Dislocation lines in the swallowtail diffraction catastrophe

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The three-dimensional distribution of amplitude and phase represented by the swallowtail catastrophe diffraction integral is based on a network of null lines for amplitude (wave dislocations or optical vortices), where the phase is singular. In the tail region there is four-wave interference, which results in an approximately repeating pattern of amplitude based on the monoclinic space group $C2/m$ and also an approximately repeating pattern of wave dislocations based on the black–white monoclinic space group $C2/m'$. Helical dislocations spring from the plane of symmetry and gradually straighten out to be parallel to the two riblines of the caustic; eventually they become the straight dislocations of the Pearcey pattern for the cusp catastrophe. In the front region, where there are zero or two points of stationary phase, each dark Airy fringe surface associated with the fold surface condenses into a single dislocation in the plane of symmetry.

Keywords: diffraction; diffraction catastrophes; wave dislocations; singular optics; optical vortices

1. Introduction

Presently, there is much interest in the relatively new subject of singular optics (e.g. Soskin & Vasnetsov 2001; Special Issue 2004). This is concerned with phase singularities in optical fields, e.g. in laser beams, or in the fields made by superposing random monochromatic waves (e.g. Freund 1995, 1998), or phase singularities in more general monochromatic diffraction fields. In three-dimensional scalar fields, the phase singularities take the form of special lines on which the amplitude is zero and the phase is indeterminate; they are called optical vortices owing to the circulation of phase around them, or wave dislocations because topologically they resemble dislocations in crystals. The lines may extend infinitely or they may be closed loops; they may also be braided (Dennis 2003) or knotted (Leach et al. 2004). The topological nature of these lines is in fact their most important property, because it ensures that they can never come to an end and that they survive perturbation; that is to say, if the field itself is slightly changed, the dislocations will generally move, but they cannot disappear; they are said to be structurally stable.

In this paper, the general concept of wave dislocations (Nye & Berry 1974) is applied to the caustics of geometrical optics. The elementary catastrophes of Thom (1975) can be realized physically as the structurally stable caustics of

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geometrical optics (Berry & Upstill 1980; Berry 1981; Nye 1999). Scalar wave
theory decorates them with diffraction detail, and the diffraction patterns for
the first five catastrophes have been computed numerically (Pearcey 1946; Wright
1977; Berry et al. 1979; Connor et al. 1983, 1984; Connor 1990; Berry & Howls
2006). It is useful to regard each of these diffraction patterns as based not only
on the structurally stable caustic, where the amplitude is infinite, but also on the
array of dislocation lines which thread through it, on which the amplitude is zero.
It is hard to epitomize the whole three-dimensional distribution of both
amplitude and phase represented by an optical catastrophe; hence, it is
attractive to concentrate, as we do here, on the structurally stable framework
of dislocation lines that supports it. The dislocation frameworks have been
elucidated for the cusp (Pearcey 1946), the elliptic umbilic (Berry et al. 1979)
and the hyperbolic umbilic (Nye 2006a). From this point of view, it remains to
study the one further catastrophe of codimension 3, namely the swallowtail,
which is the subject of this paper.

2. The diffraction integral and the caustic

The swallowtail catastrophe (Poston & Stewart 1978) is represented by the function
\[
\phi(\xi; x, y, z) = \frac{1}{5} \xi^5 + \frac{1}{3} z\xi^3 + \frac{1}{2} y^2 + x\xi,
\] (2.1)
where \(x, y, z\) are dimensionless Cartesian control variables and \(\xi\) is a
dimensionless state variable, and the corresponding diffraction integral may be
written as
\[
S(x, y, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \{i\phi(\xi; x, y, z)\} d\xi.
\] (2.2)

Reflection across the plane \(y=0\) produces the complex conjugate, thus
\[
S(x, y, z) = S^*(x, -y, z).
\] (2.3)

The caustic (figure 1a) consists of a fold that crosses itself. Thus, a line of
self-intersection emerges from the origin in the plane of symmetry, \(y=0\),
accompanied by two riblines (cusplines). The fold surface that connects the two
riblines contains the negative \(z\)-axis. The number of stationary phase points in
the various regions is indicated in the three typical sections drawn in figure 1b.
Outside the body of the swallowtail, there are either two points of stationary
phase or none, while within the tail there are four.

It is consistent with the symmetry (2.3) that the plane \(y=0\), rather than \(z=0\),
is the ‘natural’ focal plane in optical realizations of the swallowtail caustic
(Nye & Hannay 1984), i.e. the sideways orientation, rather than, as one might
perhaps have expected, the head-on orientation with \(z=0\) as the focal plane.
Some further remarks are given in appendix C about the relation between a
diffraction pattern that might be produced experimentally and the canonical
pattern analysed here.

3. Dislocations in the four-ray region

The four-ray region in the tail will be discussed first because it is the heart of the matter. Figure 2 shows a small part of it in the plane of symmetry $y=0$ not near either of the bounding caustics. The dislocations form a repeating set of wavy nodal lines that define a series of bands. At the marked points and similar equivalent points other dislocations spring out of the plane upwards and downwards. These junctions are exactly the same as those that appear in the focal plane of the hyperbolic umbilic; they can only occur in a conjugating plane of symmetry, as described by equation (2.3). The sense of a dislocation, decided by the circulation of phase around it, is
represented by an axial vector; some of these are marked by arrows in the right-hand part of the figure. The senses of the four dislocations meeting at a junction are such that there is no net inflow or outflow of dislocation strength (two opposing vectors at a given junction point inwards and two outwards).

A unit cell is outlined in figure 2, but the spatial repetition is not exact. The lattice is curved and the dimensions of the unit cell change slowly with position. In three dimensions, the whole pattern of both amplitude and dislocations is likewise approximately repeating. If the arrows on the dislocations are ignored, the symmetry of the resulting pattern of lines is the same as that of the amplitude (for they are loci of zero amplitude). To a first approximation, the dislocations form a system of lines that intersect each other, but in the real pattern, except on the symmetry plane, each intersection disconnects into two hyperbolic branches to leave separate non-intersecting helical dislocations. Such behaviour, to be described in more detail in §5, would be expected because fourfold junctions are non-generic except on a plane of symmetry. In fact, the lattice becomes exact asymptotically as $\frac{z}{K}N$, except very near the caustic. This ideal lattice is analysed in §4.

4. The ideal lattice

The lattice may be found by the method of stationary phase as a result of the interference of four waves that are locally plane. The points of stationary phase are given as the roots of the quartic equation obtained by differentiating equation (2.1), thus

$$\xi^4 + z\xi^2 + y\xi + x = 0.$$  \hfill (4.1)

For $y=0$, this has the following four solutions:

$$\xi_0 = \left[\frac{1}{2}(-z + \sqrt{z^2 - 4x})\right]^{1/2}, \quad \xi_1 = \left[\frac{1}{2}(-z - \sqrt{z^2 - 4x})\right]^{1/2},$$

$$\xi_2 = -\xi_1, \quad \xi_3 = -\xi_0. \hfill (4.2)$$

In the plane $y=0$, the four-wave region is bounded by the caustic line $x=0$ and the fold crossing line $x=\frac{1}{4}z^2$; moreover, $z<0$. Thus, $z^2-4x$ in equation (4.2) is positive and all four roots are real. The stationary phase solution for $y=0$ is accordingly

$$S = \sum_{j=0}^{3} \sqrt{\frac{i}{\phi''(\xi_j)}} \exp[i\phi(\xi_j)], \hfill (4.3)$$

where it may be noted that $\phi''(\xi_j) = 4\xi_j^3 + 2z\xi_j + y$.

Incidentally, for small $y$, an approximation to the $j$th root may be written as

$$\xi_j^{\text{approx}} \approx \xi_j - \frac{y}{4(\xi_j)^2 + 2z}, \hfill (4.4)$$

$\xi_j$ being the $j$th root for $y=0$. Thus, for small $y$, an approximate solution would be equation (4.3), but with $\xi_j^{\text{approx}}$ given by equation (4.4) in place of $\xi_j$, but we shall not, in fact, use this.
Although the four contributing waves are not plane, because the roots \( \xi_j \) depend on position, they do locally have the plane-wave form \( A_j e^{i\mathbf{k}_j \cdot \mathbf{r}} \), where \( \mathbf{k}_j = \nabla \phi |_{\xi = \xi_j} \). Thus, directly from equation (2.1), \( \mathbf{k}_j = (\xi_j, \frac{1}{2} \xi_j^2, \frac{1}{2} \xi_j^3) \). The interference of four general exact plane waves produces a three-dimensional repeating pattern of amplitude characterized by a lattice (Nye 2006a), and following that paper we can find the unit cell. It is independent of the amplitudes of the waves and we shall examine it on the symmetry plane \( y = 0 \) (the relative amplitudes of the waves decide whether there are dislocations or not, as discussed by O'Holleran et al. (2006), but not the lattice of the amplitude pattern). First form the difference vectors

\[
\mathbf{K}_1 = \mathbf{k}_1 - \mathbf{k}_0, \quad \mathbf{K}_2 = \mathbf{k}_2 - \mathbf{k}_0, \quad \mathbf{K}_3 = \mathbf{k}_3 - \mathbf{k}_0, \quad (4.5)
\]

and find their reciprocals, in the crystallographic sense, given by

\[
\mathbf{r}_1 = 2\pi(\mathbf{K}_2 \wedge \mathbf{K}_3)/(\mathbf{K}_1 \cdot (\mathbf{K}_2 \wedge \mathbf{K}_3)),
\]

and similar expressions for \( \mathbf{r}_2 \) and \( \mathbf{r}_3 \). The choice of \( \mathbf{k}_0 \) for this purpose is arbitrary; a different choice would lead to different lattice vectors, but they would still describe the same lattice. \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) are repeat vectors for amplitude, as may be verified, but they do not necessarily define a primitive unit cell of convenient shape and related to the symmetry of the lattice. Combinations of integer multiples of \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) define other points of the lattice, and we note that \( \mathbf{r}_1 + \mathbf{r}_2 \) is a vector perpendicular to the symmetry plane \( y = 0 \). In fact, it is convenient to choose a unit cell defined by the vectors

\[
\mathbf{a} = -2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{b} = -\mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{c} = -3\mathbf{r}_3 - \mathbf{r}_1 - 2\mathbf{r}_2. \quad (4.7)
\]

This sets \( \mathbf{a} \) and \( \mathbf{c} \) in the symmetry plane \( y = 0 \) and \( \mathbf{b} \) normal to it (figure 2).

Figure 2 was computed by choosing a region on \( y = 0 \), with a fairly large negative value of \( z \) and not very near a caustic, so that the pattern is close to the limiting asymptotic form. The amplitude pattern has twofold axes of symmetry parallel to \( \mathbf{b} \) and planes of symmetry normal to it. The point group symmetry is monoclinic \( 2/m \). As well as the lattice points at the corners of the unit cell, there is also one per cell at height \( \frac{1}{2} \), indicating centring. By convention, the face containing the \( \mathbf{a} \) and \( \mathbf{b} \) axes is the one taken as centred, and the \( \mathbf{a} \) axis has been chosen accordingly. To confirm the face centring, note that the position vector of the centring point, \( \frac{1}{2} (\mathbf{a} + \mathbf{b}) \), is \( -\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3 \), which is a lattice point, because it is composed of integer multiples of \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \). The possible space groups are \( C2/m \) or \( C2/c \) (\( C \) indicates the face centring), but the latter has no \( m \) planes, only glide planes. Therefore, the space group is \( C2/m \), no. 12, second setting, in International Tables (1965). In figure 2, centres of symmetry may be seen halfway along the sides of the unit cell and also in the centre of it. There are interleaving \( m \) planes at height \( \frac{1}{2} \) and also glide planes, twofold axes and screw twofold axes. The origin of the cell is taken at a centre of symmetry. The dislocations parallel to \( y \) lie on twofold axes and are thus constrained to be straight in this limiting idealization. There are dislocation intersections not only at \( 0, 0, 0 \) and, by lattice repeat, at \( \frac{1}{2}, \frac{1}{2}, 0 \), but also ones with opposite sense at \( \frac{1}{2}, 0, 0 \) and \( 0, \frac{1}{2}, 0 \).
\(a, b, c\), thus defined, may be expressed in terms of the roots, \(\xi_0, \xi_1\), defined in equation (4.2) as

\[
a = \left[\frac{4\pi}{3}\xi_1^3/T, 0, -4\pi\xi_1/T\right],
\]

\[
b = \left[0, -\frac{8\pi}{3}\xi_1\xi_0/T, 0\right],
\]

\[
c = \left[2\pi\left(\xi_1^3 - \frac{1}{3}\xi_0^3\right)/T, 0, 2\pi(-3\xi_1 + \xi_0)/T\right],
\]

where \(T = \frac{2}{3}\xi_1\xi_0(\xi_1^2 - \xi_0^2)\).

In the plane \(y=0\), the four terms in equation (4.3) group together in the pairs \(j=0, 3\) and \(1, 2\) to give a simple expression for \(S\) that is (necessarily) pure real, namely

\[
S = 2P \cos\left(\pi/4 + \frac{1}{5}\xi_0^3 + k_0 \cdot r\right) + 2Q \cos\left(\pi/4 - \frac{1}{5}\xi_1^3 - k_1 \cdot r\right),
\]

where \(P = 1/\sqrt{|4\xi_0^3 + 2\xi_0|}\) and \(Q = 1/\sqrt{|4\xi_1^3 + 2\xi_1|}\).

Values of \(r\) that make both the cosines zero provide the centres of symmetry, seen in figure 2, that lie on the nodal lines at 0 and \(\frac{1}{2}a\). Values of \(r\) that make the cosines \(\pm 1\) give the other set of centres of symmetry that lie at maxima, \(\frac{1}{2}c\), and saddles, \(\frac{1}{2}(a + c)\), of the amplitude pattern.

This completes the description of the symmetry of the amplitude pattern, and, as previously noted, the symmetry of the pattern of dislocation lines without the arrows denoting their senses is just the same. When the arrows are included, the additional notion of antisymmetry is needed to allow for operations that reverse the gradient of phase. The black–white or magnetic space groups (Bradley & Cracknell 1972) are now relevant, and out of the 1191 possibilities the one here is \(C2/m\). The conjugating mirror planes \(m\) normal to the \(y\)-axis reverse the senses of dislocations that are parallel to this axis, and allow dislocations to lie in the \(m\) plane itself.

The symmetry of the phase pattern itself is another matter, as discussed by Nye (2006a), because generally in four-wave interference the phase pattern does not repeat. On the other hand, if the phase values (the labels) on the equiphase lines (seen in cross-sections) are disregarded, the pattern of lines repeats in the same way as the amplitude.

5. Disconnections at the junctions

It has been mentioned that in the real pattern each dislocation intersection (junction) present in the ideal lattice disconnects into two branches of a hyperbola. To illustrate this, we choose a moderate value of \(z\), say \(-20\), and look near the plane of symmetry \(y=0\) and far from either of the bounding caustics. In the three-dimensional views of figure 3, one can see that the zigzag dislocations in the \(y=\text{const}\.) planes are stacked one above the other to form bands in the \(ab\) plane, but staggered owing to the face centring of the lattice.
The junctions are perfect in the plane $y=0$, but disconnected elsewhere. In the ideal lattice, the junctions occur at centres of symmetry of the amplitude pattern where twofold axes intersect mirror planes. When a junction disconnects, the local centre of symmetry remains. A junction can disconnect

Figure 3. (a) Three-dimensional isometric view of a single band of dislocations in the $ab$ plane near the plane of symmetry in the four-wave region, with $x=50$ and $z=-20$. $b$ is parallel to $y$. Axes are shown with a local origin. The courses of two selected dislocations (one up and one down), starting at two different centres of symmetry, are thickened. (b) The same from a different angle.

Figure 4. A region of the symmetry plane $y=0$ near the origin. On the right are the caustics and the nodal lines, with their senses, and also the junctions where dislocations spring normal to the plane with senses directed out of the plane (filled circles) and into the plane (open circles). On the left, the same region is seen with shaded contours of amplitude. None of the nodal lines in the region with $x<0$ penetrates the fold line.

The junctions are perfect in the plane $y=0$, but disconnected elsewhere. In the ideal lattice, the junctions occur at centres of symmetry of the amplitude pattern where twofold axes intersect mirror planes. When a junction disconnects, the local centre of symmetry remains. A junction can disconnect
in two different ways. The rule is that it does so in a way that keeps the resulting continuous dislocations moving in the $-a$ and $+b$ directions or in the $+a$ and $-b$ directions; this keeps a dislocation always within its own band. In figure 3, two of the dislocations are shown thickened (one with sense upwards and one downwards), to emphasize this continuity. Their general forms are helices made up of fairly straight segments, with a cross-section that is far from circular.

As $z$ becomes more negative, the dislocations in the four-wave region on $y=0$ become more numerous, and it is natural to ask where the new ones come from. The answer is that they come in through the caustic. In figure 4, which shows the region on $y=0$ near the origin, one sees the first three nodal lines in the plane $y=0$ entering the four-wave region across the fold crossing line. They emerge from a region where the amplitude is vanishingly small. As $z$ decreases, they become the zigzags we have been discussing. This implies that a dislocation line, where the amplitude is zero, can pierce a caustic surface, where the amplitude given by geometrical optics is infinite, e.g. at point $R$. The section through $R$ (the dashed line in figure 4) is shown in figure 5, where $R$ appears as a phase singularity.

6. Relation to the Pearcey pattern and dislocations in other regions

The dislocations in the four-wave region springing from roots in $y=0$ eventually, as $z \to -\infty$, become the dislocations of the Pearcey pattern, by straightening from helices into smooth lines running parallel to the two riblines (figure 6a). We recall that the Pearcey pattern has a single row of dislocations just outside the cusp caustic, whereas the root points we are now considering in $y=0$ all lie inside the caustics bounding the four-wave region. The outer row of root points in $y=0$ in the four-wave region (A1, B1, etc. in figure 4), in fact, have a special role. As $z \to -\infty$, the helical dislocations that spring from them pierce the caustic, many times, before finally becoming straight as dislocations of the Pearcey pattern outside the cusp caustic, on the positive $x$ side. Figure 6a, for example, shows the
two dislocations springing out of the symmetry plane from A1. All the other root points in the four-wave region, such as A2, B2, etc. produce out-of-plane dislocations that remain within the caustics bounding the tail.

On the other side of the four-wave region, the negative $x$ side, matters are very different. In the plane $y=0$, there is a series of nodal lines (figure 4) that loop back so that they never cross the fold (quite unlike the nodal lines on the positive $x$ side of the four-wave region). In fact, these lines are very nearly nodal surfaces, extending perpendicular to $y=0$, but that would be non-generic. For large positive $z$, they are the dark fringes of the Airy diffraction pattern associated with the bright side of the fold. These dark fringes contain no other dislocations. Thus, each nodal surface of the pure Airy pattern on the bright side of the fold has condensed into a single nodal line. (On the dark side of the fold at positive $z$, the amplitude is very small and any dislocations are lost.)

Alternate nodal lines for negative $x$ contain root points like P1 and P2 (figure 4) from which dislocations spring out normally in the $y$-direction. Successive $xy$ sections with decreasing $z$ direction encounter root points like P1 and P2, where they show a single dislocation splitting into a triplet, with the central member changing its sense. In this way, the number of dislocations in $xy$ sections for negative $z$ increases with distance from the origin. Figure 6b is an isometric, upside-down view that shows the nodal line through P1 coming from $z>0$ looping back on itself in the plane $y=0$. It also shows the two dislocations that spring from P1 out of the symmetry plane. They are helical and run generally parallel to the riblines. Eventually, they straighten to become outer dislocations of the Pearcey pattern on the negative $x$ side. Other out-of-plane dislocations, such as the ones starting at P2, behave in a similar way. The dislocations in figure 6a,b are very much like the ‘joined curly antelope’s horns’ in

Figure 6. Isometric projections of selected dislocations. (a) The two helical dislocations that spring from the point A1, in the plane of symmetry $y=0$, cross the fold caustic to finish on the outside, running roughly parallel to the riblines. Segments that lie inside the caustic are shown thickened and the tracks extend to $z=−8$. The senses, which reverse at A1, are indicated. (b) The caustic is shown upside down. A nodal line in the plane $y=0$ at the top right folds back on itself and passes through the point P1 seen in figure 4; two helical dislocations spring symmetrically from P1 out of the plane $y=0$ and remain outside the caustic, running roughly parallel to the riblines. Their tracks extend to $z=−6$. 

the elliptic umbilic pattern, seen in figure 11 of Berry et al. (1979); the difference is that those did not spring, as these do, from a fourfold junction at their base. The description has now covered the whole pattern of dislocations.

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**Appendix A. Computational methods**

For larger values of $x$, $y$, $z$, it is not satisfactory to evaluate the infinite integral in equation (2.2) by numerical integration along the real axis because the integrand oscillates very rapidly. Instead, we used a method of contour integration in the complex $\xi$ plane that is similar to that of Connor et al. (1984), and used their tabulated numerical results as a check on accuracy. Except very near the caustic, the stationary phase method becomes more accurate as the arguments become large. It also has the advantage of being much faster. Sufficient accuracy for the present purpose was obtained by combining these methods. Figure 7 is a comparison between the results from stationary phase and contour integration, demonstrating agreement except in narrow regions near the caustics. (In §4, describing the stationary phase method, all four roots $\xi_j$ were real. For regions where two or four of the roots are complex, a study of the contour in the complex $\xi$ plane shows that one should drop contributions where $\text{Im} \phi(\xi_j) < 0$, leading to a diverging exponential.)

The algorithm (Nye 2003b) used to track individual dislocations necessarily fails at the fourfold junctions on the plane $y=0$. Figure 6 superimposes plots of tracks from different starting points. The nodal lines in the plane $y=0$ in figures 2 and 4 are readily plotted as the loci where $\text{Re}S=0$. It is harder to locate the root points on them; they were found by plotting the equiphase lines of $S$ at height $y=1$ and lower.

Figure 3, plotting all the dislocations in a given volume, used the tetrahedron method, suggested by J. H. Hannay and described by Nye (2006b).

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Appendix B. Evolution from the pattern for a small aperture

A previous paper (Nye 2003a) described how the single slit diffraction pattern produced by a cylindrical lens in its focal plane develops into the cusp diffraction pattern as the aperture is increased. In the same way, the pattern of Airy rings produced by a circular lens of small aperture can be made to evolve into the diffraction pattern of either the elliptic (Nye 2003b) or the hyperbolic (Nye 2006b) umbilic catastrophe by enlarging the aperture. By suitable scaling of the variables, each of these cases can be regarded as depending on a single variable of evolution (rather than considering all combinations of focal length, aperture, aberration and wavelength), and this makes them readily treatable; but the same is not true for the swallowtail, and the reason is as follows.

Forming the diffraction integral over a limited, finite aperture necessarily involves integration over two state variables, say $x$, $h$. For the umbilics, there was a natural choice for the shape of the aperture, namely a circle, but for the swallowtail this is not so. One might, for example, choose a rectangle for the shape of the aperture with appropriate limits of integration for $x$, $h$. Then, to examine the evolution of the diffraction pattern from small to infinite aperture, one would have to specify not only the size of the aperture, but also its aspect ratio. It is inevitably a two-parameter sequence, and so the problem is much less tractable.

Appendix C. Relation to physical diffraction patterns

As discussed by Nye & Hannay (1984), the swallowtail caustic seen in an experiment will generally be a rotated, sheared and anisotropically stretched version of the canonical caustic. Moreover, the canonical diffraction integral (2.2) has only one state variable, whereas two are required to specify the height of an initial wavefront, so it is not immediately obvious how to construct an initial wavefront that will propagate to give this catastrophe. As an example, Nye (1999, pp. 64–66) gives an explicit expression for a wavefront that will give a sideways swallowtail and shows photographs of sections $z=\text{const.}$, made by reflecting a wave in a curved metallized plastic film on to a sloping screen. To relate observed patterns like this to the canonical pattern computed in this paper, one might argue as follows.

The canonical pattern is in terms of dimensionless control variables $x$, $y$, $z$, and the wavenumber $k$ has been removed by scaling the variables, as specified by Arnold (1973). If, instead of this canonical integral, one starts from a diffraction integral with a fixed $k$ corresponding to propagation of a (two-dimensional) wavefront of specified height, e.g. the one giving a sideways swallowtail, it is no longer possible to remove $k$ by scaling; the amplitude diffraction pattern is inherently $k$-dependent and thus not universal. However, as $k$ approaches infinity, all the diffraction detail streams anisotropically towards the focus and, with appropriate magnification, the physical amplitude pattern, with its wave dislocations, will approach the canonical pattern discussed here. If $k$ is large enough, it is the canonical pattern that will be seen; just how large $k$ has to be for this depends on the details of the experiment.
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