A new large-number coincidence and a scaling law for the cosmological constant

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An ensemble of pure numbers of the order of 10^{122} is produced naturally from the fundamental parameters of modern cosmology. This new large-number coincidence problem is resolved by demonstrating implicit physical connections that follow from the standard cosmological model. However, the occurrence of the new large-number coincidence combined with the known coincidence among pure numbers of the order of 10^{40} poses a distinct problem that is resolved with a scaling law for the cosmological constant that was originally proposed by Zel'dovich.

Keywords: large numbers; cosmological constant; cosmic coincidence

1. Introduction

With the discovery of the accelerating Universe and the re-introduction of Einstein's cosmological constant, a new mystery concerning large, pure numbers was initiated. The Planck energy density is roughly 122 orders of magnitude larger than the apparent cosmological vacuum density (Sahni & Starobinsky 2000). The discrepancy between the Planck and the vacuum densities is problematic primarily owing to its inexplicable enormity. However, the mystery surrounding the factor 10^{122} is compounded by the fact that an ensemble of distinct, pure numbers of similar order is produced naturally by the fundamental parameters of modern cosmology.

In §2 of this paper six, physically significant, pure numbers of the order of 10^{122} are presented. Those pure numbers constitute a compelling, new large-number coincidence problem. The new coincidence problem is then resolved by the analysis in §3, which demonstrates implicit physical connections among the pure numbers that follow from the standard cosmological model. However, the same basic parameters of nature also generate a coincidence problem that involves pure numbers of the order of 10^{40}. Since it is extremely unlikely for chance alone to cause two separate large-number coincidences to manifest from a few basic parameters, there is a good motivation to hypothesize that the pure numbers of the order of 10^{122} are physically scaled to the pure numbers of the order of 10^{40}. In §4 it is shown that such a relationship leads to a scaling law between the mass of the nucleon and the cosmological constant that was originally proposed by Zel'dovich (Okun 2002).

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2. The pure numbers of the order of 10^{122}

The existence of a new large-number coincidence problem is established here by presenting a collection of pure numbers of the order of 10^{122}. The pure numbers all follow naturally from basic ratios of fundamental parameters and require no arbitrary powers or coefficients. The parameters from which the pure numbers are generated are presented in table 1 and the resulting large, pure numbers are presented in table 2. The cosmological parameters result from a relativistic treatment of a flat Universe in which the current Hubble parameter H_0 is $71 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$, the fraction Ω_Λ of the total cosmic energy density due to vacuum is 0.73 and the fraction Ω_{m} due to matter is 0.27. Those parameters are consistent with nominal empirical values (Spergel et al. 2003). Although two significant figures are used for the calculations in this work, the conclusions would all still be valid even if only integer powers of 10 were used. If improved observations should eventually indicate different cosmological parameters than those employed here, this work would still be valid providing that the estimations of the parameters do not change by more than a few orders of magnitude.

The first two large numbers of the order of 10^{122} involve the putative cosmological constant, Λ. The vacuum energy density due to the cosmological constant is $\varepsilon_\Lambda = 3\Lambda c^2/(8\pi G)$, where c is the vacuum speed of light and G is the Newtonian gravitational coupling. With $\Lambda \approx 1.2 \times 10^{-35} \, \text{s}^{-2}$, the vacuum density is roughly $6.2 \times 10^{-10} \, \text{J m}^{-3}$. The Planck energy density is defined as $\varepsilon_{\text{P}} \equiv m_{\text{P}} c^2 l_{\text{P}}^{-3}$, where m_{P} is the Planck mass and l_{P} is the Planck length, and is roughly $4.5 \times 10^{113} \, \text{J m}^{-3}$. The ratio of the Planck density to the vacuum density is defined here as n_1 and is

$$n_1 \equiv \frac{\varepsilon_{\text{P}}}{\varepsilon_\Lambda} \approx 7.1 \times 10^{122}. \tag{2.1}$$

The pure number in equation (2.1) is proportional to the fundamental pure number $c^5/(G\hbar\Lambda)$, where $\hbar$ is the Planck constant. The Wesson mass $m_{\text{W}} \equiv c^{-2} h \sqrt{\Lambda/3}$, where $h \equiv 2\pi\hbar$, is roughly $1.5 \times 10^{-68} \, \text{kg}$ and is the smallest, physically significant quantum of mass that could be associated with our cosmos (Wesson 2004). The largest possible mass that could be associated with our current Universe is the observable mass, which is the mass, M_{p0}, contained within the sphere whose radius is the current particle horizon R_{p0}. The mass of the observable Universe is roughly $9.3 \times 10^{53} \, \text{kg}$. The ratio n_2 of the largest mass to the smallest mass is thus

$$n_2 \equiv \frac{M_{\text{p0}}}{m_{\text{W}}} \approx 6.3 \times 10^{121}. \tag{2.2}$$

The next three large numbers presented are related to the cosmic capacities for information and computation. According to the Bekenstein–Hawking entropy bound, the maximum number of degrees of freedom available to the Universe is one-quarter of the surface area, $4\pi R_{\text{e}}^2$, of the sphere whose radius is the cosmic event horizon R_{e}, measured in Planck units (Bousso 2002). In this current age the maximum number of degrees of freedom, defined here as n_3, is

$$n_3 \equiv \frac{\pi R_{\text{e0}}^2}{l_{\text{P}}^2} \approx 2.5 \times 10^{122}, \tag{2.3}$$

Table 1. List of cosmological parameters discussed in this work.

parameter	approximate value	description
Λ	1.2×10^{-35} s^{-2}	cosmological constant
ε_Λ	6.2×10^{-10} J m^{-3}	vacuum density
R_{e0}	1.5×10^{26} m	current event horizon
R_{p0}	4.4×10^{26} m	current particle horizon
M_{e0}	3.4×10^{52} kg	current mass of event sphere
M_{p0}	9.3×10^{53} kg	current mass of particle sphere
T_0	4.3×10^{17} s	current age of the Universe

Table 2. Pure numbers of the order of 10^{122}.

name	definition	approximate value
n_1	$\varepsilon_P/\varepsilon_\Lambda$	1.7×10^{122}
n_2	M_{p0}/m_W	6.3×10^{121}
n_3	$\pi R_{e0}^2/l_P^2$	2.5×10^{122}
n_4	$2M_{p0}c^2 T_0/(\pi\hbar)$	2.1×10^{122}
n_5	V_{e0}/V_n	1.3×10^{123}
n_6	$M_{p0}^2 l_n/(m_n^2 R_{p0})$	9.3×10^{119}

where R_{e0} is the current event horizon. According to the Margolus–Levitin theorem, the maximum rate at which logical operations could be performed by a physical system with energy E is $2E/(\pi\hbar)$ (Margolus & Levitin 1998). Thus, the maximum number of operations that could have been performed by the mass of the observable Universe is

$$n_4 \equiv \frac{2M_{p0}c^2 T_0}{\pi\hbar} \approx 2.1 \times 10^{122}, \tag{2.4}$$

where T_0 is the age of the Universe (Lloyd 2002). The fifth pure number of the order of 10^{122} presented here follows from holographic considerations applied to the nucleon (Mena Marugán & Carneiro 2002). The characteristic size of the nucleon is of the order of 10^{-15} m, which is of the order of the Compton wavelength of the nucleon $l_n \equiv h/(m_n c)$. The number of nucleon volumes $V_n \sim l_n^3$ contained within the volume $V_{e0} \sim R_{e0}^3$ of the sphere whose radius in the current event horizon is

$$n_5 \equiv \frac{V_{e0}}{V_n} \approx 1.3 \times 10^{123}. \tag{2.5}$$

Another pure number contributing to the new coincidence problem is found by considering the fundamental scales of gravitational energy. The gravitational potential energy of the cosmic mass is roughly GM_{p0}^2/R_{p0}. The scale of gravitational binding energy associated with a nucleon is roughly Gm_n^2/l_n. The ratio of the two, defined as n_6, is

$$n_6 \equiv \frac{M_{p0}^2/R_{p0}}{m_n^2/l_n} \approx 9.3 \times 10^{119}. \tag{2.6}$$

Finally, there are at least two other pure numbers of the order of 10^{122} that may be produced naturally from the parameters of the Universe,

$$\frac{GM_{e0}^2}{\hbar c} \approx 2.4 \times 10^{120} \tag{2.7}$$

and

$$\frac{M_{e0} R_{e0} c}{\hbar} \approx 1.4 \times 10^{121}. \tag{2.8}$$

The terms in (2.7) and (2.8) do not have any known physical significance and therefore should not be considered to contribute strongly to the new coincidence problem.

The pure numbers n_1–n_6 constitute a new large-number coincidence. It is unlikely for chance alone to be responsible for generating so many similar pure numbers from just several fundamental parameters. It is reasonable to hypothesize that the new coincidence problem among large, pure numbers of the order of 10^{122} results from some underlying physics rather than chance alone. That hypothesis is verified in the following section.

3. Resolving the new coincidence problem

The coincidence problem associated with the pure numbers of the order of 10^{122} is the result of a few physical scaling laws that follow from the standard model of cosmology. The Friedmann–Robertson–Lemaitre–Walker (FRLW) equations summarize the general relativistic component of the model. They provide a relativistic treatment of the expansion of the Universe by relating the dynamics of the cosmic scale factor to the average density of energy in the cosmos. In a flat Universe the primary FRLW equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\varepsilon_m + \varepsilon_\Lambda + \varepsilon_r), \tag{3.1}$$

where a is the scale factor; $\dot{a}$ its rate of change; ε_m the average density of mass energy; and ε_r the density of energy due to photons. The density of photons is negligible now since the era of radiation dominance ended very long ago. The density ε_m of matter varies with the scale factor according to $\varepsilon_m = \varepsilon_{m0} a^{-3}$, where ε_{m0} is the current density of matter (and the current scale factor a_0 is defined to be 1). The vacuum density ε_Λ is treated as if it were constant. The left side of equation (3.1) is defined to be H^2, where H is the Hubble parameter (Bergstrom & Goobar 2006).

By virtue of the cosmic coincidence, the densities ε_Λ and ε_{m0} are now of the same order. It follows immediately from the cosmic coincidence and equation (3.1) that the current Hubble parameter H_0 is near $\Lambda^{1/2}$. The age of the Universe T associated with some scale factor a can be obtained from equation (3.1) by the integration

$$T = \int_0^a \frac{d\alpha}{\dot{\alpha}}, \tag{3.2}$$

where α is the variable of integration representing the scale factor. During matter dominance, and still roughly at this time, the age of the Universe T and the inverse of the Hubble parameter, H^{-1}, are proportional to $a^{3/2}$, and thus are

mutually scaled. Since $H_0 \sim \Lambda^{1/2}$, the current age of the Universe T_0 is scaled to the cosmological constant

$$T_0 \sim \Lambda^{-1/2}. \tag{3.3}$$

The particle horizon R_p associated with some scale factor a may be obtained from equation (3.1) by the integration

$$R_p = a \int_0^a c \frac{d\alpha}{\dot{\alpha}\alpha}. \tag{3.4}$$

During matter dominance the particle horizon is proportional to $a^{3/2}$, and thus scaled to the age of the Universe T. Therefore, due to the cosmic coincidence and equation (3.4), the current particle horizon is scaled to $c\Lambda^{-1/2}$. The event horizon R_e associated with some scale factor a is given by the integration

$$R_e = a \int_a^\infty c \frac{d\alpha}{\dot{\alpha}\alpha}. \tag{3.5}$$

In times just before vacuum dominance and at all times thereafter, the event horizon is of the order of the de Sitter horizon $R_\Lambda \equiv c(3/\Lambda)^{1/2}$, which is the maximum event horizon that may exist in the Universe with a positive cosmological constant. Thus, due to the cosmic coincidence, the current event horizon R_{e0} is of the order of the particle horizon R_{p0}, and both are of the order of the de Sitter horizon

$$R_{p0} \sim R_{e0} \sim c\Lambda^{-1/2}. \tag{3.6}$$

The event horizon is asymptotically approaching the de Sitter horizon. The particle horizon, however, will increase without limit, becoming proportional to the scale factor.

The observable mass M_p is defined as the average density of matter in the Universe multiplied by the volume of the sphere whose radius is the particle horizon. During matter dominance the volume of the particle sphere is proportional to $a^{9/2}$. Since the density of matter is proportional to a^{-3} the observable mass M_p is proportional to $a^{9/2}a^{-3} = a^{3/2}$, and thus proportional to the particle horizon R_p, the age T and the inverse of the Hubble parameter, H^{-1}. Specifically, during the era of matter dominance and still roughly at this time, the following scaling law is satisfied:

$$M_p \sim \frac{c^2}{G} R_p. \tag{3.7}$$

Owing to the cosmic coincidence and equation (3.6), the current observable mass M_{p0} is scaled to the current mass M_{e0} of the sphere whose radius is the event horizon and both are scaled to the cosmological constant according to

$$M_{p0} \sim M_{e0} \sim \frac{c^3}{G\Lambda^{1/2}}. \tag{3.8}$$

Equations (3.3)–(3.8) constitute the underlying physics responsible for generating the coincidence problem associated with the pure numbers in equations (2.1)–(2.8). The number n_2 is scaled to n_1 due to equation (3.8) and n_3 is scaled to n_1 due to equation (3.6). (As a result of the cosmic coincidence, the maximum number of degrees of freedom in the Universe is currently of the order

of the maximum number that the Universe could ever contain, which is roughly $(R_\Lambda/l_{\mathrm{P}})^2 \sim c^5/(G\hbar\Lambda)$, which is proportional to n_1.) The number n_4 is scaled to n_1 due to equations (3.3) and (3.8) and n_5 reduces to n_6 due to equations (3.6) and (3.8), which were not formally considered to be part of the new large-number coincidence; the terms in equations (2.7) and (2.8) are scaled to n_1 due to equations (3.6) and (3.8). All that remains of the coincidence problem among pure numbers of the order of 10^{122} is the similarity between n_1 and n_6, which no longer constitutes a compelling problem *per se*. The new large-number coincidence is therefore the result of the physics of the standard cosmological model and the cosmic coincidence.

4. Two large-number coincidences and the cosmological constant

The new coincidence problem among pure numbers of the order of 10^{122} is resolved. However, its existence is still problematic given that there is another large-number coincidence that is generated from the fundamental parameters of the Universe. The other large-number coincidence originates with Weyl, Dirac and Eddington, and it concerns pure numbers of the order of 10^{40} (Dirac 1974). The existence of that older coincidence problem motivated the formulation of the so-called large numbers hypothesis, which suggests that the large, pure numbers that are formed by the parameters of nature must be related physically (Dirac 1974). The prevalence of two large, pure numbers among the parameters of nature poses a special problem that needs explanation. Whether or not the two large-number coincidences occur only in this epoch is not important since their coexistence would be a problem even if they occurred together in every epoch.

For the purposes of this present investigation the known, physically significant pure numbers of the order of 10^{40} are presented below and listed in Table 3. The gallery of such terms presented here differs slightly from the collection usually presented in this context (Funkhouser 2006). Some of the pure numbers of the order of 10^{40} that have been addressed historically involved arbitrary powers, and some terms bore no direct physical significance; such terms have been excluded from the ensemble presented here. Two new pure numbers of the order of 10^{40} are presented here as well.

The known, physically significant pure numbers of the order of 10^{40} are as follows. The maximum number of degrees of freedom allowed within a sphere whose radius is the Compton wavelength of the nucleon is defined to be j_1 and is

$$j_1 \equiv \pi \frac{l_{\mathrm{n}}^2}{l_{\mathrm{P}}^2} \approx 2.1 \times 10^{40}. \tag{4.1}$$

The term j_1 in equation (4.1) is also roughly the ratio of the rest energy of the nucleon to its characteristic gravitational energy, $Gm_{\mathrm{n}}^2/l_{\mathrm{n}}$. The gravitational potential energy of a nucleon due to the mass distribution of the cosmos is roughly $GM_{\mathrm{p}0}m_{\mathrm{n}}/R_{\mathrm{p}0}$. The characteristic gravitational binding energy of the nucleon is roughly $Gm_{\mathrm{n}}^2/l_{\mathrm{n}}$. The ratio of the two is defined to be j_2 and is

$$j_2 \equiv \frac{M_{\mathrm{p}0}/R_{\mathrm{p}0}}{m_{\mathrm{n}}/l_{\mathrm{n}}} \approx 1.7 \times 10^{39}. \tag{4.2}$$

Table 3. Pure numbers of the order of 10^{40}.

name	definition	approximate value
j_1	$\pi l_{\mathrm{n}}^2 / l_{\mathrm{P}}^2$	2.1×10^{40}
j_2	$M_{\mathrm{p0}} l_{\mathrm{n}} / (m_{\mathrm{n}} R_{\mathrm{p0}})$	1.7×10^{39}
j_3	$k_{\mathrm{e}} e^2 / (G m_{\mathrm{n}} m_{\mathrm{e}})$	2.3×10^{39}
j_4	$2 m_{\mathrm{n}} c^2 T_0 / (\pi \hbar)$	3.9×10^{41}
j_5	$R_{\mathrm{p0}} / l_{\mathrm{n}}$	3.3×10^{41}
j_6	$m_{\mathrm{n}} / m_{\mathrm{W}}$	1.1×10^{41}

The ratio of the electrostatic force between the electron and proton to the gravitational force between them is

$$j_3 \equiv \frac{k_{\mathrm{e}} e^2}{G m_{\mathrm{n}} m_{\mathrm{e}}} \approx 2.3 \times 10^{39}, \tag{4.3}$$

where m_{n} represents the proton mass; m_{e} the electron mass; k_{e} the Coulomb constant; and e the fundamental charge unit. The maximum number of logical operations that could have been performed by a single nucleon over the age of the Universe is, according to the Margolus–Levitin theorem,

$$j_4 \equiv \frac{2 m_{\mathrm{n}} c^2 T_0}{\pi \hbar} \approx 3.9 \times 10^{41}. \tag{4.4}$$

The term j_4 is proportional to the ratio of the age of the Universe to the time required for light to traverse the Compton wavelength of the nucleon. The ratio of the current particle horizon to the Compton wavelength of the nucleon is

$$j_5 \equiv \frac{R_{\mathrm{p0}}}{l_{\mathrm{n}}} \approx 3.3 \times 10^{41}. \tag{4.5}$$

Finally, the ratio of the nucleon mass to the smallest possible mass, being the Wesson mass, is

$$j_6 \equiv \frac{m_{\mathrm{n}}}{m_{\mathrm{W}}} \approx 1.1 \times 10^{41}. \tag{4.6}$$

The terms j_1–j_6 constitute conservatively the coincidence problem among pure numbers of the order of 10^{40}. The number j_5 is scaled to j_6 due to equation (3.6). The number j_4 is scaled to j_6 due to equation (3.3). The number j_3 is similar to j_1 since $m_{\mathrm{e}}/m_{\mathrm{n}}$ is not very different from the fine structure constant. The number j_2 reduces to j_6 due to equations (3.6) and (3.8). All that remains is the similarity between j_1 and j_6, which does not constitute a compelling coincidence problem. The Dirac–Eddington large-number coincidence is thus resolved. Note that this resolution of the problem alleviates the need to introduce time variation of parameters such as G (Funkhouser 2006).

Even though both the coincidence problem among pure numbers of the order of 10^{40} and 10^{122} have been resolved, it is yet problematic that two such coincidence problems are generated from the same set of parameters. There are also several remarkable algebraic similarities between the two sets of large, pure

numbers that compound that coincidence. First, note that j_4, j_5 and j_6 are all scaled to $n_6^{1/3}$ due to equations (3.3), (3.6) and (3.8). That does not mean that the two coincidence problems are identical, since n_6 does not scale to n_1, n_2, n_3 or n_4, which are all mutually scaled. However, the unresolved similarity between j_1 and j_6 among the pure numbers of the order of 10^{40} is algebraically equivalent to the unresolved similarity between n_1 and n_6 among the pure numbers of the order of 10^{122}. (The numbers j_1, j_2 and j_3 do not scale to any of the pure numbers of the order of 10^{122} through the standard cosmological model.) Furthermore, the two coincidence problems are resolved through the same basic physics from the standard cosmological model, being equations (3.1)–(3.8). Most suggestive is the fact that all of the known pure numbers of the order of 10^{40} would be scaled to all of the known pure numbers of the order of 10^{122} if j_1 were scaled to j_6, or, equivalently, if n_1 were scaled to n_6.

In order to resolve the host of coincidences associated with the large pure numbers, it is well motivated to propose that the pure numbers of the order of 10^{122} are proportional to the third power of the pure numbers of the order of 10^{40}. Conveniently, that is equivalent to hypothesizing that $j_1 \sim j_6$, or, rather, $n_1 \sim n_6$. Such a relationship would cause the two large-number coincidences to be identical and would amount to the scaling law

$$m_{\mathrm{n}} \sim \left(\frac{h^4 \Lambda}{G^2 c^2} \right)^{1/6}. \qquad (4.7)$$

The term on the r.h.s. of equation (4.6) is roughly 4.2×10^{-28} kg, which is within an order of magnitude of the nucleon mass scale. The relation in equation (4.6) was proposed to explain why the coincidence among pure numbers of the order of 10^{40} occurs simultaneously with the cosmic coincidence (Funkhouser 2006). Equation (4.7) is essentially the same scaling law first proposed by Zel'dovich (Okun 2002) based on the considerations of field theory. Mena Marugán & Carneiro (2002) have proposed the same relationship based on holographic considerations. The scaling law in equation (4.6) also follows naturally from applying the Bekenstein–Hawking bound to a quantum-cosmological model of a three-dimensional Universe inflating from the collapse of seven extra dimensions (Funkhouser in press).

Lloyd has already shown that the maximum number N_Λ of allowed bits in our cosmos is related to certain products of the pure numbers of the order of 10^{40} regardless of any underlying relationship (Lloyd 2002). If the scaling law in equation (4.6) should be physical then N_Λ would be scaled to the product of any three of the known pure numbers of the order of 10^{40}. That is because all of the known pure numbers of the order of 10^{40} would be mutually scaled if equation (4.6) should be physically meaningful.

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