Blow up and instability of solitary wave solutions to a generalized Kadomtsev–Petviashvili equation and two-dimensional Benjamin–Ono equations

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Let \( p \geq 2 \) with \( p \) being the ratio of an even to an odd integer. For the generalized Kadomtsev–Petviashvili equation, coupled with Benjamin–Ono equations, in the form

\[
( u_t + u_{xxx} + \beta \mathcal{H} u_{xx} + w^p u_x )_x = u_{yy}, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0,
\]

it is proved that the solutions blow up in finite time even for those initial data with positive energy. As a by-product, it is proved that for all \( c > (\beta/2) \), the solitary waves \( \varphi(x - ct, y) \) are strongly unstable if \( 2 \leq p < 4 \). This result, even in a special case \( \beta = 0 \), improves a previous work by Liu (Liu 2001 Trans. AMS 353, 191–208) where the instability of solitary waves was proved only in the case of \( 2 < p < 4 \).

Keywords: Kadomtsev–Petviashvili equation; Benjamin–Ono equations; blow-up solutions; strong instability of solitary waves

1. Introduction

This paper is concerned with the following generalized Kadomtsev–Petviashvili (KP) equation and two-dimensional Benjamin–Ono (BO) equations

\[
\begin{align*}
( u_t &+ u_{xxx} + \beta \mathcal{H} u_{xx} - v_y + w^p u_x ) = 0, \\
u_y &= v_x, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}_+
\end{align*}
\]

(1.1)

\[
u(x, y, t)\bigg|_{t=0} = u_0,
\]

(1.2)

where \( \beta \geq 0 \), \( p > 0 \) and \( \mathcal{H} \) is the Hilbert operator defined by

\[
\mathcal{H} u(x, y, t) = p.v. \int_{\mathbb{R}} \frac{u(z, y, t)}{x-z} \, dz.
\]

Here, p.v. denotes the Cauchy principal value.

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System (1.1) has been raised in many branches of physics. For example, when \( \beta = 0 \) and \( p=1 \), (1.1) is the well known KP-I equation (Kadomtsev & Petviashvili 1970; Ablowitz & Clarkson 1991). When \( \beta = 0 \) and \( p>0 \), (1.1) is called the generalized KP (GKP) equation. Mathematical interests are concentrated on the existence of solitary waves (i.e. solutions of (1.1) with the form \( u(x, y, t) = \phi(x - ct, y) \)), existence of solutions of Cauchy problem (1.1) and (1.2) as well as the possible blow-up solutions of (1.1) and (1.2). But one of the most important and challenging issues is to investigate the existence of solutions that blow up in a finite time.

Recently, Guo & Han (1996) proved the existence of blow-up solutions of (1.1) and (1.2) if \( p \geq 4 \) and the initial energy

\[
E_\beta(u_0) = \int \left( \frac{u_{0x}^2 + v_0^2 + \beta u_{0x} H u_0}{2} - \frac{u_0^{p+2}}{(p+2)(p+1)} \right) < 0, \quad \text{where } u_{0x} = v_{0x}.
\]

This is one motivation of the present paper. Another motivation is the recent results for (1.1) and (1.2) in the case of \( \beta = 0 \). When \( \beta = 0 \), Liu & Wang (1997) and Liu (2001) proved the existence of solitary waves of (1.1) for \( 0 < p < 4 \) and the existence of blow-up solutions of (1.1) and (1.2) for \( 4/3 \leq p < 4 \), as well as the stability of solitary waves of (1.1) for \( 0 < p \leq 4/3 \) and the instability of solitary waves of (1.1) for \( 2 < p < 4 \). The main ingredients used in Liu & Wang (1997) and Liu (2001) are based on solving several complicated minimization problems and constructing several invariant sets (e.g. Liu & Wang 1997, theorem 1 and 3; Liu 2001, lemma 3.1). In particular, they use essentially the property of invariance of the problem under scaling \( c^{1/p} u(\sqrt{c} \cdot, c \cdot) \) and \( u(\cdot / \mu) \). But the case of \( \beta \neq 0 \) is quite different from those of \( \beta = 0 \) since the scaling invariance does not hold any more due to the term \( \int u_x H u \ dx dy \). The method used in Liu (2001) seems not to be applied in the case of \( \beta \neq 0 \).

There are two main purposes of the present paper. One is to prove the existence of solitary waves \( u(x, y, t) = \phi(x - ct, y) \) of (1.1) in the case of \( \beta \geq 0 \) and \( 0 < p \leq 4 \), which covers and generalizes the previous works in de Bouard & Saut (1997) and Liu & Wang (1997). In particular, we do not require the wave speed \( c \) satisfying \( c > (\beta/2) \), which is usually assumed in finding solitary waves of the BO equations. The other is to prove that (1.1) and (1.2) has a solution that has positive initial energy and blows up in a finite time. As a by-product, it is proved that the solitary wave is strongly unstable for \( 2 \leq p < 4 \), \( \beta \geq 0 \) and \( c > \beta/2 \). This result improves a previous work by Liu (2001) where the strong instability of a solitary wave is proved only in the case of \( 2 < p < 4 \) and \( \beta = 0 \).

This paper is organized as follows. In §2, we use the Nehari method to prove that for any \( c > 0 \), (1.1) has a solitary wave solution \( \phi(x - ct, y) \). In §3, we use several minimization problems with multiple constraints to construct sets \( R_\pm \) and \( Q_\pm \) that are invariant under the flow generated by the Cauchy problem (1.1) and (1.2). In §4, we prove that the solutions of (1.1) and (1.2) with initial data in \( R_+ \cap Q_- \) must blow up in a finite time. In §5, we prove that for any \( c > (\beta/2) \), the solitary wave \( \phi(x - ct, y) \) is strongly unstable in the sense of definition 5.1.

We end this introduction by notations where \( || \cdot ||_q \) and \( || \cdot ||_s \) will denote the norm in \( L^q(\mathbb{R}^2) \) and Sobolev space \( H^s(\mathbb{R}^2) \). Throughout this paper, \( f \) denotes the Fourier transform of \( f \), defined as \( \hat{f}(\xi) = \int \exp(-i(\xi_1 x + \xi_2 y)) f(x, y) dx \ dy \). All integrals are taken over \( \mathbb{R}^2 \) unless stated otherwise and \( dx \ dy \) will be omitted if no
confusion occurs. Let $Y$ be the closure of $\partial_x( C_0^\infty (\mathbb{R}^2) )$ under the norm
\[ ||u||_Y = ||\partial_x u||_Y = (||\nabla\psi||_{L^2}^2 + ||\partial_x^2 \psi||_{L^2}^2)^{1/2}, \]
for $u \in Y$ and $u = \partial_x \psi$, where $\psi \in L^q_{\text{loc}}(\mathbb{R}^2), \forall 2 \leq q < \infty$. $Y'$ is the dual space of $Y$. We also have $v = \partial_y \psi$ by a choice of $\psi \in L^q_{\text{loc}}$. Let
\[ X_s = \{ f \in H^s(\mathbb{R}^2); (\xi_1^{-1} \hat{f}(\xi))' \in H^s(\mathbb{R}^2) \}, \]
with the norm $||u||_{X_s} = ||u||_s + ||(\xi_1^{-1} \hat{u})'||_s$, where '$\cdot'$ is the Fourier inverse transform. We define the operator $D_x^{-k}$ by
\[ D_x^{-k} \hat{f}(\xi) = (i\xi_1)^{-k} \hat{f}(\xi), \quad k > 0. \]
$D^s = (-\partial_x^2)^{s/2}$ is the Riesz potential of order $s$ defined by $\widehat{D^s f}(\xi) = |\xi_1|^s \hat{f}(\xi)$. $S(\mathbb{R}^2)$ is the Schwartz class in $\mathbb{R}^2$.

Throughout this paper, we only consider the case of $\beta \geq 0$ and $p = n_1/n_2$, where $n_1$ is any even integer and $n_2$ any odd integer so that $\int u^{p+2} = \int |u|^{p+2}$. Different positive constants might be denoted by the same letter $C$ or $C_j$. If necessary, by $C(\cdot, \cdot)$ we denote the constant depending only on the quantities appearing in parentheses.

2. Solitary wave solutions

In this section, we will use the Nehari method to prove that (1.1) possesses a solitary wave solution. Before doing this, we recall the following well-known properties of the Hilbert operator $\mathcal{H}$ (Guo & Tan 1992):
\[ \mathcal{H} f(\xi) = \text{isgn} \xi_1 \hat{f}(\xi); \quad (I) \]
\[ |\mathcal{H} f|_2 = |f|_2; \quad (II) \]
\[ \mathcal{H} f_x = \frac{\partial}{\partial x} (\mathcal{H} f); \quad (III) \]
\[ f \mathcal{H} g = - \int g \mathcal{H} f; \quad (IV) \]
\[ \int f_x \mathcal{H} f \, dx \geq 0; \quad \text{and} \]
\[ |D^{1/2} f|_2 \leq C|f|_2^{1/2} |f_x|_2^{1/2}, \quad \text{for any } f \in S(\mathbb{R}^2). \quad (VI) \]

For any $c > 0$, a solitary wave of (1.1) with wave speed $c$ is a solution of (1.1) of the form $u(x, y, t) = \varphi(x - ct, y)$ and decaying to zero at infinity. More precisely, $\varphi$ is a solution of
\[ -c \varphi + \varphi_{xx} + \beta \mathcal{H} \varphi_x - D_x^{-2} \varphi_{yy} - \frac{u^{p+1}}{p+1} = 0, \quad \text{in } Y'. \quad (2.1) \]
We say that $\varphi \in Y$ is a solution of (2.1) if, and only if, $L_c'(\varphi) = 0$, where $L_c(u) = E_\beta(u) + c V(u)$ with $V(u) = 1/2 \int u^2$ and
\[ E_\beta(u) = \int \left( \frac{u_x^2 + u_y^2 + \beta u_x \mathcal{H} u}{2} - \frac{u^{p+2}}{(p+2)(p+1)} \right). \]
Let

\[ N(u) = \int \left( \frac{u_x^2 + v^2 + \beta u_x u + cu^2}{p+1} - \frac{w^{p+2}}{p+1} \right). \]

Then \( N(\phi) = \langle L'_c(\phi), \phi \rangle = 0 \) whenever \( \phi \) is a solution of (2.1). Here, and after, \( \langle \cdot, \cdot \rangle \) denotes the dual product between \( Y' \) and \( Y \).

**Definition 2.1.** We say that \( \phi \) is a groundstate solution of (2.1) if \( L_0 c(\phi) \leq 0 \) and for any \( \psi \) satisfying \( L_c(\psi) = 0 \) there holds \( L_c(\phi) \leq L_c(\psi) \).

Next we will prove that the following minimum

\[ d = \inf_{u \in \mathcal{N}} L_c(u), \quad \mathcal{N} = \{ u \in Y; u \neq 0, N(u) = 0 \}, \tag{2.2} \]

is achieved by some \( \phi \), which is a groundstate solution of (2.1).

**Lemma 2.2.** Suppose that both \((u_n)\) and \((u_{nx})\) are bounded in \( L^2(\mathbb{R}^2) \). Assuming \( u_n \rightharpoonup u \) a.e. in \( \mathbb{R}^2 \), then

\[ \int u_{nx} \mathcal{H} u_n = \int (u_n - u)_x \mathcal{H}(u_n - u) + \int u_x \mathcal{H} u + o(1). \tag{2.3} \]

Here, and after, \( o(1) \) goes to zero as \( n \) goes to infinity.

**Proof.** We firstly claim that if \( u_n \rightharpoonup u \) in \( L^2(\mathbb{R}^2) \), then \( \mathcal{H} u_n \rightharpoonup \mathcal{H} u \) in \( L^2(\mathbb{R}^2) \). Indeed, for any \( w \in L^2(\mathbb{R}^2) \),

\[ \int w \mathcal{H} u_n = - \int u_n \mathcal{H} w \rightarrow - \int w \mathcal{H} w = \int w \mathcal{H} u, \]

proving the claim.

Since \( u_n \rightharpoonup u \) in \( L^2(\mathbb{R}^2) \) and \( u_{nx} \rightharpoonup u_x \) in \( L^2(\mathbb{R}^2) \), we have, from the properties of \( \mathcal{H} \), that

\[ \int (u_n - u)_x \mathcal{H}(u_n - u) = \int u_{nx} \mathcal{H} u_n - \int u_x \mathcal{H} u_n - \int u_{nx} \mathcal{H} u + \int u_x \mathcal{H} u \]

\[ = \int u_{nx} \mathcal{H} u_n - \int u_x \mathcal{H} u + o(1) \quad (n \rightarrow \infty). \]

The proof is complete. \( \blacksquare \)

**Lemma 2.3.** If \( c > 0 \) and \( 0 < p < 4 \), then for any \( 0 \neq u \in Y \), there exists a unique \( \theta_u > 0 \) such that \( \theta_u u \in \mathcal{N} \). Moreover if \( N(u) < 0 \), then \( 0 < \theta_u < 1 \).

**Proof.** For any \( 0 \neq u \in Y \), define

\[ \gamma(\theta) = L_c(\theta u) = \int \left( \frac{1}{2} \theta^2 (u_x^2 + v^2 + \beta u_x u + cu^2) - \frac{\theta^{p+2} u^{p+2}}{(p+2)(p+1)} \right). \]

Direct computations show that
\[ \theta_u = \left( \frac{\int (u_x^2 + v^2 + \beta u_x H u + cu^2)}{\int u^{p+2}} \right)^{1/p}. \]

Clearly, from the expression of \( \theta_u \) we know that if \( N(u) < 0 \), then \( 0 < \theta_u < 1 \).

**Lemma 2.4.** Suppose that \( c > 0 \) and \( 0 < p < 4 \). Then

(i) \( \mathcal{N} \neq \emptyset \) and \( \mathcal{N} \) is manifold. Moreover, there exists \( \alpha > 0 \) such that for any \( u \in \mathcal{N}, \|u\|_Y \geq \alpha > 0 \) and

(ii) if \( u \) achieves the minimum \( d \), then \( u \) is a critical point of \( L_c \) in \( Y \), i.e. \( L'_c(u) = 0 \).

**Proof.** \( \mathcal{N} \neq \emptyset \) follows from lemma 2.3. For any \( u \in \mathcal{N} \),

\[ \langle N'(u), u \rangle = \int \left( 2(u_x^2 + v^2 + \beta u_x H u + cu^2) - \frac{p+2}{p+1} u^{p+2} \right) = -\frac{p}{p+1} \int u^{p+2} < 0, \]

this implies that \( \mathcal{N} \) is manifold. For any \( u \in \mathcal{N} \), denote \( \rho = \int (u_x^2 + v^2 + cu^2) \), then using the anisotropic Sobolev embedding theorem (Besov et al. 1978, p. 323) we have

\[ \rho + \int \beta u_x H u = \int \frac{u^{p+2}}{p+1} \leq C_2 |u|_2^{(4-p)/2} |v|_2^{p/2} |u_x|_2 \leq C_3 \rho^{(2+p)/2}, \]

which implies that \( \rho \geq C_3^{-2/p} : = \alpha > 0 \). Thus (i) is proved.

Since \( u \) achieves the minimum \( d \), we know that there exists \( \theta \in \mathbb{R} \) such that \( L'_c(u) = \theta N'(u) \). Noticing that \( \langle L'_c(u), u \rangle = N(u) = 0 \) and

\[ \langle N'(u), u \rangle = -\frac{p}{p+1} \int u^{p+2} < 0, \]

we have that \( \theta = 0 \). It follows that \( L'_c(u) = 0 \). Thus (ii) is proved.

Next we prove that \( d \) is achieved.

**Theorem 2.5.** Suppose that \( c > 0 \) and \( 0 < p < 4 \). Then the minimum \( d \) is achieved by some \( \varphi \in Y \), which is a groundstate solution of (2.1).

**Remark 2.6.** If \( u \neq 0 \) is a solution of (2.1), then \( L'_c(u) = 0 \) and hence \( N(u) = 0 \). The definition of \( d \) implies that \( L_c(u) \geq L_c(\varphi) \) for any \( \varphi \in \mathcal{N} \) and \( L_c(\varphi) = d \).

**Proof of theorem 2.5.** From remark 2.6 and (ii) of lemma 2.4, it suffices to prove that \( d \) is achieved by some \( \varphi \in \mathcal{N} \). Let \( (u_n) \subset Y \) be such that \( N(u_n) = 0 \) and \( d + o(1) = L_c(u_n) \). We know from (i) of lemma 2.4 that

\[ \liminf_{n \to \infty} \int u_n^{p+2} > 0. \]

Since the functionals \( L_c \) and \( N \) are invariant under translation, i.e. for any \( (x_1, y_1) \in \mathbb{R}^2 \), \( L_c(u(\cdot + x_1, \cdot + y_1)) = L_c(u) \) and \( N(u(\cdot + x_1, \cdot + y_1)) = N(u) \), we obtain from concentration compactness lemma of Lions (1984a,b) (see also
Ambrosetti & Wang (2003) and Willem (1996)) that there exist \((x_n, y_n) \in \mathbb{R}^2\) such that \(\varphi_n = u_n(\cdot + x_n, \cdot + y_n)\) satisfies \(N(\varphi_n) = 0\) and \(d + o(1) = L_c(\varphi_n)\).

Moreover, \(\varphi_n \to \varphi\) a.e. in \(\mathbb{R}^2\) and \(\varphi \neq 0\).

If \(N(\varphi) < 0\), then by lemma 2.3 there exists \(0 < \theta_\varphi < 1\) such that \(\vartheta_\varphi \varphi \in \mathcal{N}\). Using \(\varphi_n \in \mathcal{N}\) and the Fatou lemma, we have that

\[
d + o(1) = L_c(\varphi_n) = \left(1 - \frac{1}{p+2}\right) \int \frac{\varphi_n^{p+2}}{p+1} + o(1)
\geq \frac{p}{2(p+2)} \int \frac{\varphi_n^{p+2}}{p+1} + o(1)
= \frac{p\vartheta_\varphi\varphi^{p+2}}{2(p+2)} \int \frac{(\vartheta_\varphi \varphi)^{p+2}}{p+1} + o(1).
\]

Since \(0 < \theta_\varphi < 1\), (2.4) implies that \(d \geq L_c(\vartheta_\varphi \varphi)\), which is a contradiction because \(\vartheta_\varphi \varphi \in \mathcal{N}\).

If \(N(\varphi) > 0\), then using lemma 2.2 and the Brezis–Lieb lemma (Brezis & Lieb 1983) we have that

\[
0 = N(\varphi_n) = N(\varphi) + N(w_n) + o(1),
\]

with \(w_n = \varphi_n - \varphi\) and hence \(\limsup_{n \to \infty} N(w_n) < 0\). From lemma 2.3 we know that there are \(\vartheta_n := \vartheta_{w_n}\) such that \(\vartheta_n w_n \in \mathcal{N}\). Moreover, we claim that \(\limsup_{n \to \infty} \vartheta_n (0, 1)\). Indeed, if otherwise there exists a subsequence still denoted by \(\vartheta_n\) such that \(\vartheta_n \to 1 (n \to \infty)\), then \(N(w_n) = N(\vartheta_n w_n) + o(1) = o(1)\). That is a contradiction, since

\[
d + o(1) = L_c(\varphi_n) = \frac{p}{2(p+2)(p+1)} \int \varphi_n^{p+2} + o(1)
\geq \frac{p}{2(p+2)} \int w_n^{p+2} + o(1)
\geq \frac{p\vartheta_n^{p+2}}{2(p+2)} \int \frac{(\vartheta_n w_n)^{p+2}}{p+1} + o(1),
\]

one has again \(d \geq L_c(\vartheta_n w_n)\) for large \(n\).

Thus \(N(\varphi) = 0\). Using lemma 2.2 and the Brezis–Lieb lemma again, we easily get that \(\|w_n\|_Y \to 0\), i.e. \(\varphi_n \to \varphi \neq 0\) in \(Y\). The proof is complete.

**Remark 2.7.**

(i) When \(\beta = 0\), de Bouard & Saut (1997) proved that any solitary wave solutions of (1.1) belong to \(C^\infty(\mathbb{R}^2)\) for \(p = 1, 2, 3\) (see de Bouard & Saut 1997, p. 227, theorem 4.1.).

(ii) Observing the proofs of de Bouard & Saut (1997, p. 227, theorem 4.1), we know that if we replace the \(|\xi|^2 + \xi_1^4\) by \(c \xi_1^2 + \xi_2^2 + \xi_1^4 - \beta |\xi_1|^3\) in the multipliers \(\Phi_1, \Phi_2\) and \(\Phi_3\), then all the proofs contained in de Bouard & Saut (1997, p. 227, theorem 4.1) hold provided \(c > (\beta/2)\). Therefore if \(c > (\beta/2)\), then the solitary wave \(\varphi(x - ct, y)\) of (1.1) belongs to \(C^\infty(\mathbb{R}^2)\) for \(p = 1, 2, 3\).
We end this section by another characterization of the minimum $d$.

**Proposition 2.8.** Let $s \geq 3$ and

$$
\tilde{d} = \inf_{u \in \tilde{N}} L_c(u), \quad \tilde{N} = \{ u \in X_s; \ u \neq 0, \ N(u) = 0 \}.
$$

Then $\tilde{d} = d$.

**Proof.** Clearly $\tilde{d} \geq d$. To see $d \geq \tilde{d}$, it suffices to prove that for any $\epsilon > 0$ and any $u^* \in \tilde{N}$ there holds

$$
L_c(u^*) \geq \inf_{u \in \tilde{N}} L_c(u) - \epsilon.
$$

Since $X_s$ is dense in $Y$, we find a sequence $(u_n) \subset X_s$ such that

$$
u = u_n + w_n \ \text{with} \ \ w_n \to 0 \ \text{in} \ Y \ \text{as} \ n \to \infty.
$$

From $u^* \neq 0$, $N(u^*) = 0$ and $w_n \to 0$ in $Y$, we know that $\lim \inf_{n \to \infty} \int |u_n|^{p+2} \neq 0$ and $\lim_{n \to \infty} N(u_n) = 0$. The proof of lemma 2.3 implies that there exists a sequence $(\rho_n)$ such that $\rho_n u_n \in \tilde{N}$ and $\lim_{n \to \infty} \rho_n = 1$.

Next, denoting $u^* = \rho_n u_n + (1 - \rho_n) u_n + w_n$, we compare $L_c(u^*)$ with $L_c(\rho_n u_n)$. Since $1 - \rho_n \to 0$ and $w_n \to 0$ strongly in $Y$, we have immediately that

$$
\frac{1}{2} ||u^*||_Y^2 \geq \frac{1}{2} ||\rho_n u_n||_Y^2 - \frac{\epsilon}{3}
$$

and

$$
- \frac{1}{(p + 2)(p + 1)} \int (u^*)^{p+2} \geq - \frac{1}{(p + 2)(p + 1)} \int (\rho_n u_n)^{p+2} - \frac{\epsilon}{3}.
$$

We now calculate the term

$$
\frac{\beta}{2} \int (\rho_n u_n + (1 - \rho_n) u_n + w_n) \mathcal{H}(\rho_n u_n + (1 - \rho_n) u_n + w_n).
$$

Indeed, using the fact that $(w_n)_x \to 0$ in $L^2(\mathbb{R}^2)$, $w_n \to 0$ in $L^2(\mathbb{R}^2)$, $1 - \rho_n \to 0$ and interpolation inequality (VI), we know that all other terms go to zero as $n \to \infty$ but $\frac{\beta}{2} \int (\rho_n v_n)_x \mathcal{H}(\rho_n v_n)$. Hence we get that for large $n$

$$
\frac{\beta}{2} \int (u^*)_x \mathcal{H}u^* \geq \frac{\beta}{2} \int (\rho_n u_n)_x \mathcal{H}(\rho_n u_n) - \frac{\epsilon}{3}.
$$

Therefore

$$
L_c(u^*) \geq L_c(\rho_n u_n) - \epsilon \geq \inf_{u \in \tilde{N}} L_c(u) - \epsilon,
$$

and equation (2.7) holds. The proof is complete. \hfill \blacksquare

### 3. Invariant sets

In this section, we will use the minimization problem defined in (2.1) and the property of the ground-state solution $\phi$ to construct several sets that are invariant under the flow generated by the Cauchy problem (1.1) and (1.2). We emphasize that the method used in Liu (2001) seems not to be applied here due to the term $\int u_x \mathcal{H}u \, dx \, dy$. Hence we need to develop further the techniques in Chen & Guo (submitted) to get the invariant sets. First, we have the following local well-posed result.

Lemma 3.1 Guo & Han (1996), theorem 2.1. Suppose \( u_0 \in X_s(\mathbb{R}^2) \) and \( s \geq 3 \). Then \( T > 0 \) such that (1.1) and (1.2) have a unique solution

\[
u \in C(0, T; X_s) \cap C^1(0, T; H^{s-3}(\mathbb{R}^2)), \quad v \in C(0, T; H^{s-1}(\mathbb{R}^2)),
\]

and if \( \partial_x^{-2}u_{0yy} \in L^2(\mathbb{R}^2) \), one has

\[
u_t \in L^\infty(0, T; X_0), \quad v_t \in L^\infty(0, T; H^{-1}(\mathbb{R}^2)).
\]

Moreover there holds

\[
\frac{d}{dt} V(u) = \frac{d}{dt} \left[ \frac{1}{2} \left| u(x, y, t) \right|^2 \right] dx dy = 0,
\]

\[
E_\beta(u) = \int \left( \frac{u_x^2 + v^2 + \beta u_x \mathcal{H}u}{2} - \frac{u^{p+2}}{(p+2)(p+1)} \right) dx dy \equiv E_\beta(u_0).
\]

Next, we define another two functionals

\[
Q(u) = \int \left( u_x^2 + v^2 + \frac{\beta}{2} u_x \mathcal{H}u - \frac{3p}{2(p+2)(p+1)} u^{p+2} \right),
\]

\[
R(u) = \int \left( u_x^2 + \frac{\beta}{2} u_x \mathcal{H}u - \frac{p}{(p+2)(p+1)} u^{p+2} \right).
\]

We denote that

\[
\mathcal{R} = \{ u \in X_s; \ u \neq 0, \ N(u) < 0, \ R(u) = 0 \},
\]

\[
Q = \{ u \in X_s; \ u \neq 0, \ N(u) < 0, \ Q(u) = 0 \}.
\]

We set that

\[
d_{\mathcal{R}} = \inf \{ L_c(u); u \in \mathcal{R} \}, \quad d_Q = \inf \{ L_c(u); u \in Q \}.
\]

Lemma 3.2. If \( c > 0 \) and \( 2 \leq p < 4 \), then \( d_{\mathcal{R}} \geq d \).

Proof. Keep the definition of \( L_c(u), N(u) \) and \( R(u) \) in mind. For any \( u \in \mathcal{R} \), we want to find \( \tilde{u} \in \hat{\mathcal{N}} \) such that \( L_c(u) \geq L_c(\tilde{u}) \). Firstly, for any \( u \in \mathcal{R}, N(u) < 0 \) and \( R(u) = 0 \). We denote that \( u_\lambda(x) = \lambda u(\lambda x, \lambda y) \), then

\[
N(u_\lambda) = \int \left( \lambda^2 u_x^2 + v^2 + \lambda \beta u_x \mathcal{H}u + cu^2 - \frac{\lambda^p u^{p+2}}{p+1} \right).
\]

Noticing that \( N(u_\lambda) \to \int (v^2 + cu^2) > 0 \) as \( \lambda \to 0 \) and \( N(u_\lambda) \to N(u) < 0 \) as \( \lambda \to 1 \), we find a \( \lambda_1 \in (0, 1) \) such that \( N(u_{\lambda_1}) = 0 \), i.e. \( u_{\lambda_1} \in \hat{\mathcal{N}} \). Therefore \( L_c(u_{\lambda_1}) \geq d \).

Secondly, we want to prove that \( L_c(u) \geq L_c(u_\lambda) \) for any \( \lambda \in (0, 1) \). In fact, from

\[
L_c(u_\lambda) = \int \left( \frac{\lambda^2 u_x^2 + v^2 + \lambda \beta u_x \mathcal{H}u + cu^2}{2} - \frac{\lambda^p u^{p+2}}{(p+2)(p+1)} \right),
\]

we get that

\[
L_c(u) - L_c(u_\lambda) = \int \left( \frac{(1-\lambda^2) u_x^2 + (1-\lambda) \beta u_x \mathcal{H}u}{2} - \frac{(1-\lambda^p) u^{p+2}}{(p+2)(p+1)} \right).
\]

\( R(u) = 0 \) implies that
\[
L_c(u) - L_c(u_\lambda) = \int \left( g_1(\lambda)u_x^2 + g_2(\lambda)\beta u_x u_t \right),
\]
where \( g_1(\lambda) = \frac{1}{2} (1 - \lambda^2) - \frac{1}{p} (1 - \lambda^p) \) and \( g_2(\lambda) = \frac{1}{2} (1 - \lambda) - \frac{1}{3p} (1 - \lambda^p) \). Since \( g_1(\lambda) \rightarrow \frac{1}{2} - \frac{1}{p} \geq 0 \) for \( p \geq 2 \) as \( \lambda \rightarrow 0 \), \( g_1(\lambda) \rightarrow 0 \) as \( \lambda \rightarrow 1 \) and \( g_1'(\lambda) = -\lambda(1 - \lambda^{p-2}) \) for \( \lambda \in (0, 1) \), we know that \( g_1(\lambda) \geq 0 \) for \( \lambda \in (0, 1) \). Similar arguments arrive at \( g_2(\lambda) \geq 0 \) for \( \lambda \in (0, 1) \). Thus \( L_c(u) \geq L_c(u_\lambda) \) for any \( \lambda \in (0, 1) \). In particular, \( L_c(u) \geq L_c(u_{\lambda_1}) \). Taking \( w = u_{\lambda_1} \), we conclude the proof. \( \square \)

**Lemma 3.3.** If \( c > 0 \) and \( 2 \leq p < 4 \), then \( d_Q \geq d \).

**Proof.** The idea is similar to those used in the proof of lemma 3.2, but we need a quite different scaling argument. Our purpose is to show that for any \( u \in Q \), there is \( w \in \mathcal{N} \) such that \( L_c(u) \geq L_c(w) \). Firstly, for any \( u \in Q \), \( N(u) < 0 \) and \( Q(u) = 0 \). We denote that \( u^\lambda(x) = \lambda^2 u(\lambda x, \lambda^2 y) \), then
\[
N(u^\lambda) = \int \left( \lambda^2 u_x^2 + \lambda^2 v^2 + \lambda \beta u_x u_t + cu^2 - \frac{\lambda^{2p} u^{p+2}}{p+1} \right).
\]
Noticing that \( N(u^\lambda) \rightarrow c \int u^2 > 0 \) as \( \lambda \rightarrow 0 \) and \( N(u^\lambda) \rightarrow N(u) < 0 \) as \( \lambda \rightarrow 1 \), we find a \( \lambda_2 \in (0, 1) \) such that \( N(u^{\lambda_2}) = 0 \), i.e. \( u^{\lambda_2} \in \mathcal{N} \). Therefore \( L_c(u^{\lambda_2}) \geq d \).

Secondly, we prove that \( L_c(u) \geq L_c(u^\lambda) \) for any \( \lambda \in (0, 1) \). In fact, from
\[
L_c(u^\lambda) = \int \left( \frac{\lambda^2 u_x^2 + \lambda^2 v^2 + \lambda \beta u_x u_t + cu^2}{2} - \frac{\lambda^{3p/2} u^{p+2}}{(p+2)(p+1)} \right),
\]
and \( Q(u) = 0 \), we obtain that
\[
L_c(u) - L_c(u^\lambda) = \int \left( g_3(\lambda)u_x^2 + g_4(\lambda)\beta u_x u_t + g_3(\lambda)v^2 \right),
\]
where \( g_3(\lambda) = \frac{1}{2} (1 - \lambda^2) - \frac{2}{3p} (1 - \lambda^{3p/2}) \) and \( g_4(\lambda) = \frac{1}{2} (1 - \lambda) - \frac{1}{3p} (1 - \lambda^{3p/2}) \). Using an argument similar to those in the proofs of \( g_i(\lambda) \geq 0 \) for \( i = 1, 2 \) for any \( \lambda \in (0, 1) \), we have that \( g_j(\lambda) \geq 0 \) for \( j = 3, 4 \) for any \( \lambda \in (0, 1) \). Thus \( L_c(u) \geq L_c(u^\lambda) \) for all \( \lambda \in (0, 1) \). In particular, \( L_c(u) \geq L_c(u^{\lambda_2}) \) and we complete the proof. \( \square \)

Now we are in a position to define several invariant sets.

\[
\mathcal{R}_+ = \{ u \in X; L_c(u) < d, N(u) < 0, R(u) > 0 \};
\]
\[
\mathcal{R}_- = \{ u \in X; L_c(u) < d, N(u) < 0, R(u) < 0 \};
\]
\[
\mathcal{Q}_+ = \{ u \in X; L_c(u) < d, N(u) < 0, Q(u) > 0 \};
\]
\[
\mathcal{Q}_- = \{ u \in X; L_c(u) < d, N(u) < 0, Q(u) < 0 \}.
\]

**Lemma 3.4.** For any \( c > 0 \) and \( 2 \leq p < 4 \), \( \mathcal{R}_+ \) and \( \mathcal{Q}_+ \) are invariant under the flow generated by the Cauchy problem (1.1) and (1.2).

**Proof.** We only prove that \( \mathcal{Q}_- \) is invariant under the flow generated by the Cauchy problem (1.1) and (1.2) since the proofs of the others are similar. Let \( u(x, y, t) \) (\( t \in [0, T] \)) be the solution of (1.1) and (1.2) with initial data \( u_0 \). We
simply denote the function $t \mapsto u(x, y, t)$ by $u(t)$. First, by lemma 3.1, we know that if $L_c(u_0) < d$, then $L_c(u(t)) \equiv L_c(u_0) < d$.

Second, we show that $N(u(t)) < 0$ for $t \in [0, T)$. If this is not true, from the continuity, $t_0 \in (0, T)$ such that $N(u(t_0)) = 0$. Then since $u(t_0) \neq 0$, we know that $L_c(u(t_0)) \geq d$. This contradicts with $L_c(u(t)) < d$ for all $t \in (0, T)$. Therefore $N(u(t)) < 0$ for $t \in [0, T)$.

Finally, we show that $Q(u(t)) < 0$ for $t \in [0, T)$. If this is not true, from the continuity, $t_1 \in (0, T)$ such that $Q(u(t_1)) = 0$. Since we have proved that $N(u(t)) < 0$, we can have that $u(t_1) \in Q$. So that $L_c(u(t_1)) \geq d_Q \geq d$, which contradicts with $L_c(u(t)) < d$ for all $t \in (0, T)$. Therefore $Q(u(t)) < 0$ for $t \in [0, T)$.

\[ \text{4. Improved blow up} \]

Recalling that, in a different setting, if $S(t)$ is a solution of the nonlinear Schrödinger equation

\[ iS_t + \Delta S + |S|^\sigma S = 0, \quad S(0) = S_0, \quad x \in \mathbb{R}^m, \]

then it is shown that $S(t)$ either exists globally in the energy space $H^1(\mathbb{R}^m)$ or blows up in a finite time in the energy space, i.e. $\exists t_1 > 0$ such that

\[ \lim_{t \to t_1} ||S(t)||_{H^1(\mathbb{R}^m)} = \infty \]

(e.g. Berestycki & Cazenave 1981; Weinstein 1983; Zhang 2001). For the problem considered here, we know from lemma 3.4, (3.1) and (3.2) that if $u_0 \in \mathcal{R}_+$, then the solution $u(t)$ with $u(0) = u_0$ satisfies $||u(t)||_Y \leq C$. Consequently, $u(t)$ exists globally in $Y$ for $t \geq 0$ and hence blow up cannot occur in a finite time in $Y$. But as we will see below, we do have a blow-up result that is only due to the transverse dispersion.

Guo & Han (1996) proved that the solution of (1.1) and (1.2) blows up in a finite time if the initial data $u_0$ satisfies $E_\beta(u_0) < 0$ and $p \geq 4$. However, using the invariant sets constructed in §3, we are able to extend this blow-up result to allow the energy $E_\beta$ to be positive and $2 \leq p < 4$.

**Theorem 4.1.** If $2 \leq p < 4$, $u_0 \in \mathcal{R}_+ \cap Q_-$ and $yu_0 \in L^2(\mathbb{R}^2)$, then the solution $u(x, y, t)$ of (1.1) and (1.2) with initial data $u_0$ blows up at finite time.

**Proof.** For any $u_0 \in \mathcal{R}_+ \cap Q_-$, we have that $u(x, y, t) \in \mathcal{R}_+ \cap Q_-$. Now, for any fixed $t$, we simply denote $u(x, y, \cdot)$ by $u$. Thus $L_c(u) < d$, $N(u) < 0$, $Q(u) < 0$ and $R(u) > 0$. For any $\lambda > 0$, we denote that $u^\lambda(x, y) = \lambda^{3/2} u(\lambda x, \lambda^2 y)$, then similar to the computations used in the proof of lemma 3.3, we have that $N(u^\lambda) \to c \int u^2 > 0$ as $\lambda \to 0$ and $N(u^\lambda) \to N(u) < 0$ as $\lambda \to 1$. Hence there is $\lambda^* \in (0, 1)$ such that $N(u^\lambda) = 0$ and $N(u^\lambda) < 0$ for $\lambda \in (\lambda^*, 1]$. For $\lambda \in [\lambda^*, 1]$, $Q(u^\lambda)$ has the following three possibilities:

(i) $Q(u^\lambda) < 0$ for $\lambda \in [\lambda^*, 1]$,

(ii) $Q(u^\lambda) = 0$, and

(iii) there is $\mu$ such that $\lambda^* < \mu < 1$ and $Q(u^\mu) = 0$.
For cases (i) and (ii), we all have that \( N(u^\lambda) = 0 \) and \( Q(u^\lambda) \leq 0 \). It follows that \( L_c(u^\lambda) \geq d \). Moreover we have that

\[
L_c(u) - L_c(u^\lambda) = \int \left( (1 - \lambda^2)u_x^2 + (1 - \lambda^2)\beta u_x Hu + (1 - \lambda^2)v^2 \right)
- \left( \frac{(1 - \lambda^2 p/2)}{(p + 2)(p + 1)} u^{p+2} \right)
\geq \int \left( \frac{2}{3p} \left[ (1 - \lambda^2)u_x^2 + (1 - \lambda^2)\beta u_x Hu + (1 - \lambda^2)v^2 \right]
- \left( \frac{(1 - \lambda^2 p/2)}{(p + 2)(p + 1)} u^{p+2} \right) \right)
= \frac{2}{3p} (Q(u) - Q(u^\lambda)) \geq \frac{2}{3p} Q(u).
\]

(4.1)

For case (iii), we have \( N(u^\mu) < 0 \) and \( Q(u^\mu) = 0 \), i.e. \( u^\mu \in Q \). Thus \( L_c(u^\mu) \geq d_Q \geq d \) and we can also obtain from a similar computation that

\[
L_c(u) - L_c(u^\mu) \geq \frac{2}{3p} (Q(u) - Q(u^\mu)) \geq \frac{2}{3p} Q(u).
\]

In all cases, we have that

\[
Q(u) \leq \frac{3p}{2} (L_c(u_0) - d) = -\delta_0 < 0.
\]

Note that from Guo & Han (1996, theorem 2.4), \( yu \in L^\infty(0, T; L^2(\mathbb{R}^2)) \) as long as \( yu_0 \in L^2(\mathbb{R}^2) \). Moreover by Guo & Han (1996, theorem 3.1), for \( I(t) = \int y^2 u^2(x, y, t)dx \), there holds

\[
I''(t) = 8 \int \left( v^2 - \frac{pu^{p+2}}{2(p + 2)(p + 1)} \right) = 8(Q(u) - R(u)).
\]

On the other hand, \( R(u) > 0 \) since \( u_0 \in \mathcal{R}_+ \). Thus we find that there exists \( t_0 > 0 \) such that \( I(t_0) = 0 \). The conserved momentum \( \frac{1}{2} \int |u|^2 dxdy \) and the classical inequality \( |u|^2 \leq 2|yu|^2|u_y|^2 \) imply that there exists a blow-up time \( T_1 \leq t_0 \) such that

\[
\lim_{t \to T_1^-} |\partial_y u|^2 = +\infty.
\]

Remark 4.2. If \( 2 \leq p < 4 \) and \( u_0 \in \mathcal{R}_+ \), then

\[
E_\beta(u_0) = \int \left( \frac{u_{0x}^2 + v_0^2 + \beta u_{0x} Hu_0}{2} - \frac{u_0^{p+2}}{(p + 2)(p + 1)} \right)
> \int \left( \left( \frac{1}{2} - \frac{1}{p} \right) u_{0x}^2 + \frac{1}{2} v^2 + \left( \frac{1}{2} - \frac{1}{2p} \right) \beta u_{0x} Hu_0 \right) > 0.
\]

5. Instability of solitary waves

In this section, we will prove that the solitary wave of (1.1) is strongly unstable. We assume throughout this section that \( c > (\beta/2) \) and if \( \varphi \) is a solitary wave of (1.1), then \( y\varphi \in L^2(\mathbb{R}^2) \).

**Definition 5.1.** Suppose for \( c > 0 \), \( \varphi \in Y \) is a groundstate solution of (2.1) (of course a solitary wave of (1.1) with wave speed \( c \) in the \( x \)-direction). We say that \( \varphi \) is strongly unstable if for any \( \delta > 0 \) there is \( u_0 \in X_s(s \geq 3) \), with \( ||u_0 - \varphi||_Y < \delta \) such that the solution \( u \) of (1.1) with initial data \( u_0 \) blows up in a finite time. More precisely, there is \( 0 < T_2 < \infty \) such that

\[
\lim_{t \to T_2^-} |\partial_y u|^2 = \infty.
\]

**Theorem 5.2.** Suppose that \( \varphi \) is a solitary wave solution of (1.1) corresponding to the groundstate solution of (2.1) and \( 2 \leq p < 4 \). If \( c > (\beta/2) \), then \( \varphi \) is strongly unstable in the sense of definition 5.1.

Before proving theorem 5.2, we need the following lemmas.

**Lemma 5.3.** Suppose that \( \varphi \) is the groundstate solution of (2.1) obtained in \( \S \) 2. Then \( Q(\varphi) = 0 \) and \( R(\varphi) = 0 \).

**Proof.** Since \( \varphi \) is a groundstate solution of (2.1), \( \partial_x E_\beta(\varphi^k)|_{\lambda = 1} = 0 \) with \( \varphi^k(x, y) = \lambda^{3/2} \varphi(\lambda x, \lambda^2 y) \) and \( \partial_x E_\beta(\varphi^k)|_{\lambda = 1} = 0 \) with \( \varphi^k(x, y) = \lambda \varphi(\lambda x, \lambda y) \). Direct computations arrive at \( Q(\varphi) = \partial_x E_\beta(\varphi^k)|_{\lambda = 1} = 0 \) and \( R(\varphi) = \partial_x E_\beta(\varphi^k)|_{\lambda = 1} = 0 \). The proof is complete.

**Lemma 5.4.** There exists \( \lambda > 0 \) and \( \eta > 0 \) such that for \( w_0(x, y) = \lambda \varphi(x, \eta y) \),

\[
L_c(w_0) < d, \quad N(w_0) < 0, \quad Q(w_0) < 0 \quad \text{and} \quad R(w_0) > 0.
\]

**Proof.** Let \( \varphi \) be the groundstate solution of (2.1). We denote that \( \varphi_y = \psi_x \). Then

\[
N(\varphi) = \int \left( \psi_x^2 + \psi^2 + \beta \varphi_x \mathcal{H} \varphi + cp^{p+2} \frac{\varphi^{p+2}}{p + 1} \right) = 0.
\]

Lemma 5.3 implies that

\[
R(\varphi) = \int \left( \psi_x^2 + \frac{\beta}{2} \varphi_x \mathcal{H} \varphi - \frac{p}{(p + 2)(p + 1)} \varphi^{p+2} \right) = 0,
\]

\[
Q(\varphi) = \int \left( \psi_x^2 + \psi^2 + \frac{\beta}{2} \varphi_x \mathcal{H} \varphi - \frac{3p}{2(p + 2)(p + 1)} \varphi^{p+2} \right) = 0.
\]

From (5.2) and (5.4), we have that

\[
\int c \varphi^2 = \int \left( \frac{4 - p}{2(p + 2)(p + 1)} \varphi^{p+2} - \frac{\beta}{2} \varphi_x \mathcal{H} \varphi \right).
\]
By (5.3) and (5.4), we get that
\[
\int \psi^2 = \frac{p}{2(p + 2)(p + 1)} \int \varphi^{p+2}.
\] (5.6)

Let \( w_0(x, y) = \lambda \varphi(x, \eta y) \). We want to find \( \lambda \) and \( \eta \) such that
\[
L_c(w_0) < d, \quad N(w_0) < 0, \quad Q(w_0) < 0 \quad \text{and} \quad R(w_0) > 0.
\] (5.7)

Since
\[
L_c(w_0) = \int \left( \frac{\lambda^2}{2\eta} \varphi_x^2 + \frac{\lambda^2}{2\eta} \psi^2 + \frac{\lambda^2}{2\eta} \beta \varphi_x \mathcal{H} \varphi + \frac{\lambda^2}{2\eta} \varphi_x^2 - \frac{\lambda^{p+2} \varphi_x^{p+2}}{\eta(p + 2)(p + 1)} \right),
\] (5.8)
\[
N(w_0) = \int \left( \frac{\lambda^2}{\eta} \varphi_x^2 + \lambda^2 \psi^2 + \frac{\lambda^2}{\eta} \beta \varphi_x \mathcal{H} \varphi + \frac{\lambda^2}{\eta} \varphi_x^2 - \frac{\lambda^{p+2} \varphi_x^{p+2}}{\eta(p + 2)(p + 1)} \right),
\] (5.9)
\[
Q(w_0) = \int \left( \frac{\lambda^2}{\eta} \varphi_x^2 + \lambda^2 \psi^2 + \frac{\lambda^2}{\eta} \beta \varphi_x \mathcal{H} \varphi - \frac{3p \lambda^{p+2} \varphi_x^{p+2}}{2\eta(p + 2)(p + 1)} \right) \quad \text{and}
\] (5.10)
\[
R(w_0) = \int \left( \frac{\lambda^2}{\eta} \varphi_x^2 + \lambda^2 \psi^2 + \frac{\lambda^2}{\eta} \beta \varphi_x \mathcal{H} \varphi - \frac{p \lambda^{p+2} \varphi_x^{p+2}}{\eta(p + 2)(p + 1)} \right),
\] (5.11)

we have from (5.3), (5.5), (5.6) and delicate computations that condition (5.7) is equivalent to conditions
\[
R(w_0) = \left( \frac{\lambda^2}{\eta} - \frac{\lambda^{p+2}}{\eta} \right) \int \frac{p \varphi_x^{p+2}}{(p + 2)(p + 1)} > 0,
\] (5.12)
\[
Q(w_0) = \left( \frac{\lambda^2}{\eta} + \frac{\lambda^2 \psi^2}{2} - \frac{3p \lambda^{p+2} \varphi_x^{p+2}}{2\eta} \right) \int \left( \varphi_x^2 + \frac{\beta}{2} \varphi_x \mathcal{H} \varphi \right) < 0,
\] (5.13)
\[
N(w_0) = \left( \frac{\lambda^2 \eta}{2} + \frac{(4 + p) \lambda^2}{2p\eta} - \frac{(p + 2) \lambda^{p+2}}{p\eta} \right) \int \left( \varphi_x^2 + \frac{\beta}{2} \varphi_x \mathcal{H} \varphi \right) < 0 \quad \text{and}
\] (5.14)
\[
L_c(w_0) = \left( \frac{\lambda^2 \eta}{2} + \frac{(4 + p) \lambda^2}{2p\eta} - \frac{2p \lambda^{p+2}}{p\eta} \right) \int \left( \varphi_x^2 + \frac{\beta}{4} \varphi_x \mathcal{H} \varphi \right)
\] < \int \left( \varphi_x^2 + \frac{\beta}{4} \varphi_x \mathcal{H} \varphi \right).
\] (5.15)
Conditions (5.12) and (5.15) are equivalent to
\[
\begin{align*}
\lambda^p &< 1, \\
3\lambda^p &> \eta^2 + 2, \\
\frac{2(p + 2)}{p} \lambda^p &> \eta^2 + \frac{4 + p}{p}, \\
\frac{4}{p} \lambda^p &> \eta^2 + \frac{4}{p} - 1.
\end{align*}
\]
(5.16)

Taking \(\eta_0^2 = 1 - \varepsilon_0\) (where \(\varepsilon_0 > 0\) but is still small enough) and \(\lambda_0\) with
\[
1 > \lambda_0^p > \max \left\{ 1 - \frac{\varepsilon_0}{3}, 1 - \frac{p}{2(p + 2)} \varepsilon_0 \right\},
\]
we know that (5.7) holds for \(w_0(x, y) = \lambda_0 \varphi(x, \eta_0 y)\). The proof is complete. \[\blacksquare\]

**Lemma 5.5.** For any fixed \(u, w \in Y\) with \(||w||_Y \leq K\), there exist positive constants \(K_j\) \((j = 1, 2, 3, 4)\) independent of \(w\), such that
\[
\begin{align*}
L_c(u + w) &< L_c(u) + K_1||w||_Y, \\
N(u + w) &< N(u) + K_2||w||_Y, \\
Q(u + w) &< Q(u) + K_3||w||_Y, \quad \text{and} \\
R(u + w) &> R(u) - K_4||w||_Y.
\end{align*}
\]  
(5.17)  
(5.18)  
(5.19)  
(5.20)

**Proof.** We prove (5.19) in detail and indicate the differences in the proofs of the others. First using the elementary inequality
\[
|u + w|^{p+2} \geq |u|^{p+2} + |w|^{p+2} - C(p)(|u||w|^{p+1} + |u|^{p+1}|w|),
\]
and the anisotropic Sobolev imbedding theorem (Besov *et al.* 1978), we have that
\[
\int (|u|^{p+2} - |u + w|^{p+2}) \leq ||w||_{Y}^{p+2} + C(||u||_Y||w||_{Y}^{p+1} + ||u||_{Y}^{p+1}||w||_Y)
\]
\[
\leq C||w||_Y. 
\]
(5.21)

Second using the properties (I)–(V) of the Hilbert operator \(\mathcal{H}\), we have that
\[
\int (u_x + w_x) \mathcal{H}(u + w) \leq \int u_x \mathcal{H} u + C||w||_Y. 
\]
(5.22)

Finally, we have that
\[
Q(u + w) \leq Q(u) + C_1||w||_Y + C_2||w||_Y^2 + C_3 \int (|u|^{p+2} - |u + w|^{p+2}) 
\]
\[
\leq Q(u) + K_3||w||_Y,
\]
proving (5.19). The proofs of (5.17) and (5.18) are almost the same and omitted here. For (5.20), firstly using the properties (I)–(V) of the Hilbert operator \(\mathcal{H}\),
we have that
\[
\int (u_x + w_x) \mathcal{H}(u + w) \geq \int u_x \mathcal{H}u - C_4 ||w||_Y.
\]
Secondly, using the anisotropic Sobolev imbedding theorem and
\[
|u + w|^{p+2} \leq |u|^{p+2} + |w|^{p+2} + C_5 (|u||w|^{p+1} + |u|^{p+1}|w|),
\]
we have that
\[
\int (u + w)^{p+2} \leq \int u^{p+2} + C_6 ||w||_Y.
\]
Thus
\[
R(u + w) \geq R(u) + \int (2u_x w_x + w_x^2) - C_4 ||w||_Y - C_6 ||w||_Y
\]
\[
\geq R(u) - K_4 ||w||_Y.
\]
We complete the proof of lemma 5.5. 

Now we are in a position to prove theorem 5.2.

Proof of theorem 5.2. For any \( \delta > 0 \), we choose sufficiently small \( \varepsilon_0 \),
\[
\eta_0^2 = 1 - \varepsilon_0 \quad \text{and} \quad 1 > \lambda_0^p > \max \left\{ 1 - \frac{\varepsilon_0}{3}, 1 - \frac{p}{2(p+2)} \varepsilon_0 \right\},
\]
such that
\[
||w_0 - \varphi||_Y < \frac{\delta}{2},
\]
where \( w_0 \) is defined as in lemma 5.4. Since \( X_s \) is dense in \( Y \), we find \( u_0 \in X_s \) such that
\[
||u_0 - w_0||_Y < \min \left\{ \frac{\delta}{2}, K_1 + K_2 + K_3 + K_4 \right\},
\]
where \( K_j \) (\( j = 1, 2, 3, 4 \)) are chosen as in lemma 5.5 and \( \delta_1 \) is defined as
\[
\delta_1 = \min \left\{ d - L_c(w_0), -N(w_0), -Q(w_0), R(w_0), \frac{\delta}{2} \right\}.
\]
Therefore
\[
||u_0 - \varphi||_Y \leq ||u_0 - w_0||_Y + ||w_0 - \varphi||_Y < \delta,
\]
and we obtain from lemmas 5.4 and 5.5 that
\[
u_0 \in R_+ \cap Q_-. 
\]
Since \( \delta \) is arbitrary and \( y\varphi \in L^2(\mathbb{R}^2) \), we have that \( yu_0 \in L^2(\mathbb{R}^2) \). Theorem 4.1 implies that the solution \( u(x, y, t) \) of (1.1) with initial data \( u(x, y, 0) = u_0 \) blows
up in a finite time. Hence for any \( c > (\beta/2) \), the solitary wave \( \varphi(x - ct, y) \) is strongly unstable in the sense of definition 5.1.

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References


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