Adhesion of an elastic plate to a sphere

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Stationary principles and the von Kármán plate theory are used to study the adhesion of thin elastic plates to a rigid sphere. Contact requires both flexural and membrane strains that can lead to partial or complete delamination. Interestingly, whereas a large area plate might spontaneously delaminate from the sphere, dividing this plate into many smaller plates with equivalent thickness eliminates membrane strains and may allow complete contact. The theoretical predictions are compared to experimental results for low density polyethylene on a smooth glass sphere. The peel strength is estimated with a modified Kendall peel equation.

Keywords: plate theory; thin-film delamination; peeling; non-developable

1. Introduction

Theoretical models for plate or membrane delamination are widely used in adhesion and thin film sciences. Recently, these models have been extended to study silicon wafer bonding (Turner & Spearing 2002; Pamp & Adams 2007) and biological and bio-inspired adhesives, in particular the peeling or attachment of plate-like *spatulae* or *lamellae* in wall-climbing insects and lizards. These latter structures are treated as a thin elastic plate in contact with a rigid, randomly rough substrate (Persson & Gorb 2003; Carbone *et al.* 2004; Filippov & Popov 2007).

Previous analyses of biological and bio-inspired adhesives are limited to two dimensions, and so issues related to contact with non-developable surface features do not arise. In practice, however, most rough or non-planar surfaces are non-developable (i.e. contain non-zero Gaussian curvature), and so complete contact requires both bending and stretching of the plate. This is exemplified by adhesion to a sphere that may represent a surface asperity or a large rounded protrusion.

A dual case is the adhesion of a spherically bowed plate to a flat substrate. This has been studied in the context of silicon wafer bonding (Turner & Spearing 2002; Pamp & Adams 2007). Such analyses use the Kirchhoff plate theory and only consider elastic strain energy by bending, which is reasonable since wafer bonding is governed by small angle, small strain deformations. In the current analysis, deformation involves moderate angle changes and so nonlinear, von

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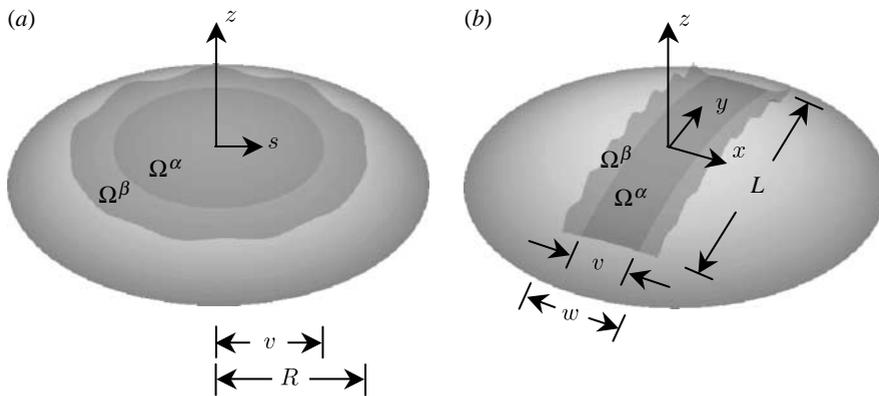
Figure 1. Elastic plate adhering to a rigid sphere of radius ρ: (a) circular plate of radius R; (b) rectangular plate of width w and length L. Ω^α (light grey) and Ω^β (dark grey) denote the contact and delaminated zones, respectively.

Kármán plate theory is required (Reddy 2007). This introduces membrane strain that, for the geometries of interest, often governs the elastic strain energy of the plate.

In §2, stability conditions are derived for the delamination of thin circular and rectangular plates from a rigid sphere. At equilibrium, the plate assumes a configuration that minimizes the total potential energy, which equals the elastic strain energies for bending and stretching minus the interfacial energy of adhesion. For prescribed sphere radius and material and interfacial properties, complete adhesion is limited to a range of plate geometries, which are presented in §3. In §4, the theoretical predictions are compared to empirical measurements obtained from the adhesion of low density polyethylene (LDPE) to smooth glass spheres. In the case of rectangular plates, the peel strength may be estimated using the Kendall peel theory (Kendall 1975), which has been used to model the detachment of insect and lizard spatulae (Persson & Gorb 2003; Spolenak *et al.* 2005; Pugno 2007). In §5, the theory is modified for peeling from a sphere.

2. Analysis

Plate geometries are presented in figure 1. Each plate has an elastic modulus E, Poisson's ratio ν and thickness H. The work of adhesion per area of contact between the plate and the sphere is denoted by W_{ad}. The plates make contact with a rigid sphere that has a radius of curvature ρ. The radius or the width of the contact zone is assumed to be several times smaller than ρ, and so whenever convenient the sphere may be modelled as a paraboloid.

(a) Free circular plate

As illustrated in figure 1a, a circular plate Ω of radius R adheres to a sphere over a contact zone of radius $v \leq R$. Because the system is axisymmetric, Ω may be parametrized by the radial arc length s and the distance z from the midplane. When the plate adheres to the substrate, points on the midplane displace by an amount $u_s = u_s(s)$ and $u_z = u_z(s)$ in the radial and vertical directions, respectively.

For $3R < \rho$, the sphere may be approximated as a paraboloid and so $u_z = -s^2/2\rho$. Hence, for a von Kármán plate, the radial, hoop and shear strains on Ω have the form

$$\epsilon_{ss} = \frac{\partial u_s}{\partial s} + \frac{1}{2}\left(\frac{s}{\rho}\right)^2 + \frac{z}{\rho}, \qquad \epsilon_{\theta\theta} = \frac{u_s}{s} + \frac{z}{\rho} \quad \text{and} \quad \epsilon_{s\theta} = 0, \qquad (2.1)$$

respectively. The elastic strain energy density is

$$\phi = \frac{E}{2(1-\nu^2)}\left(\epsilon_{ss}^2 + 2\nu\epsilon_{ss}\epsilon_{\theta\theta} + \epsilon_{\theta\theta}^2\right). \qquad (2.2)$$

As described for an adhering rod (Majidi 2007), the configuration Ω may be decomposed into a contacting portion Ω^α $(0 \le s \le v)$ and non-contacting portion Ω^β $(v < s \le R)$. Let Φ^α and Φ^β denote the total elastic strain energy contained in Ω^α and Ω^β, respectively. So, for example,

$$\Phi^\alpha = \int_0^v \int_{-H/2}^{H/2} 2\pi s \phi \, \mathrm{d}z \, \mathrm{d}s. \qquad (2.3)$$

Following this convention, the total potential energy of the system is

$$U = \Phi^\alpha + \Phi^\beta - \pi v^2 W_{\mathrm{ad}}. \qquad (2.4)$$

At equilibrium, U is stationary with respect to the contact radius, which implies that $\mathrm{d}U/\mathrm{d}v = 0$. Since Φ^β decreases monotonically with increasing v, a sufficient condition for the stability of complete contact is that $(\mathrm{d}\Phi^\alpha/\mathrm{d}v)_{v=R} \le 2\pi R W_{\mathrm{ad}}$.

The function $u_s = u_s(s)$ at equilibrium is determined by solving the boundary-value problem

$$\frac{\partial \mathcal{L}}{\partial u_s} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}}{\partial u_s'}\right) = 0 \quad : \quad u_s(0) = 0 \quad \left(\frac{\partial \mathcal{L}}{\partial u_s'}\right)_{s=v} = 0, \qquad (2.5)$$

for the Lagrangian density

$$\mathcal{L} = \int_{-H/2}^{H/2} 2\pi s \phi \, \mathrm{d}z, \qquad (2.6)$$

obtained from the expression for Φ^α (Lanczos 1970). This yields

$$u_s = \frac{(1-\nu)sv^2 - (3-\nu)s^3}{16\rho^2}. \qquad (2.7)$$

Substituting this into (2.1) and computing Φ^α, it follows that complete contact is stable if

$$\frac{R^4}{128\rho^4} + \frac{H^2}{12(1-\nu)\rho^2} \le \frac{W_{\mathrm{ad}}}{EH}. \qquad (2.8)$$

The first and second terms are the strain energy release rate associated with plate stretching and bending, respectively. For smaller plates such that $R \ll \rho$, only bending is a significant factor in determining contact stability.

(b) Rectangular plate

A rectangular plate Ω of width w and length L adheres to a sphere in the manner illustrated in figure 1b. It is assumed that $3w < \rho$ and so the sphere may again be approximated as a paraboloid. Let the coordinates x and y denote the distance from the plate centre along the intermediate (widthwise) and major (lengthwise) axes, respectively. When performing the experiment described in §4, it is observed that the delamination front is parallel to the major axis and so the contact zone Ω^α is assumed to be a rectangle of length L and width $v \leq w$.

At equilibrium, we postulate that points on the midplane displace by an amount

$$u_x = -\frac{x^3}{6\rho^2}, \qquad u_y = -\frac{y^3}{6\rho^2} - \frac{x^2 y}{2\rho^2} \quad \text{and} \quad u_z = -\frac{x^2}{2\rho} - \frac{y^2}{2\rho}. \qquad (2.9)$$

The von Kármán strains are thus

$$\epsilon_{xx} = \frac{z}{\rho}, \qquad \epsilon_{yy} = -\frac{1}{2}\left(\frac{x}{\rho}\right)^2 + \frac{z}{\rho} \quad \text{and} \quad \epsilon_{xy} = 0. \qquad (2.10)$$

The expression for elastic strain energy density is the same as in (2.2), with the subscripts s and θ substituted by x and y, respectively. Hence,

$$\Phi^\alpha = \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \int_{-v/2}^{v/2} \phi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \frac{EHv^5 L}{640(1-\nu^2)\rho^4} + \frac{EH^3 vL}{12(1-\nu)\rho^2}. \qquad (2.11)$$

Complete contact is stable when $(\mathrm{d}\Phi^\alpha/\mathrm{d}v)_{v=w} \leq W_{\mathrm{ad}} L$, which implies

$$\frac{w^4}{128(1-\nu^2)\rho^4} + \frac{H^2}{12(1-\nu)\rho^2} \leq \frac{W_{\mathrm{ad}}}{EH}. \qquad (2.12)$$

As in (2.8), the first and second terms correspond to plate stretching and bending, respectively.

3. Results

The stability criteria (2.8) and (2.12) determine whether a thin plate of known geometry (R, w, H) will adhere to a hemisphere with radius of curvature ρ for prescribed elastic modulus E, Poisson's ratio ν and interfacial work of adhesion W_{ad}. These equations may be rearranged to express the critical thickness H_{cr} above which the plate will spontaneously delaminate from the hemisphere,

$$H_{\mathrm{cr}} = \frac{2^{1/3}\kappa^{2/3} - 24^{1/3}\alpha\beta}{6^{2/3}\beta\kappa^{1/3}}, \qquad (3.1)$$

where

$$\alpha = \begin{cases} R^4/128\rho^4, & \text{circular,} \\ w^4/128(1-\nu^2)\rho^4, & \text{rectangular,} \end{cases}$$
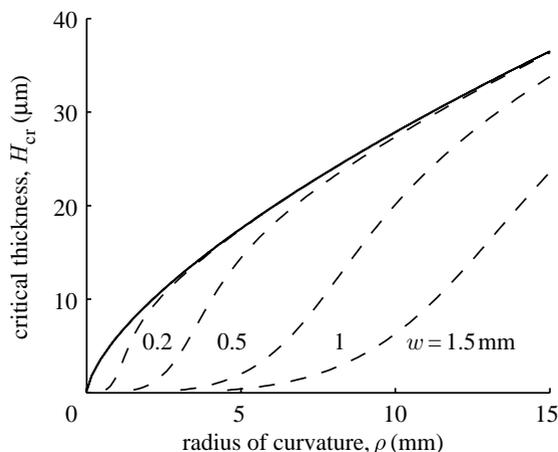$$\beta = 1/12(1-\nu)\rho^2$$

Figure 2. Critical thickness H_{cr} for the adhesion of a rectangular plate to a sphere of radius ρ; varying plate width w (dashed lines); narrow plate (solid line), i.e. $(w/\rho)^2 \ll H/\rho$. The paraboloid approximation for a sphere may not be valid for $3w > \rho$ and so values are not plotted in this range; $E = 1$ GPa, $\nu = 0.4$, $W_{ad} = 30$ mJ m^{-2}.

and

$$\kappa = 9\beta^2 \left(\frac{W_{ad}}{E} \right) + \sqrt{3\beta^3 \left\{ 4\alpha^3 + 27\beta \left(\frac{W_{ad}}{E} \right)^2 \right\}}.$$

For a narrow rectangular plate or small radius circular plate, this reduces to

$$H_{cr} = \left\{ \frac{12(1-\nu)\,W_{ad}\rho^2}{E} \right\}^{1/3}. \tag{3.2}$$

Values of H_{cr} for a rectangular plate are plotted in figure 2. As the ratio w/ρ increases, values for wide and narrow plates diverge to the extent that the critical thicknesses may differ by an order of magnitude. That is, whereas a wide rectangular plate might spontaneously delaminate from a sphere, dividing this plate into multiple narrow strips with the same thickness eliminates membrane strain and allows complete adhesion. For example, consider an elastic plate of modulus $E = 1$ GPa, Poisson's ratio $\nu = 0.4$, thickness $H = 25$ μm and width $w = 1.5$ mm in contact with a sphere of radius $\rho = 10$ mm and let $W_{ad} = 30$ mJ m^{-2}. As shown in figure 2, $H > H_{cr}$ and so the plate will partially delaminate from the sphere. For complete contact, H must be reduced by a factor of five. Alternatively, the plate may be divided into three 0.5 mm wide strips, such that $H_{cr} > H$.

4. Experiment

Adhesion experiments are performed between the LDPE and the spherical ends of glass laboratoryware, which have radii ρ of 8, 12.5, 20 and 25 mm. Prior to the experiment, the spherical glass surfaces are scrubbed with soap and water, submerged in a beaker of sodium hydroxide (5 mol l^{-1}, EM Science, Merck),
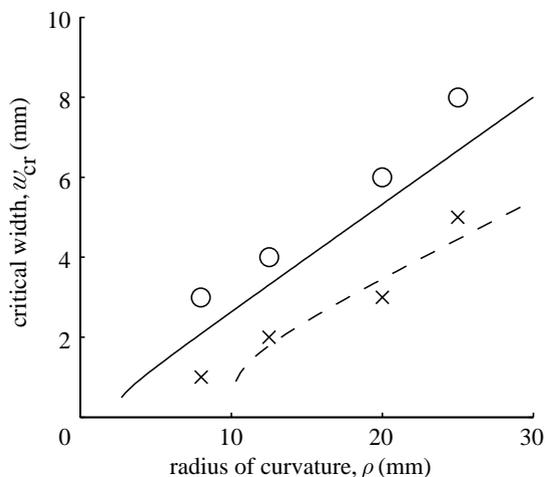
Figure 3. Comparison of experimental and theoretical values for critical width w_{cr} versus sphere radius ρ. Experiment: $H=45\,\mu m$ (circles) and $H=90\,\mu m$ (crosses); theory: $H=45\,\mu m$, $W_{ad}=0.32\,J\,m^{-2}$ (solid line) and $H=90\,\mu m$, $W_{ad}=0.15\,J\,m^{-2}$ (dashed line); W_{ad} measured with peel angle tests; $E=0.15\,GPa$ (Rabinowitz & Brown 2001).

rinsed in distilled water and then air dried. Next, sheets of LDPE (Film-Gard, Carlisle Plastic, Inc.) and polyester are rinsed in distilled water and air dried. The LDPE sheets are then stacked, placed between the polyester covers and passed through a heated roller laminator (Catena 35, General Binding Corporation) at 110°C and at a speed of $6.35\times10^{-4}\,m\,s^{-1}$. This process yields 45 and 90 µm thick rectangular LDPE plates of 1 cm length that adhere well to the glass.

Peel angle tests performed on flat glass slides with weights of 10, 20 and 50 g suggest a work of adhesion W_{ad} of 0.32 and 0.15 $J\,m^{-2}$ for the 45 and 90 µm plates, respectively. These values should not be regarded as the *true* work of adhesion since the interfacial contact is not complete but limited by the microscale non-planarity of the LDPE. This may explain why the thicker plate, which has a greater bending stiffness and hence greater resistance to complete planar contact, exhibits a lower measured W_{ad}.

The rectangular plates are carefully pressed to the glass spheres. Scissors are used to reduce the plate width 1 mm at a time until the edges of the plate cease to delaminate in the manner illustrated in figure 1. This corresponds to the critical width w_{cr} below which complete adhesion is possible.

Measurements for w_{cr} are plotted versus ρ in figure 3. Also shown is the theoretical prediction for w_{cr} that is obtained by solving (2.12) for w,

$$w_{cr} = \rho \left\{ \frac{128(1-\nu^2)W_{ad}}{EH} - \frac{32(1+\nu)H^2}{3\rho^2} \right\}^{1/4}. \qquad (4.1)$$

Theoretical w_{cr} is calculated for $E=0.15\,GPa$ (Rabinowitz & Brown 2001) and $\nu=0.4$. Reasonable agreement is obtained between the theoretical prediction and the experimental measurement without the aid of data fitting. Stronger agreement can be achieved by including the term $d\Phi^\beta/dv$ into the l.h.s. of (2.12).
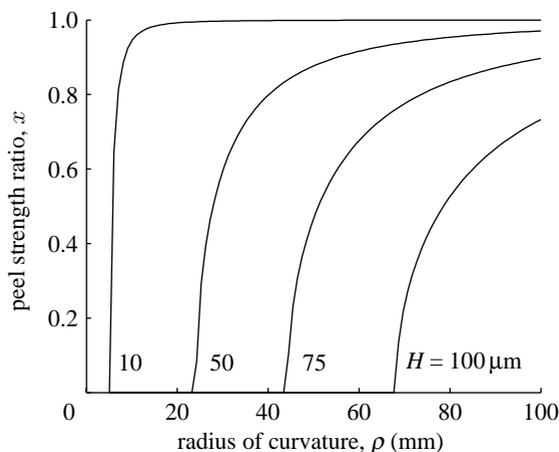
Figure 4. Ratio χ of peel strength for adhesion to a sphere of radius ρ versus a flat substrate for $w=1$ mm and varying plate thickness H; $E=1$ GPa, $\nu=0.4$, $W_{\mathrm{ad}}=30$ mJ m^{-2}.

For the geometries of interest, w_{cr} is linearly dependent on ρ. This indicates that the stability is governed by membrane strain and that the role of bending stiffness is negligible. In contrast, adhesion to a cylinder is governed by bending alone and is independent of plate width.

5. Peel strength

The force necessary to peel a rectangular plate from a rigid sphere is estimated using a modified version of the Kendall peel theory (Kendall 1975). A stress σ is applied at an angle θ along the edge $y=L/2$. The peel strength is the solution to the equilibrium condition

$$\frac{\sigma}{E}(1-\cos\theta) + \frac{1}{2}\left(\frac{\sigma}{E}\right)^2 = \frac{W_{\mathrm{eff}}}{EH}, \tag{5.1}$$

where W_{eff} is the 'effective' interfacial work of adhesion, which accounts for the elastic strain energy necessary for adhesion to a sphere. Following from (2.11):

$$W_{\mathrm{eff}} = W_{\mathrm{ad}} - \left\{\frac{EHw^4}{640(1-\nu^2)\rho^4} + \frac{EH^3}{12(1-\nu)\rho^2}\right\}. \tag{5.2}$$

For contact with a flat substrate, $W_{\mathrm{eff}} = W_{\mathrm{ad}}$, and (5.1) reduces to the Kendall peel equation. When peeling from a sphere, $W_{\mathrm{eff}} < W_{\mathrm{ad}}$ which implies that less force is required to peel from a sphere than from a flat substrate.

For small peel angles ($\theta \approx 0$), delamination occurs when the applied stress reaches $\sigma = \sqrt{2 W_{\mathrm{eff}} E/H}$. In this case, we let $\chi = \sqrt{W_{\mathrm{eff}}/W_{\mathrm{ad}}}$ denote the ratio of peel strength for adhesion to a sphere versus a flat substrate. The ratio χ is plotted in figure 4 for various values of plate thickness H and sphere radius ρ. As ρ decreases, the peel strength drops dramatically and vanishes when H exceeds H_{cr}.

Other possible detachment modes include interfacial sliding that occurs when the peeling force σHw exceeds the total interfacial shear strength τLw, where τ is the interfacial shear strength per unit area of contact. Also, for wide but thin

plates, stretching induced wrinkling relaxes the elastic strain energy Φ^β stored in the delaminated portion of the plate. Following the arguments (Cerda & Mahadevan 2003), Φ^β is approximately invariant to the delamination length and so the coefficient $1/2$ will drop from the second term in (5.1). This leads to, at most, a factor of $\sqrt{2}$ reduction in peel strength, but wrinkling only occurs when a critical strain is exceeded, prior to which other detachment modes may have already been activated.

6. Concluding remarks

A thin elastic plate must bend and stretch in order to adhere to a rigid sphere. For contact to be stable, the elastic strain energy associated with deformation must be balanced by the interfacial energy of adhesion. Since both bending and membrane (stretching) strains are involved, the von Kármán plate theory is adopted. Stability criteria relating geometry, elasticity and interfacial properties are derived using stationary principles.

The analysis yields critical values for plate thickness (H) and width (w) or radius (R) below which complete adhesion is possible for prescribed material elasticity (E, ν), sphere radius (ρ) and work of adhesion (W_{ad}). These results suggest that for some plates that do not adhere to a sphere, dividing into multiple smaller plates of the same thickness will eliminate membrane strain and allow complete adhesion.

For rectangular plates capable of complete adhesion to a sphere, peel strength is estimated with a modified Kendall peel equation. Specifically, the work of adhesion (W_{ad}) for contact with a flat substrate is replaced by an effective value (W_{eff}) that accounts for bending and membrane strain energy necessary to conform to a sphere over the contact zone. This modified term leads to a reduction in peel strength compared with peeling from a flat substrate.

The aim of this analysis is to aid in the design of synthetic lamellar and spatular structures for biologically inspired adhesives. However, the results are general enough to be applicable to a wide range of issues in adhesion and thin film sciences that involve non-developable substrates or surface features. The analysis might also be extended to partial contact by including the effects of buckling in the delaminated portion (Ω^β) of the plate.

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