summarized graphically in figure 5.

Figure 3. Behaviour of fixed points of equation (3.1) in parameter space for $\gamma = 1$. $\mu \in (-\infty, -1) \leftrightarrow \mu \in (-1, 0) \leftrightarrow \mu \in (0, \infty)$.

Figure 4. Equation (3.1) for $\gamma = -1$ with $\alpha \in (0, 1)$, $\beta \in (-1, 0)$ as $\mu$ varies. Note the absence of solutions when $\mu \in (0, 1)$. Horizontal axis $x_n$, vertical axis $x_{n+1}$, dotted line $x_{n+1} = x_n$. The fixed point $x^*$ exists for $\mu > 0$, if $\alpha > 1$ when it is unstable (A), and exists for $\mu < 0$, if $\alpha < 1$ when it is stable (A). The fixed point $x^{**}$ is given by

$$x^{**} = \frac{\mu + 1}{\beta - 1}. \quad (3.5)$$

Since $x^{**} < 0$, it exists only for $\mu + 1 > 0$ because $\beta < 0$. It has eigenvalue $\beta$ and hence stable (B) for $\beta > -1$ and unstable (b) for $\beta < -1$. These results are summarized graphically in figure 3.

When $\gamma = -1$, the map is shown in figure 4 for the case $\alpha \in (0, 1)$, $\beta \in (-1, 0)$ for three ranges of $\mu$. For $\mu \in (1, \infty)$, there is only one fixed point $x^*$ (in blue), again given by equation (3.3). As $\mu$ decreases, the two arms of the map move upward until (when $\mu = 1$) the left-hand branch no longer intersects $x_{n+1} = x_n$. For $\mu \in (0,1)$, neither branch of the map intersects $x_{n+1} = x_n$. Only when $\mu = 0$ the right-hand branch gives rise to a second fixed point $x^{**}$ (in red) in the range $\mu \in (-\infty, 0)$ given by

$$x^{**} = \frac{\mu - 1}{\beta - 1}. \quad (3.6)$$

Since $x^{**} < 0$, it exists only for $\mu - 1 > 0$ because $\beta < 0$. It has eigenvalue $\beta$ and hence stable (B) for $\beta > -1$ and unstable (b) for $\beta < -1$. These results are summarized graphically in figure 5.
4. Period 2 solutions

Since both parts of equation (3.1) are linear, period 2 solutions cannot exist with all iterates either positive or negative. Therefore, the only period 2 solution that equation (3.1) can support is \( AB = ab \). In this section, we present a detailed analysis of period 2 solutions to equation (3.1) when \( g \in (1, \infty) \). The results are sometimes counter intuitive. We shall show, for example, that an \( AB \) solution can occur without a coexisting \( a \) or \( b \) solution so that this solution can bifurcate, under parameter variation, from a region with no period 1 solutions.

(a) Period 2 solutions with \( \gamma = 1 \)

From equation (3.1) with \( \gamma = 1 \), the period 2 solution is given by \((x_1^{**}, x_2^{**})\), where \( x_1^{**} < 0, x_2^{**} > 0 \) and

\[
x_1^{**} = \frac{\mu(1 + \alpha) + \alpha}{\alpha \beta - 1}, \tag{4.1}
\]

\[
x_2^{**} = \frac{\mu(1 + \beta) + 1}{\alpha \beta - 1}. \tag{4.2}
\]

Here, \( \alpha \beta < 0 \), hence \( \alpha \beta - 1 < 0 \) and so equations (4.1) and (4.2), respectively, imply that

\[
\mu(1 + \alpha) + \alpha > 0, \tag{4.3}
\]

\[
\mu(1 + \beta) + 1 < 0. \tag{4.4}
\]

If we take \( \mu > 0 \), then equation (4.3) (together with the fact that \( \alpha > 0 \)) implies that

\[
\alpha > 0, \tag{4.5}
\]

and from equation (4.4) we find that

\[
\beta < -\left(1 + \frac{1}{\mu}\right). \tag{4.6}
\]

Hence both equations (4.5) and (4.6) are required for a period 2 solution to exist for \( \mu > 0 \) when \( \gamma = 1 \). On the other hand, if \( \mu < 0 \), it is straightforward to show from equations (4.3) and (4.4) that no period 2 solution is possible when \( \gamma = 1 \).
Figure 6. Period 1 and period 2 solutions of (3.1) for $\gamma=1$ with (a) $\mu \in (-\infty, -1)$, (b) $\mu \in (-1, 0)$ and (c) $\mu \in (0, \infty)$.

Figure 7. Period 1 and period 2 solutions of (3.1) for $\gamma=-1$ with (a) $\mu \in (-\infty, 0)$, (b) $\mu \in (0, 1/2)$, (c) $\mu \in (1/2, 1)$ and (d) $\mu \in (1, \infty)$.
From equation (3.1) with $\gamma = 1$, the period 2 solution is given by $(x_1^*, x_2^*)$, where $x_1^* < 0$, $x_2^* > 0$ and

$$x_1^* = \frac{\mu(1 + \alpha) - \alpha}{\alpha \beta - 1}, \quad \alpha > 1,$$  \hspace{1cm} (4.7)

$$x_2^* = \frac{\mu(1 + \beta) - 1}{\alpha \beta - 1}.$$

(b) Period 2 solutions with $\gamma = 1$

Figure 8. Existence and stability boundaries of higher order solutions of the form $A^{k-1}B/a^{k-1}b$ that exist for $\gamma = 0$ with $\mu > 0$. Regions that have one stable solution are highlighted in shades of blue; the darker the shade, the lower the periodicity of the stable solution.

Figure 9. Existence and stability boundaries of higher order solutions of the form $A^{k-1}B/a^{k-1}b$ that exist for $\gamma = 1$ with $\mu = 1/4$. Regions that have one stable solution are highlighted in shades of blue; the darker the shade, the lower the periodicity of the stable solution.
Since $\alpha \beta - 1 < 0$, equations (4.7) and (4.8) imply that

$$\mu (1 + \alpha) - \alpha > 0,$$

(4.9)

and

$$\mu (1 + \beta) - 1 < 0.$$

(4.10)
If $\mu > 0$, then analysis of equations (4.9) and (4.10) (together with $\alpha > 0$, $\beta < 0$) shows that $AB/ab$ exists for $\mu \in (0,1)$ when both

$$\alpha < -\frac{\mu}{1-\mu}, \quad \beta < 0,$$

(4.11)

are satisfied and for $\mu \in (1,\infty)$ when both

$$\alpha > 0, \quad \beta < -1 + \frac{1}{\mu},$$

(4.12)

are satisfied. If $\mu < 0$, equations (4.9) and (4.10) imply that no period 2 solution is possible when $\gamma = -1$.

(c) Stability of period 2 solutions

For any period 2 solution of the map (3.1) for all values of $\gamma$, the Jacobian is given by $\alpha \beta$. Hence that solution, if it exists, is stable/unstable when

$$\alpha \beta \geq -1.$$

(4.13)

5. Period 1 and period 2 solutions

We summarize graphically our results for the existence and stability of period 1 and period 2 solutions present in equation (3.1) for $\gamma = \pm 1$. When $\gamma = 1$, we have three cases to consider. These are shown in figure 6. When $\gamma = -1$ we have four cases to consider. These are shown in figure 7. We shall discuss these results in subsequent sections.

6. Higher periodic solutions

We wish to find the existence and stability boundaries for higher periodic solutions of equation (3.1). In particular, we shall consider solutions of the form $A^{k-p}B, a^{k-p}b$, where $k > 1, p \geq 1, k > p$. It can be shown for $\gamma = 0$ (Higham 2000) that solutions of this form with $p > 1$ are unstable, if they exist. An extension of this straightforward argument shows that this is also true for $\gamma = -1$. There may be small parameter windows for which such solutions exist and are stable when $\gamma = 1$, but each value of $p$ requires a separate analysis. We did not observe any such solutions in simulations, which implies that any basin of attraction must be very small. So, in what follows, we shall only consider the case $p = 1$.

(a) Existence and stability of periodic solutions of the form $A^{k-1}B, a^{k-1}b$

By a simple extension of the argument in Di Bernardo et al. (1999), it is straightforward to show that, for $\gamma = 0, \pm 1$, an $A^{k-1}B, a^{k-1}b$ solution exists for $\mu > 0$ if and only if the following two conditions are satisfied

$$0 < \left(1 + \frac{\gamma}{\mu} + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{k-1}}\right),$$

(6.1)

$$\beta < -\left(1 + \frac{\gamma}{\mu} + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{k-2}}\right).$$

(6.2)

Note that if $\gamma \geq 0$, equation (6.2) $\Rightarrow$ equation (6.1) since $\beta < 0$. 

It is clear that $\forall \gamma$ the eigenvalue of the $A^{k-1}B$, $a^{k-1}b$ solution is $\alpha^{k-1}\beta$ and so is stable, if and only if
\[ \beta > \frac{1}{\alpha^{k-1}}. \] 
(6.3)

(b) Bifurcations between periodic solutions

We plot the above existence and stability curves in $(\alpha, \beta)$ space. As before, we begin by recapping the work of Di Bernardo et al. (1999) when $\gamma = 0$. This is shown in figure 8 (a redrawing of fig. 22 of that paper). As with all the other diagrams in this section, the upper, or left hand, curve of each shaded region is always the existence boundary for that particular higher order solution, whereas the lower, or right hand, curve is always the stability boundary.

When $\gamma \neq 0$, we have a different bifurcation diagram for each value of $\mu$, since $\gamma$ and $\mu$ occur together in equations (6.1) and (6.2). Thus, for $\gamma = 1$, we plot bifurcation diagrams for $\mu = 1/4, 3/2$ in figures 9 and 10. Note that we have shown in §6 that no higher periodic solutions are possible in this parameter range for $\mu \leq 0$. Figures 9 and 10 are to be compared with figure 6c. The upper bound for the existence of the $AB, ab$ solution is given by $\beta = -(1 + (1/\mu))$. Thus, as $\mu$ increases from zero, at first no higher order solutions are possible since that boundary is at (minus) infinity, but then as $\mu$ increases, the boundary tends towards $\beta = -1$ and at which point the bifurcation diagram looks like figure 8.

The stability boundaries are given by $\alpha^{k-1}\beta = -1$ and these do not change as $\mu$ is varied. It is only the existence boundaries (6.1) and (6.2) that depend on $\mu$. Thus, if we fix $\alpha$ and $\beta$ then in general, we will see the rise of a stable higher periodic solution as $\mu$ increases. In the unstable regions, we expect to see chaotic solutions (Homer et al. 2004).

In general, we note that the effect of a positive ($\gamma = 1$) discontinuity is to delay the onset of stable high-order periodic solutions for fixed $\alpha, \beta$ as $\mu$ varies in the range $(0, \infty)$.

When $\gamma = -1$, we take $\mu = 1/6, 5/3$. The bifurcation diagrams, shown in figures 11 and 12, are to be compared with figure 7b,d, respectively. The upper bound for the existence of the $AB, ab$ solution is given by $\beta = (1 - \mu)/\mu$. Because we have restricted $\beta < 0$, this means that this bound is effectively $\beta = 0$ for $\mu \in (0,1)$. Thus, as $\mu$ increases from 0, at first no higher order solutions are possible, since that boundary is at infinity, but then as $\mu$ increases, the boundary tends towards $\beta = -1$ and at which point the bifurcation diagram looks like figure 8.

In general, the effect of a negative ($\gamma = -1$) discontinuity is to accelerate the onset of stable high-order periodic solutions for fixed $\alpha, \beta$ as $\mu$ varies.

7. Comparison with other work

In Jain & Banerjee (2003) and Banerjee et al. (2004), the map
\[ x_{n+1} = \begin{cases} 
  x_n - 1.5, & x_n > 0, \\
  \beta x_n - 0.5, & x_n < 0,
\end{cases} \] 
(7.1)
was considered with $\beta \in (-0.3, -2)$. We shall use the results from §5 to explain the simulations of Banerjee et al. (2004). In their fig. 13, they showed a stable
period 1 solution that went unstable at $\beta = -1$, a stable period 2 solution that appeared around $\beta = 0.33$, which also lost stability at $\beta = -1$, and chaos for $\beta < -1$. The period 2 map corresponding to equation (7.1) is given by

$$x_{n+2} = \begin{cases} 
\beta x_n - 2, & x_n \in (-\infty, \frac{1}{2\beta}), \\
\beta^2 x_n - \frac{1}{2}(1 + \beta), & x_n \in \left(\frac{1}{2\beta}, 0\right), \\
\beta x_n - \frac{3}{2} \beta - \frac{1}{2}, & x_n \in \left(0, \frac{3}{2}\right), \\
x_n - 3, & x_n \in \left(\frac{3}{2}, \infty\right). 
\end{cases} \quad (7.2)$$

This is shown in figure 13 for $\beta = -(1/4)$, where we see only the (stable) period 1 solution. The other three sections of the map do not intersect the (dashed) line $x_{n+2} = x_n$, so no other solutions exist. Figure 14 shows the cases $\beta = -(1/3), -(1/2), -1, -(3/2)$. In figure 14a, with $\beta = (1 - \mu)/\mu = -(1/3)$, the period 2 solution of the map arises as the first and third sections of the map, which have identical slope $\beta$ and touch the line $x_{n+2} = x_n$ for the first time. It is stable since $\beta > -1$. Figure 14b shows the coexistence of both the period 1 and period 2 solutions. In figure 14c, $\beta = -1$ and both solutions are about to go unstable together and in figure 14d, we have no stable periodic attractors. Figures 13 and 14 provide the graphical explanation for the behaviour of periodic solutions in fig. 13 of Banerjee et al. (2004).

Note that while the coexistence of the period 1 and period 2 solutions of this map is generic, the loss of stability of these solutions together is not. This happens only when $\alpha = 1$. In general, the period 2 solution can remain stable for $\beta \in (-(1/\alpha), -1)$. 

---

Figure 13. Equation (7.2) with $\beta = -(1/4)$.

Figure 14. Equation (7.2) with (a) $\beta = -(1/3)$, (b) $\beta = -(1/2)$, (c) $\beta = -1$ and (d) $\beta = -(3/2)$.
when the period 1 solution is unstable. Also, for $\alpha \in (1, \mu/(\mu - 1))$ and $\beta \in (-1, (1 - \mu)/\mu)$, the period 1 solution is stable but the period 2 solution is unstable for $\beta \in (-1, -(1/\alpha))$. This situation is shown in figure 7d.

8. Conclusions

We have shown how the fixed points of a general piecewise linear map with a discontinuity (‘a map with a gap’) of the form (2.1) behave under parameter variation. Depending on the sign of the product of two characteristic polynomials, there are two pairs of possibilities, given by equations (2.10)–(2.13). In the specific case of the one-dimensional map (3.1), we have shown that there is a simple scaling which means that we need to consider only maps with either positive ($\gamma = 1$) or negative ($\gamma = -1$) gaps. What is more, the maps have a simple invariance, given by equation (3.2), which means that the amount of analysis required to understand the full ($\alpha, \beta$) parameter space is considerably reduced. We have also shown all the possible behaviours of period 1 and period 2 solutions of equation (3.1) under parameter variation. For positive gaps, we find that period 2 solutions can exist independently of period 1 solutions. Conversely, for negative gaps, period 1 and period 2 solutions can coexist. Higher periodic solutions of equation (2.1) can also exist and be stable and we have given several examples of how these solutions behave under parameter variation. Finally, we have compared our results with those of Jain & Banerjee (2003) and Banerjee et al. (2004) and were able to explain the results of their numerical simulations for which parameter values are available.

This work was partly funded by the EPSRC grant GR/72020/01.

References


