We consider a financial contract that delivers a single cash flow given by the terminal value of a cumulative gains process. The problem of modelling such an asset and associated derivatives is important, for example, in the determination of optimal insurance claims reserve policies, and in the pricing of reinsurance contracts. In the insurance setting, aggregate claims play the role of cumulative gains, and the terminal cash flow represents the totality of the claims payable for the given accounting period. A similar example arises when we consider the accumulation of losses in a credit portfolio, and value a contract that pays an amount equal to the totality of the losses over a given time interval. An expression for the value process of such an asset is derived as follows. We fix a probability space, together with a pricing measure, and model the terminal cash flow by a random variable; next, we model the cumulative gains process by the product of the terminal cash flow and an independent gamma bridge; finally, we take the filtration to be that generated by the cumulative gains process. An explicit expression for the value process is obtained by taking the discounted expectation of the future cash flow, conditional on the relevant market information. The price of an Arrow–Debreu security on the cumulative gains process is determined, and is used to obtain a closed-form expression for the price of a European-style option on the value of the asset at the given intermediate time. The results obtained make use of remarkable properties of the gamma bridge process, and are applicable to a wide variety of financial products based on cumulative gains processes such as aggregate claims, credit portfolio losses, defined benefit pension schemes, emissions and rainfall.

Keywords: asset pricing; insurance claim reserves; credit portfolio risk; gamma bridge process; beta distribution; reinsurance

1. Introduction

There are a number of problems in finance and insurance that involve the analysis of processes representing cumulative gains or losses. The typical setup is as follows. We fix an accounting period \([0, T]\), where time 0 is the present. At time \(T\) a contract pays a random cash flow \(X_T\), which is assumed to be positive and given by the terminal value of a process of accumulation. In the case of an insurance contract, for example, we consider the situation where claims are made

* Author for correspondence (dorje@imperial.ac.uk).
over the accounting period, and are then paid at $T$. The variable $X_T$ represents the totality of the payments made at $T$ in settlement of claims arising over $[0, T]$. Let us write $\{S_t\}$ for the value process of the contract that pays $X_T$ at $T$, $\{\mathcal{F}_t\}$ for the filtration representing the flow of information available to market participants, and $\mathcal{Q}$ for the pricing measure. Then the value at $t$ of the contract that pays $X_T$ at $T$ is $S_t = P_{tT} \mathbb{E}[X_T | \mathcal{F}_t]$, where $\mathbb{E}[\cdot]$ denotes expectation with respect to $\mathcal{Q}$, and $P_{tT}$ denotes the discount factor, which for simplicity we take to be deterministic. One can interpret $S_t$ as the reserve the insurance firm requires at $t$ to ensure that $X_T$ will be payable at $T$. Alternatively, one can view $S_t$ as the amount that would have to be paid at $t$ for the insurance firm to relieve itself of the obligation to pay $X_T$, that is, to commute the relevant claims. Similarly, the cost $C_{tT}$ at $t$ of a stop-loss reinsurance contract that pays $(X_T - K)^+$ at $T$ for some fixed threshold $K$ is given by $C_{tT} = P_{tT} \mathbb{E}[(X_T - K)^+ | \mathcal{F}_t]$. We assume that $\{\mathcal{F}_t\}$ is generated by an aggregate claims process $\{\xi_t\}$, where for each $t$ the variable $\xi_t$ represents the totality of claims known at $t$ to be payable at $T$. The problem can then be stated as follows: given the history of claims over the interval $[0, t]$, what is the appropriate reserve to allocate for settlement of these and any future claims arising in the accounting period? To obtain a solution we need to specify $\{\xi_t\}$, then work out the reserve process $\{S_t\}$. Once we have the reserve process, we can value reinsurance contracts.

Another example of an accumulation process comes from credit risk management. We consider a large credit portfolio and let $X_T$ denote the accumulated losses at $T$. For instance, at time 0 a credit card firm has a large number of customers, each with an outstanding balance payable in the accounting period. If a customer does not pay the balance by the required date, they will be deemed to be in default, and a loss will be registered. The random variable $X_T$ will denote the totality of such losses. We assume that once a customer is in default, no further payments are made by that customer (this assumption can be relaxed in a more sophisticated model). The problem facing the firm is to determine what reserve policy to maintain, and what premium to charge over the base interest rate, to ensure that funds will be in hand to cover the default losses.

The purpose of this paper is to present a modelling framework for accumulation processes, and to establish explicit formulae for the associated valuation processes. We shall assume that $\{\xi_t\}$ takes the form

$$\xi_t = X_T \gamma_{tT},$$

(1.1)

where $\{\gamma_{tT}\}$ is a gamma bridge over $[0, T]$, independent of $X_T$. The idea that the gamma process might be used for describing the losses associated with insurance claims dates to the work of Hammersley (1955), Moran (1956), Gani (1957), Kendall (1957), Kingman (1963) and others (see, for example, Bingham 1975; Basawa & Brockwell 1978), in connection with the theory of storage and dams. Moran (1956), in particular, observed that the amount of rainfall accumulating in a dam can be modelled by a gamma process, and Gani (1957) pointed out the relevance to insurance, that if the portfolio of events insured is large, one can think of the arrival of claims as analogous to the accumulation of dam rain. The gamma process has since then been investigated by Dufresne et al. (1991), Dickson & Waters (1993), Norberg (1999) and others as a model for aggregate
claims. Let us therefore consider in more detail what results if we model the aggregate claims as a $\mathbb{Q}$-gamma process. In other words, suppose we set $x_t = k \gamma_t$, where $k$ is a constant and $\{\gamma_t\}$ is a standard gamma process under $\mathbb{Q}$, with mean and variance $m\tau$ (see §2 for definitions). It follows that $x_t = X_T \gamma_t$, where $X_T = k \gamma_T$ and the process $\{\gamma_t\}$, defined by $\gamma_t = \gamma_t / \gamma_T$, is a gamma bridge. Moreover, by virtue of the special properties of the gamma process, we find that $X_T$ is independent of $\{\gamma_t\}$. In the $\mathbb{Q}$-gamma model the aggregate claims process is thus the product of a gamma-distributed terminal cash flow and an independent gamma bridge. One can think of the gamma bridge as representing that part of the aggregate claims process that has no bearing on the terminal result. We are thus led to a multiplicative decomposition of the claims process into the product of a ‘signal’ $X_T$ and an independent ‘noise’ $\{\gamma_t\}$.

For such processes we are able to apply the techniques of information-based asset pricing developed in Macrina (2006), Brody et al. (2007, 2008), Rutkowski & Yu (2007) and Hughston & Macrina (2008). Through this line of enquiry one is led to consider the situation where the terminal cash flow has a generic a priori distribution and the claims process takes the form (1.1). The additive decomposition of the information process in the case of the Brownian bridge noise considered in the references cited above is natural from the viewpoint of nonlinear filtering theory. The product representation of the gamma information process is also natural, since many properties of the Brownian bridge that hold additively have multiplicative analogues for gamma bridges (Yor & Émery 2004; Yor 2007). The resulting model for aggregate claims is remarkably tractable, and we are able to derive explicit formulae both for the claims reserve process and for the valuation of reinsurance contracts. These insurance products have analogues in various other financial markets, for which the same modelling techniques are applicable.

The paper is organized as follows. In §§2 and 3 we review the properties of the gamma process and the gamma bridge. At the same time, we establish our notation and some techniques that will be applied later. In §4 we derive an expression for the value process of a contract that delivers the cash flow $X_T$ at time $T$, when the market filtration is generated by the accumulation process (1.1). We show in proposition 4.1 that $\{\xi_t\}$ has the Markov property, then use the Bayes theorem to determine the conditional density of $X_T$, and finally the value process, which is given in proposition 4.2. By use of the conditional density we are also able to obtain an expression for the value process of a simple stop-loss reinsurance contract. In §5 we consider the valuation of general reinsurance contracts. We derive a formula for the value at time 0 of a contract that at some fixed time $t$ gives the contract holder the option to commute the claim $X_T$ by paying a fixed amount $K$ at $t$. Such a contract takes the form of a European call option on the value of the reserve at $t$. An Arrow–Debreu method is introduced to perform the calculations. The resulting formula for the option value is expressed in terms of the cumulative beta distribution. In §6 we examine the case where $X_T$ takes discrete values. When $X_T$ is a binary random variable, the problem of option pricing can be solved completely. In §7 the material of §5 is extended to determine the price process of an option on the value of an aggregate claim. This result allows one to value various types of reinsurance instruments. In §8 we conclude by returning to the case where $X_T$ has a $\mathbb{Q}$-gamma distribution.
The key results of the paper include: (i) the valuation of an aggregate claim, given by (4.4), (ii) the valuation of a stop-loss reinsurance policy, given by (4.11), (iii) the price of an Arrow–Debreu security based on the level of the cumulative gains at some intermediate time, given by (5.7), (iv) the price of a general reinsurance policy, given by (5.14) and (7.8), and (v) the price of the Arrow–Debreu security in the case of a gamma-distributed cumulative gain, given by (8.4).

2. Gamma processes and associated martingales

We fix a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\). In our applications \(\mathbb{Q}\) will always denote the pricing (risk-neutral) measure, but the material in this section, and the next, does not depend on this interpretation. Equalities and inequalities among random variables are to be understood as holding except possibly on sets of measure zero. By a standard gamma process \(\{\gamma_t\}_{0 \leq t < \infty}\) on \((\Omega, \mathcal{F}, \mathbb{Q})\) with growth rate \(m\), we mean a process with independent increments such that \(\gamma_0 = 0\) and such that the random variable \(\gamma_t\) has a gamma distribution with mean and variance \(mt\). More precisely, writing \(\mathbb{Q}[\gamma_t \in dx] = g(x)dx\), we have

\[
g(x) = \mathbb{Q}_{\{x>0\}} \frac{x^{mt-1}e^{-x}}{\Gamma(mt)}
\]  

(2.1)

for the density of \(\gamma_t\). Here, \(\Gamma[a]\) is the gamma function, which for \(a > 0\) has the Eulerian representation \(\Gamma[a] = \int_0^\infty x^{a-1}e^{-x} \, dx\). The identity \(\Gamma[a+1] = a\Gamma[a]\) implies that \(\mathbb{E}[\gamma_t] = mt\), justifying the interpretation of \(m\) as the mean growth rate.

A straightforward calculation shows that the characteristic function for the gamma process is given by

\[
\mathbb{E}[e^{i\lambda \gamma_t}] = \frac{1}{(1-i\lambda)^{mt}},
\]  

(2.2)

valid for \(t \geq 0\) and for \(\lambda \in \mathbb{C}\) such that \(\text{Im}(\lambda) > -1\), from which the higher moments of \(\gamma_t\) can be deduced. Writing \((a)_0 = 1\) and \((a)_k = a(a+1)(a+2)\cdots(a+k-1)\) for the Pochhammer symbol, we find that \(\mathbb{E}[\gamma_t^n] = (mt)_n\). We note that \(\mathbb{E}[\gamma_t^2] = mt + m^2t^2\), and hence \(\text{var}[\gamma_t] = mt\). It follows from the independent increments property that \(\text{cov}[\gamma_t, \gamma_u] = mt\) for \(u \geq t\).

An alternative expression for the characteristic function is given by the Lévy–Khintchine representation \(\mathbb{E}[e^{i\lambda \gamma_t}] = e^{-t\psi(\lambda)}\) for \(\text{Im}(\lambda) > -1\), where

\[
\psi(\lambda) = m \ln (1 - i\lambda) = \int_0^\infty mx^{-1}e^{-x}(1-e^{ix}) \, dx,
\]  

(2.3)

which shows that the Lévy density associated with the gamma process is given by \(mx^{-1}e^{-x}\) for \(x > 0\) (see, for example, Protter 2005).

By use of the independent increments property we deduce that for \(u \geq t \geq 0\) and for \(a,b \in \mathbb{C}\) with \(\text{Im}(a+b) > -1\) and \(\text{Im}(b) > -1\) we have

\[
\mathbb{E}[\exp(i a \gamma_t + ib \gamma_u)] = \mathbb{E}[\exp(i(a+b) \gamma_t + ib(\gamma_u - \gamma_t))] \\
= \mathbb{E}[\exp(i(a+b) \gamma_t)] \mathbb{E}[\exp(ib(\gamma_u - \gamma_t))] \\
= \frac{1}{[1-i(a+b)]^{mt}} \frac{1}{(1-i b)^{m(u-t)}}.
\]  

(2.4)

In particular, if we set \( a = b = \lambda \) we see that \( \gamma_u - \gamma_t \) is gamma distributed with parameter \( m(u-t) \). It follows that the increments of \( \{\gamma_t\} \) have a stationary probability law in the sense that \( \gamma_{u+h} - \gamma_{t+h} \) has the same distribution as \( \gamma_u - \gamma_t \).

It is straightforward to deduce that \( \{\gamma_t - mt\} \) and \( \{(\gamma_t - mt)^2 - mt\} \) are martingales. More generally, for \( \alpha > -1 \) the process \( \{L_t\} \) defined by

\[
L_t = (1 + \alpha)^{mt} e^{-\alpha\gamma_t}
\]

is a martingale, which can be verified by use of (2.4). We refer to this process as the exponential gamma martingale. It follows, by consideration of the corresponding power series in \( \alpha \), that for each term we obtain a martingale involving a polynomial expression in the gamma process. Suppose for \( n \in \mathbb{N} \) and \( k \in \mathbb{R} \) we define the so-called associated Laguerre polynomials \( \{L_n^k(z)\} \) by setting

\[
L_n^k(z) = \frac{z^{-k} e^z}{n!} \frac{d^n}{dz^n}(z^{n+k} e^{-z}).
\]

The standard Laguerre polynomials \( L_n(z) = L_n^0(z) \) have the property that if \( Z \) is a standard exponentially distributed random variable then \( \mathbb{E}[L_n(Z)L_n'(Z)] = 0 \) for \( n \neq n' \) (cf. Wiener 1949). More generally, if \( Z \) has a gamma distribution with parameter \( k+1 \), i.e. such that \( \mathbb{Q}[Z < z] = \int_0^z x^k e^{-x} dx / \Gamma[k+1] \) for \( k > -1 \), then \( \mathbb{E}[L_n^k(Z)L_n^k(Z)] = 0 \) for \( n \neq n' \). The significance of the associated Laguerre polynomials arises from the identity

\[
(1 + \alpha)^k e^{-z} \mathbb{E}[\alpha^n] = \sum_{n=0}^{\infty} L_n^{h-n}(z) \alpha^n,
\]

for \( |\alpha| < 1 \) and \( h \geq 0 \) (Erdélyi 1953), which gives us the required series expansion. By setting \( h = mt \) and \( z = \gamma_t \) in equation (2.7), we deduce that for each \( n \) the process \( \{L_n^{mt-n}(\gamma_t)\} \), \( t \geq 0 \), is a martingale (cf. Schoutens 2000). For example, we have \( L_n^{mt-1}(\gamma_t) = -(\gamma_t - mt) \) and \( L_2^{mt-2}(\gamma_t) = [(\gamma_t - mt)^2 - mt]/2 \).

So far we have considered the case of the ‘standard’ gamma process, for which \( \mathbb{E}[\gamma_t] = mt \) and \( \text{var} [\gamma_t] = mt \), for some \( m \). We note that \( (\mathbb{E}[\gamma_t])^2 / \text{var} [\gamma_t] \) is dimensionless, and hence that \( m \) has the units of inverse time. For fixed \( m \), we can choose the units of time so that \( m = 1 \) in those units (this is done implicitly, e.g., in Yor 2007). We shall take the units of time as fixed, and \( m \) as a model parameter.

For many applications it is useful to consider a broader family of processes, labelled by two parameters, which we shall call ‘scaled’ gamma processes. By a scaled gamma process with growth rate \( \mu \) and spread \( \sigma \) we mean a process \( \{\Gamma_t\}_{0 \leq t < \infty} \) with independent increments such that \( \Gamma_0 = 0 \) and such that \( \Gamma_t \) has a gamma distribution with mean \( \mu t \) and variance \( \sigma^2 t \), where \( \mu \) and \( \sigma \) are parameters. Defining \( m = \mu^2 / \sigma^2 \) and \( \kappa = \sigma^2 / \mu \), we have \( \mu = km \) and \( \sigma^2 = \kappa^2 m \). One can think of \( m \) as a ‘standardized’ growth rate, and \( \kappa \) as a scale. The density of \( \Gamma_t \) is given by

\[
\mathbb{Q}[\Gamma_t \in dx] = \mathbb{1}_{\{x > 0\}} \frac{\kappa^{-mt} x^{mt-1} e^{-x/\kappa}}{\Gamma[mt]} dx.
\]

One can check that if \( \{\Gamma_t\} \) is a scaled gamma process with standardized growth rate \( m \) and scale \( \kappa \), then \( \{\kappa^{-1} \Gamma_t\} \) is a gamma process with growth rate \( m \).
Now suppose that \{\gamma_t\} is a standard gamma process on \((\mathcal{Q}, \mathcal{F}, \mathcal{Q})\), let \{\mathcal{G}_t\} denote the filtration generated by \{\gamma_t\}, and let \(\mathcal{Q}^*\) denote the measure on \((\mathcal{Q}, \mathcal{G}_T)\), for some fixed \(T\), defined by the likelihood ratio

\[
\frac{\mathrm{d}\mathcal{Q}^*}{\mathrm{d}\mathcal{Q}} \bigg|_T = \kappa^{-mT} \exp \left( \frac{1-k}{k} \gamma_T \right)
\]  
(2.9)

for some \(k>0\). Then \(\gamma_{t|0\leq t\leq T}\) is a scaled gamma process on \((\mathcal{Q}, \mathcal{G}_T, \mathcal{Q}^*)\), with scale parameter \(k\). Thus, \(\mathbb{E}[\gamma_t] = mt\), \(\text{var}[\gamma_t] = mt\), \(\mathbb{E}^{*}[\gamma_t] = kmt\) and \(\text{var}^{*}[\gamma_t] = k^2mt\). This can be established by working out the joint characteristic function under \(\mathcal{Q}^*\) of the increments \(\gamma_{t}-\gamma_{s}, \gamma_{s}-\gamma_{s_1}, \ldots, \gamma_{s_{n-1}} - \gamma_{s_n}\) for \(T\geq t\geq s\geq s_1\geq \cdots \geq s_n\) for each \(n\in\mathbb{N}\). The change-of-measure density martingale arising in this example is obtained by taking the process (2.5) and setting \(\alpha=(1-k)/k\).

The gamma process has been used as the basis of a number of different asset pricing models; see, for example, Madan & Seneta (1990), Madan & Milne (1991), Heston (1995), Carr et al. (2002) and Baxter (2007a,b).

3. Properties of gamma bridge processes

Let \(\gamma_{t|0\leq t<\infty}\) be a standard gamma process with growth rate \(m\), and for fixed \(T\) define the process \(\gamma_{t|0\leq t\leq T}\) by setting \(\gamma_{t|0\leq t\leq T} = \gamma_t/\gamma_T\). Then clearly \(\gamma_{0|0\leq t\leq T} = 0\) and \(\gamma_{T|0\leq t\leq T} = 1\). We refer to \(\gamma_{t|0\leq t\leq T}\) as the standard gamma bridge over \([0,T]\) associated with \(\gamma_t\). More generally, we refer to any process having the law of \(\gamma_{t|0\leq t\leq T}\) as a standard gamma bridge over \([0,T]\). It can be shown that the random variable \(\gamma_{t|0\leq t\leq T}\) has a beta distribution.

**Proposition 3.1.** Let \(\mathbb{Q}[\gamma_{t|0\leq t\leq T} \in dy] = f(y)\,dy\). Then

\[
f(y) = \mathbf{1}_{\{0 < y < 1\}} \frac{ym^{-1}(1-y)^{m(T-t)-1}}{B[mt, m(T-t)]},
\]  
(3.1)

where \(B[a, b] = \Gamma[a] \Gamma[b] / \Gamma[a+b]\).

**Proof.** Since \(\gamma_t\) and \(\gamma_T - \gamma_t\) are independent, we see that

\[
\mathbb{Q} \left[ \frac{\gamma_t}{\gamma_T} \leq y \right] = \mathbb{Q} \left[ \gamma_t \leq \frac{y}{1-y} (\gamma_T - \gamma_t) \right]
\]

\[
= \mathbb{E} \left[ \mathbb{Q} \left[ \gamma_t \leq \frac{y}{1-y} (\gamma_T - \gamma_t) \left| \gamma_T - \gamma_t \right. \right] \right]
\]

\[
= \frac{1}{\Gamma[mt]} \mathbb{E} \left[ \int_0^y (\gamma_T - \gamma_t)^{(1-y)} x^{mt-1} e^{-x} \, dx \right].
\]  
(3.2)

The corresponding density is therefore given by

\[
f(y) = \mathbf{1}_{\{0 < y < 1\}} \frac{ym^{-1}(1-y)^{-mt-1}}{\Gamma[mt]} \mathbb{E} \left[ (\gamma_T - \gamma_t)^{mt} \exp \left( -\frac{y}{1-y} (\gamma_T - \gamma_t) \right) \right].
\]  
(3.3)

Now, since $\gamma_T - \gamma_t$ has a gamma distribution with parameter $m(T-t)$, for the expectation appearing in the line above we obtain
\[
\mathbb{E}\left[ (\gamma_T - \gamma_t)^m \exp\left( -\frac{y}{1-y} (\gamma_T - \gamma_t) \right) \right] = \frac{I[mT]}{I[m(T-t)]} (1-y)^mT. \tag{3.4}
\]

Putting this result back into (3.3), we obtain (3.1).

Recalling that $B[a, b] = \int_0^1 y^{a-1}(1-y)^{b-1} \, dy$, we deduce for $n \in \mathbb{N}$ that $\mathbb{E}[\gamma_i^n] = B[mt + n, m(T-t)]/B[mt, m(T-t)] = (mt)_n/(mT)_n$. We see that $\mathbb{E}[\gamma_iT] = t/T$ and $\mathbb{E}[\gamma^2_iT] = t(mt+1)/T(mT+1)$, and hence $\text{var}[\gamma_iT] = t(T-t)/T^2(1+mT)$. Note that the expectation of $\gamma_iT$ does not depend on the growth rate $m$, and the variance of $\gamma_iT$ decreases with increasing $m$.

**Proposition 3.2.** Let $\{\gamma_t\}_{0 \leq t < \infty}$ be a standard gamma process. Then for $T \geq t \geq 0$ the random variables $\gamma_t/\gamma_T$ and $\gamma_T$ are independent.

**Proof.** Writing $F(y, z)$ for the joint distribution of $\gamma_t/\gamma_T$ and $\gamma_T$, we have
\[
F(y, z) = \mathbb{Q}\left[ \frac{\gamma_t}{\gamma_T} \leq y \cap \gamma_T \leq z \right]
= \mathbb{Q}\left[ \gamma_t \leq \frac{y}{1-y} (\gamma_T - \gamma_t) \cap \gamma_T \leq z - (\gamma_T - \gamma_t) \right]. \tag{3.5}
\]
Conditioning with respect to $\gamma_T - \gamma_t$, we use the independence of $\gamma_t$ and $\gamma_T - \gamma_t$ to deduce that
\[
F(y, z) = \frac{1}{I[mt]} \mathbb{E}\left[ \int_0^\infty \mathbb{I}_{\{x \leq y(\gamma_T - \gamma_t)/(1-y)\}} \mathbb{I}_{\{x \leq z - (\gamma_T - \gamma_t)\}} x^{mt-1} e^{-x} \, dx \right], \tag{3.6}
\]
and hence the following expression for the joint density:
\[
f(y, z) = \frac{1}{I[mt]} \mathbb{E}\left[ \int_0^\infty \frac{\gamma_T - \gamma_t}{(1-y)^2} \delta\left( x - \frac{y}{1-y} (\gamma_T - \gamma_t) \right) \times \delta(x - [z - (\gamma_T - \gamma_t)]) x^{mt-1} e^{-x} \, dx \right]. \tag{3.7}
\]
Here we have used the relation $\partial_x \mathbb{I}_{\{x \leq a\}} = -\delta(x-a)$, where $\delta(z)$ denotes the Dirac distribution. Integrating out the first delta function we have, for $0<y<1$ and $z>0$,
\[
f(y, z) = \frac{1}{I[mt]} \mathbb{E}\left[ \frac{\gamma_T - \gamma_t}{(1-y)^2} \delta\left( \frac{y}{1-y} (\gamma_T - \gamma_t) - [z - (\gamma_T - \gamma_t)] \right) \times \left( \frac{y}{1-y} (\gamma_T - \gamma_t) \right)^{mt-1} \exp\left( -\frac{y}{1-y} (\gamma_T - \gamma_t) \right) \right]
= \frac{y^{mt-1}(1-y)^{-mt-1}}{I[mt]} \mathbb{E}\left[ (\gamma_T - \gamma_t)^{mt} \exp\left( -\frac{y}{1-y} (\gamma_T - \gamma_t) \right) \right] \times \delta\left( \frac{\gamma_T - \gamma_t}{1-y} - z \right). \tag{3.8}
\]
Now we introduce the Fourier representation \( \delta(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\lambda x} \, d\lambda \) for the delta function, interpreted as a distribution, from which we deduce that
\[
f(y, z) = \frac{y^{mt-1}(1-y)^{-mt-1}}{\Gamma[mt]} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda z} \mathbb{E} \left[ (\gamma_T - \gamma_t)^m \exp \left( -\frac{y}{1-y} (\gamma_T - \gamma_t) \right) \right] \, d\lambda.
\]
(3.9)

When one works out the expectation appearing in the integrand above the result is
\[
\frac{\Gamma[mT]}{\Gamma[m(T-t)]} \frac{(1-y)^{mT}}{(1-\lambda)^{mT}}.
\]
Substituting this expression into (3.9) we see that
\[
f(y, z) = \frac{\Gamma[mT]}{\Gamma[mt]\Gamma[m(T-t)]} \frac{y^{mt-1}(1-y)^{m(T-t)-1}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda z}}{z^{mT-1} e^{-z}} \, d\lambda
\]
for \( 0 < y < 1 \) and \( z > 0 \). Thus, the joint density for \( \gamma_t/\gamma_T \) and \( \gamma_T \) factorizes into the product of a beta density for \( \gamma_t/\gamma_T \) and a gamma density for \( \gamma_T \). \( \square \)

More generally, it follows from proposition 3.2 and the independent increments property of the gamma process that \( \{ \gamma_u \}_{u \geq T} \) and \( \{ \gamma_{tT} \}_{0 \leq t \leq T} \) are independent. This allows us to verify that \( \{ \gamma_t \} \) has the Markov property (which one knows is the case, since \( \{ \gamma_t \} \) is a Lévy process). We need to check for \( a > 0 \) that
\[
\mathbb{Q}[\gamma_t < a | \gamma_s, \gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n}] = \mathbb{Q}[\gamma_t < a | \gamma_s]
\]
(3.11)
for all \( t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 0 \) and for all \( n \geq 1 \). But clearly,
\[
\mathbb{Q}[\gamma_t < a | \gamma_s, \gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n}] = \mathbb{Q}[\gamma_t < a | \gamma_s, \gamma_{s_1}, \gamma_{s_2}, \gamma_{s_3}, \ldots, \gamma_{s_n}] = \mathbb{Q}[\gamma_t < a | \gamma_s],
\]
(3.12)
since \( \gamma_{s_1}/\gamma_s, \gamma_{s_2}/\gamma_{s_1}, \ldots \) are independent of \( \gamma_s \) and \( \gamma_t \). A similar argument shows that the gamma bridge is Markovian. In particular, we see that
\[
\mathbb{Q} \left[ \frac{\gamma_t}{\gamma_T} < a \left| \frac{\gamma_s}{\gamma_T}, \frac{\gamma_{s_1}}{\gamma_T}, \frac{\gamma_{s_2}}{\gamma_T}, \ldots \right. \right] = \mathbb{Q} \left[ \frac{\gamma_t}{\gamma_T} < a \left| \frac{\gamma_s}{\gamma_T}, \frac{\gamma_{s_1}}{\gamma_T}, \frac{\gamma_{s_2}}{\gamma_T}, \ldots \right. \right]
\]
(3.13)
since the random variables \( \gamma_{s_1}/\gamma_s, \gamma_{s_2}/\gamma_{s_1}, \ldots \) are independent of \( \gamma_t/\gamma_T \) and \( \gamma_s/\gamma_T \).

The following lemma is a classical result (Lukacs 1955) which can be used as the basis of an alternative proof of proposition 3.2.
Lemma 3.3. Let $A$ and $B$ be independent gamma variables with parameters $p$ and $q$, respectively. Then $A/(A+B)$ and $A+B$ are independent, $A/(A+B)$ has a beta$(p, q)$ distribution and $A+B$ has a gamma$(p+q)$ distribution.

Proof. It suffices to show that the joint Laplace transform of $A/(A+B)$ and $A+B$ factorizes. In particular, for positive constants $\alpha$ and $\beta$ we have

$$
\mathbb{E}
\left[
\exp
\left(-\alpha \frac{A}{A+B} - \beta (A+B)\right)
\right] = \int_0^\infty \int_0^\infty \frac{a^{p-1}e^{-a}}{\Gamma[p]} \frac{b^{q-1}e^{-b}}{\Gamma[q]} \exp\left(-\alpha \frac{a}{a+b} - \beta (a+b)\right) da \, db.
$$

(3.14)

Setting $x=a/(a+b)$ and $y=a+b$, we have $a=xy$ and $b=(1-x)y$, and hence $da \, db = y \, dx \, dy$. We see that

$$
\mathbb{E}
\left[
\exp\left(-\alpha \frac{A}{A+B} - \beta (A+B)\right)
\right] = \left[ \int_0^1 x^{p-1}(1-x)^{q-1} e^{-\alpha x} dx \int_0^\infty y^{p+q-1}e^{-\beta y} dy \right] = \mathbb{E}\left[ \exp\left(-\beta (A+B)\right)\right].
$$

(3.15)

Proposition 3.2 follows if we set $A=\gamma_t$ and $B=\gamma_T - \gamma_t$. More generally, if we set $A=\gamma_u - \gamma_t$ and $B=\gamma_T - \gamma_u$ for $0 \leq u \leq T$, we deduce that $(\gamma_u - \gamma_t)/(\gamma_T - \gamma_t)$ and $\gamma_T - \gamma_t$ are independent. A straightforward extension of lemma 3.3 establishes that for $n \geq 1$ and $T \geq t \geq s \geq s_1 \geq \cdots \geq s_n$ the variables $\gamma_T, \gamma_{tT}, \gamma_{st}, \gamma_{s_1s_1}, \cdots, \gamma_{s_n s_n-1}$ are independent.

4. Valuation of aggregate claims

Our objective is to calculate the value at $t$ of a contract that pays $X_T$ at $T$. We assume that $X_T$ is strictly positive and integrable. For simplicity of exposition, in this section we take $X_T$ to be a continuous random variable; the adjustments required for the more general situation are straightforward (a discrete example follows in §7). We assume that the default-free interest rate system is deterministic, that $Q$ is the risk-neutral measure, and that the market filtration is generated by an aggregate claims process $\{\xi_t\}_{0 \leq t \leq T}$ of the form $\xi_t = X_T \gamma_{tT}$, where $\{\gamma_{tT}\}$ is a standard gamma bridge under $Q$, with parameter $m$, which we take to be independent of $X_T$. The value $S_t$ of the contract at $t \leq T$ is given under these assumptions by $S_t = P_t \mathbb{E}[X_T | \mathcal{F}_t]$, where $\mathcal{F}_t = \sigma(\{\xi_s\}_{0 \leq s \leq t})$. In our modelling considerations we are to some extent looking towards the future, where we may presume a rich range of insurance-related derivative products will be trading in liquid financial markets; it is then reasonable to assume that $Q$ will be established. In contrast to the actuarial approach (Arjas 1989; Dickson & Waters 1993; Norberg 1999), we are concerned with valuation rather than prediction or estimation. The assumptions that interest rates are deterministic, that the gains process is of the specific simple form (1.1) and that the filtration is generated by a single information process, are model simplifications that can be relaxed in various ways, and are not essential.
Proposition 4.1. The aggregate claims process \( \{ \xi_t \}_{0 \leq t \leq T} \) has the Markov property.

Proof. For the Markov property we must verify that \( \mathbb{Q}[\xi_t < a|\mathcal{F}_s] = \mathbb{Q}[\xi_t < a|\xi_s] \) for all \( s, t \) such that \( 0 \leq s \leq t \leq T \). It suffices to establish that

\[
\mathbb{Q}[\xi_t < a|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_n}] = \mathbb{Q}[\xi_t < a|\xi_s]
\]

for all \( t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_n \) and for all \( n \geq 1 \). We use the representation \( \{ \gamma_{tT} \} = \{ \gamma_t/\gamma_T \} \), where \( \{ \gamma_t \} \) is a standard gamma process with rate \( m \). Then,

\[
\mathbb{Q}[\xi_t < a|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \gamma_{s_1}/\gamma_s, \gamma_{s_2}/\gamma_s, \ldots] = \mathbb{Q}[\xi_t < a|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots].
\]

But \( \gamma_{s_1}/\gamma_s, \gamma_{s_2}/\gamma_s, \ldots \) are independent of \( \xi_t \) and \( \xi_s \), which gives the result.

By virtue of the fact that \( \{ \xi_t \} \) has the Markov property and that \( X_T \) is \( \mathcal{F}_T \)-measurable, we are able to simplify the expression for \( S_t \) so that it takes the form

\[
S_t = P_{tT} \mathbb{E}[X_T|\xi_t].
\]

The conditional expectation appearing here can be carried out in closed form, leading to the following pricing formula.

Proposition 4.2. The value \( S_t \) at time \( t < T \) of the aggregate claims process \( \{ \zeta_t \}_{0 \leq t \leq T} \) at time \( T \) is given by

\[
S_t = P_{tT} \int_{\xi_t}^{\infty} \frac{p(x) x^{2-mT} (x - \xi_t)^{m(T-t)-1}}{\int_{\xi_t}^{\infty} p(x) x^{1-mT} (x - \xi_t)^{m(T-t)-1}} \, dx,
\]

where \( p(x) \) is the density of \( X_T \).

Proof. The conditional expectation (4.3) can be written in the form

\[
\mathbb{E}[X_T|\xi_t] = \int_{0}^{\infty} x \pi_t(x) \, dx,
\]

where \( \{ \pi_t(x) \} \) is the conditional density process for \( X_T \), which by virtue of the Markov property of \( \{ \xi_t \} \) is given by \( \pi_t(x) \, dx = d \mathbb{Q}[X_T \leq x|\xi_t] \). We compute \( \pi_t(x) \) by use of the following form of the Bayes formula:

\[
\pi_t(x) = \frac{\rho(\xi_t|X_T = x) x}{\int_{0}^{\infty} \rho(\xi_t|X_T = x) \, dx},
\]

where \( \rho(\xi_t|X_T = x) \) is the conditional density for \( \xi_t \), valued at \( \xi_t \), given by

\[
\rho(\xi|X_T = x) = \frac{d \mathbb{Q}[\xi \leq x|X_T = x]}{d \xi} = \frac{d \mathbb{Q}[\gamma_{tT} \leq \frac{\xi}{x}]}{d \xi}.
\]

Therefore, writing \( f(y) \) for the density of \( \gamma_{tT} \), we find

\[
\rho(\xi|X_T = x) = \frac{d \xi}{d \xi} \int_{0}^{\xi/x} f(y) \, dy = \frac{1}{x} f\left(\frac{\xi}{x}\right).
\]
Hence by proposition 3.1 we have
\[
\rho(\xi | X_T = x) = \mathbb{I}_{\{x > \xi\}} \xi^{mt-1} \frac{x^{1-mT} (x-\xi)^{m(T-t)-1}}{B(mt, m(T-t))}. \tag{4.9}
\]
The conditional density is thus given by
\[
\pi_t(x) = \mathbb{I}_{\{x > \xi_t\}} \frac{p(x)x^{1-mT} (x-\xi_t)^{m(T-t)-1}}{\int_{\xi_t}^{\infty} p(x)x^{1-mT} (x-\xi_t)^{m(T-t)-1} \, dx}, \tag{4.10}
\]
from which (4.4) follows at once.

With these results at hand, we are also in a position to price a simple stop-loss reinsurance policy. For such a policy the value process is given by
\[
C_{tT} = P_{tT} \int_{0}^{\infty} (x - K)^+ \pi_t(x) \, dx
\]
\[
= P_{tT} \int_{\xi_t}^{\infty} (x - K)^+ p(x)x^{1-mT} (x-\xi_t)^{m(T-t)-1} \, dx
\]
\[
= \int_{\xi_t}^{\infty} p(x)x^{1-mT} (x-\xi_t)^{m(T-t)-1} \, dx
\]
for \( t < T \), and \( C_{T} = (X_T - K)^+ \). Once a time \( t \) has been reached such that \( \xi_t \geq K \), then \( C_{uT} = S_u - P_{uT} K \) for \( t \leq u \leq T \); that is, when a sufficient number of claims have accumulated, the option is sure to expire in-the-money. It should be evident that although the language of the results of this section, and those following, is mainly that of insurance, the relevant concepts are equally applicable to a variety of financial products. In the case of a large credit portfolio, for example, \( C_{tT} \) has the interpretation of being the value at \( t \) of a contract that pays at \( T \) an amount equal to the total loss incurred by the portfolio, in excess of some threshold \( K \). It is then straightforward to see that the pay-off function can be adjusted to accommodate a segregation of the loss allocation into tranches.

5. Valuation of general reinsurance contracts

In §4 we showed how one works out the reserve process for an aggregate claim that pays \( X_T \) at \( T \); and we were also able to determine the value process of a stop-loss reinsurance contract that pays \( (X_T - K)^+ \) at \( T \). In this section we consider the more general situation of a contract that at a fixed time \( t < T \) allows the firm the option of commuting the claim \( X_T \) in exchange for a pre-fixed settlement \( K \). Let us write \( C_{0t} \) for the value at time 0 of such an option; then clearly we have
\[
C_{0t} = P_{0t} \mathbb{E}[(S_t - K)^+], \tag{5.1}
\]
where \( S_t \) is the value at \( t \) of the claim that pays \( X_t \) at \( T \). In the context of a credit portfolio, \( S_t \) represents the value at \( t \) of a contract that pays an amount equal to the accumulated losses in the portfolio at time \( T \); then \( C_{0t} \) is the price at time 0 of a contract that pays at time \( t \) the excess of \( S_t \) over \( K \). With reference to proposition 4.2, we introduce a function \( S(t, y) \) for \( t < T \) and \( y \geq 0 \) by setting
\[
S(t, y) = P_{tT} \int_{y}^{\infty} p(x)x^{2-mT} (x-y)^{m(T-t)-1} \, dx
\]
\[
= \int_{y}^{\infty} p(x)x^{1-mT} (x-y)^{m(T-t)-1} \, dx. \tag{5.2}
\]
Then the value of the claim is $S_t = S(t, \xi_t)$, and the value of the option is

$$C_{0t} = P_{0t}[ E[(S(t, \xi_t) - K)^+]]. \quad (5.3)$$

Before proceeding with the calculation of this expectation let us check that the integrals in (5.2) converge. Let $T > t > 0$ and $y > 0$ be given, set $a = mt$ and $b = m(T - t)$, fix a constant $z > y$, and for the numerator term in (5.2) write $I = I^+ + I^-$, where $I^+ = \int_z^\infty p(x)x^{2-(a+b)}(x-y)^{b-1} \, dx$ and $I^- = \int_y^z p(x)x^{2-(a+b)}(x-y)^{b-1} \, dx$. It follows that

$$I^+ = \int_z^\infty p(x)x^{1-a}(1/y-x)^{b-1} \, dx \leq c \int_z^\infty p(x)x^{1-a} \, dx, \quad (5.4)$$

where $c = 1$ if $b > 1$, and $c = (1-y/z)^{b-1}$ if $b \leq 1$. In either case we have $I^+ < c \int_z^\infty p(x)x \, dx < \infty$, since $X_T$ is assumed to be integrable. On the other hand, we have $I^- = b^{-1} \int_y^z p(x)x^{2-(a+b)}d(x-y)^b$; writing $w = (x-y)^b$ we obtain

$$I^- = b^{-1} \int_{x=y}^{z-y} p(y + w^{1/b})(y + w^{1/b})^{2-(a+b)} \, dw, \quad (5.5)$$

which is finite. Similar arguments show that the denominator in (5.2) is finite.

Since the payout of the option is a function of $\xi_t$, one way of working out the expectation in (5.3) is to obtain an expression for the price $A_{0t}(y)$ of an Arrow–Debreu security that pays $\delta(\xi_t - y)$ at $t$, where $y \geq 0$ is a parameter. Thus, we have $A_{0t}(y) = P_{0t}[ \delta(\xi_t - y)]$, and for the option we can write

$$C_{0t} = \int_0^\infty A_{0t}(y)[S(t, y) - K]^+ \, dy. \quad (5.6)$$

We shall calculate $A_{0t}(y)$ and use the result to determine the expectation (5.3). We state the result first, the proof of which is given at the end of this section.

**Proposition 5.1.** The price $A_{0t}(y)$ at time 0 of an Arrow–Debreu security that pays $\delta(\xi_t - y)$ at $t$ is given by

$$A_{0t}(y) = P_{0t} \frac{y^{mt-1}}{B[mt, m(T-t)]} \int_y^\infty p(x)x^{1-mT}(x-y)^{m(T-t)-1} \, dx. \quad (5.7)$$

By comparing (5.2) and (5.7) we observe that the integral term in (5.7) cancels with the denominator in the expression for $S(t, y)$. We thus obtain

$$C_{0t} = \int_0^\infty \frac{P_{0t}y^{mt-1}}{B[mt, m(T-t)]} \left[ \int_y^\infty p(x)(xp_{tT} - K)x^{1-mT}(x-y)^{m(T-t)-1} \, dx \right]^+ \, dy. \quad (5.8)$$

We are now left with the task of finding the critical values at which the argument of the max function in (5.8) vanishes. Suppose $S(t, y)$ is monotonic in $y$; then there is at most a single critical value $y^*$, obtained by solving

$$\int_{y^*}^\infty p(x)(xp_{tT} - K)x^{1-mT}(x-y)^{m(T-t)-1} \, dx = 0. \quad (5.9)$$

The lower limit of the outer integration in the expression for $C_{0t}$ above can then be changed, and we have

$$C_{0t} = \int_y^\infty \frac{P_{0t}y^{mt-1}}{B[mt, m(T-t)]} \left[ \int_y^\infty p(x)(xp_{tT} - K)x^{1-mT}(x-y)^{m(T-t)-1} \, dx \right] \, dy. \quad (5.10)$$
This expression simplifies further if we swap the order of integration as follows:

\[
C_{0t} = \frac{P_{0t}}{B[mt, m(T-t)]} \int_0^\infty \int_0^y p(x)(xP_t T - K) y^{mT-1} x^{1-mT} (x-y)^{m(T-t)-1} \, dy \, dx.
\]

Making the substitution \(y = xz\), we then obtain

\[
C_{0t} = \frac{P_{0t}}{B[mt, m(T-t)]} \int_0^\infty \int_0^1 z^{mT-1} (1-z)^{m(T-t)-1} \, dz \, dx. \tag{5.12}
\]

Let us now define the complementary beta distribution function \(B(u)\) with parameters \(mt\) and \(m(T-t)\) by the expression

\[
B(u) = \frac{\int_u^1 z^{mT-1} (1-z)^{m(T-t)-1} \, dz}{\int_0^1 z^{mT-1} (1-z)^{m(T-t)-1} \, dz}.
\]

Clearly, the denominator in (5.13) is \(B[mt, m(T-t)]\). We thus find that the integration over the variable \(z\) in (5.12) combines with the factor \(B[mt, m(T-t)]\) to give a cumulative beta distribution function, and for the option price we have

\[
C_{0t} = P_{0t} \int_0^\infty p(x)(xP_t T - K)B \left( \frac{y}{x} \right) \, dx. \tag{5.14}
\]

We remark, incidentally, that a sufficient condition for \(S(t, y)\) to be monotonic in \(y\) for fixed \(t\) is \(m(T-t) > 1\). To see this, we differentiate \(S(t, y)\) with respect to \(y\), assuming the stated condition, and after some rearrangement we obtain

\[
\frac{\partial S(t, y)}{\partial y} = P_t T [m(T-t) - 1] \left( \int_y^\infty p(x) \alpha^2(x) \, dx \int_y^\infty p(x) \beta^2(x) \, dx \right) - 1, \tag{5.15}
\]

where \(\alpha^2(x) = x^{1-mT} (x-y)^{m(T-t)}\) and \(\beta^2(x) = x^{1-mT} (x-y)^{m(T-t)-2}\). Then if \(m(T-t) > 1\), the integrals exist, and \(\partial S(t, y)/\partial y > 0\) by the Schwartz inequality.

**Proof of proposition 5.1.** Using the Fourier representation for the delta function we can write

\[
\mathbb{E}[\delta(\xi_t - y)] = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda y} \mathbb{E}[\exp(i\lambda \xi_t)] \, d\lambda. \tag{5.16}
\]

Since \(X_T\) and \(\gamma_T\) are independent, it follows from the tower property that

\[
\mathbb{E}[\exp(i\lambda X_T \gamma_T)] = \mathbb{E}[\exp(i\lambda X_T)] \mathbb{E}[\exp(i\lambda \gamma_T)] = \int_0^\infty p(x) \mathbb{E}[\exp(i\lambda x \gamma_T)] \, dx
\]

\[
= \int_0^\infty p(x) \phi(\lambda x) \, dx,
\]

where \(\phi(\nu) = \mathbb{E}[\exp(i\nu \gamma_T)]\) is the characteristic function of \(\gamma_T\). We deduce that

\[
\mathbb{E}[\delta(\xi_t - y)] = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda y} \int_0^\infty p(x) \phi(\lambda x) \, dx \, d\lambda. \tag{5.18}
\]

Thus, by interchanging the order of integration and using the fact that the inverse Fourier transform of the characteristic function is the density function.
we have

\[
\mathbb{E}[\delta(\xi_t - y)] = \int_{x=0}^{\infty} p(x) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} \phi(\lambda) d\lambda \right] dx
\]

\[
= \int_{x=0}^{\infty} p(x) \frac{1}{x} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} \phi(\nu) d\nu \right] dx
\]

\[
= \int_{0}^{\infty} p(x) \frac{1}{x} f\left( \frac{y}{x} \right) dx,
\]

where \(f\) is the density function of \(\gamma_{tT}\). Substituting (3.1) into (5.19) we find that

\[
\mathbb{E}[\delta(\xi_t - y)] = \int_{0}^{\infty} p(x) \frac{1}{x} \mathbb{1}_{\{x>y\}} \frac{(y/x)^{mt-1}(1-y/x)^{m(T-t)-1}}{B(mt, m(T-t))} dx
\]

\[
= \frac{y^{mt-1}}{B(mt, m(T-t))} \int_{y}^{\infty} p(x) x^{1-mT} (x-y)^{m(T-t)-1} dx,
\]

which verifies the claim.

We remark that the price of the Arrow–Debreu security can be put in the form

\[
A_{0t}(y) = P_{0t} \int_{0}^{1} \left( \frac{y}{u} \right)^{mt-2} (1-u)^{m(T-t)-1} du,
\]

by use of which the normalization \(\int_{0}^{\infty} A_{0t}(y) dy = P_{0t}\) can be checked. It follows also from (5.21) that the characteristic function \(\Phi_{\xi}(\lambda)\) of \(\xi_t\) is given by the beta average of the characteristic function \(\Phi_x\) of \(X_T\):

\[
\Phi_{\xi}(\lambda) = \frac{\int_{0}^{1} \Phi_x(\lambda u) u^{mt-1}(1-u)^{m(T-t)-1} du}{\int_{0}^{1} u^{mt-1}(1-u)^{m(T-t)-1} du}.
\]

It is worthwhile noting that \(\lim_{t \to T} A_{0t}(y) = P_{0T} p(y)\), a result that is intuitively reasonable, though not immediately evident from (5.7) or (5.21). This shows that if we are given a set of call option prices with maturity \(T\) for all strikes, then the density function \(p(x)\) can be constructed. It follows more generally that if we are given the values of call options for all strikes for any specific maturity date \(t\), then the density can be constructed, and hence option prices for all other maturities.

6. Discretely distributed cash flows

In this section we consider the example for which \(X_T\) takes values in a discrete set \(\{x_i\}_{i=1,\ldots,n}\). The corresponding \textit{a priori} probabilities will be denoted \(\{p_i\}\). The calculation presented in §4 holds and we obtain, instead of (4.4), the following expression for the value process:

\[
S_t = P_{tT} \sum_{i} p_i x_i^{2-mT} (x_i - \xi_t)^{m(T-t)-1} \mathbb{1}_{\{\xi_t < x_i\}}.
\]

It is straightforward to verify that expression (6.1) converges to the correct terminal value as \(t\) approaches \(T\). To see this, suppose that for some \(\omega \in \Omega\) the

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value of $X_T$ is $x_k$. Then for that choice of $\omega$ we have, after some rearrangement,

$$S_t = P_{tT} \frac{p_k x_k^{1-mT} + \sum_{i \neq k} p_i x_i^{2-mT} \left( \frac{1-\gamma_{i,t}}{x_i-x_k y_{i,t}} \right)^{1-m(T-t)} q_{\{x_i>0\}}}{p_k x_k^{1-mT} + \sum_{i \neq k} p_i x_i^{1-mT} \left( \frac{1-\gamma_{i,t}}{x_i-x_k y_{i,t}} \right)^{1-m(T-t)} q_{\{x_i>0\}}}.$$ (6.2)

It follows at once that $S_{t^+} = x_k$. See figure 1 for typical sample paths of $\{S_t\}$.

We proceed now to value a reinsurance contract that pays $(S_t - K)^+$ at time $t$. For this purpose we need the price of an Arrow–Debreu security with pay-off $\delta(\xi_i - y)$ at $t$. In the discrete case the Arrow–Debreu price is given by

$$A_{0t}(y) = P_{0t} \frac{y^{mt-1}}{B[mt, m(T-t)]} \sum_{i=0}^{n} p_i x_i^{1-mT} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}}.$$ (6.3)

Substituting (6.3) and the function

$$S(t, y) = P_{tT} \frac{\sum_{i} p_i x_i^{2-mT} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}}}{\sum_{i} p_i x_i^{1-mT} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}}}$$ into (5.6), we obtain

$$C_{0t} = \int_{0}^{\infty} \frac{P_{0t} y^{mt-1}}{B[mt, m(T-t)]} \left[ \sum_{i=1}^{n} p_i x_i^{1-mT} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}} (P_{tT} x_i - K) \right]^+ dy.$$ (6.4)

A discrete version of formula (5.15) shows that $S(t, y)$ is increasing in $y$ if $m(T-t) > 1$, and decreasing in $y$ for $y \in (x_k, x_{k+1})$ for each $k = 1, \ldots, n-1$ if $m(T-t) < 1$. For fixed $t$ there is at most a single critical value $y = y^*$ for which $S(t, y) = K$, when $y \neq x_k$ for all $k$. We thus have three scenarios to consider: (i) $S(t, y)$ is increasing in $y$ at $y = y^*$, (ii) the critical value $y^*$ is at $y = x_k$ for some $k$, and (iii) $S(t, y)$ is decreasing in $y$ at $y = y^*$.

In case (i) the integrand in (6.5) is nonzero when $y \in (y^*, \infty)$, and for the price of the reinsurance contract we have

$$C_{0t} = P_{0t} \sum_{i=1}^{n} \frac{p_i x_i^{1-mT} (P_{tT} x_i - K)}{B[mt, m(T-t)]} \int_{y^*}^{\infty} y^{mt-1} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}} dy.$$ (6.6)

The $y$ integration in (6.6) can be carried out by observing that for $x_i > y^*$ we have

$$\int_{y^*}^{\infty} y^{mt-1} (x_i - y)^{m(T-t)-1} q_{\{x_i>y\}} dy = x_i^{mt-1} \int_{y^*/x_i}^{1} z^{mt-1} (1-z)^{m(T-t)-1} dz.$$ (6.7)

Therefore, the price of the reinsurance contract can be expressed in terms of the complementary beta distribution function with parameters $mt$ and $m(T-t)$:

$$C_{0t} = P_{0t} \sum_{i=1}^{n} q_{\{x_i>y^*\}} p_i (P_{tT} x_i - K) B(y^*/x_i).$$ (6.8)

If there is no critical value in the range $(x_k, x_{k+1})$, then $y^* = x_k$ for some $k$. Hence the pricing formula in case (ii) is identical to the result obtained in (6.8), with $y^* = x_k$. In case (iii) there are two distinct regions for which the integrand in (6.5) is nonzero. These are given by $y \in (x_k, y^*)$ and $y \geq x_{k+1}$ for some $k$. Thus, the pricing formula is similar to that obtained in (6.8), except there are additional terms arising from the integration over the range $[x_k, y^*)$.

As an example of a discrete cash flow, we consider the case where $X_T$ can take the values $x_0, x_1$. The critical value $y^* < x_0$ can be worked out by solving

$$x_0^{1-mT} (x_0 - y^*)^{m(T-t)-1} = \frac{p_1(K - P_{0T}x_1)}{p_0(P_{0T}x_0 - K)} x_1^{1-mT} (x_1 - y^*)^{m(T-t)-1}$$

for $y^*$. A short calculation shows that

$$y^* = \frac{\theta x_1 - x_0}{\theta - 1}, \quad \text{where} \quad \theta = \left[ \frac{p_1(K - P_{0T}x_1)}{p_0(P_{0T}x_0 - K)} \left( \frac{x_1}{x_0} \right)^{1-mT} \right]^{1/[m(T-t)-1]}.$$

It follows that the price of a reinsurance contract in the case of a binary pay-off is

$$C_{0t} = p_0(P_{0T}x_0 - P_{0t}K) \mathbb{B}(y^* / x_0) + p_1(P_{0T}x_1 - P_{0t}K) \mathbb{B}(y^* / x_1).$$

### 7. Price processes for options on reserves

We now generalize the analysis of §5 to derive the price process of a call option on the value of the reserve $S_t$ at time $t$ associated with the claim $X_T$. As before, let $K$ be the strike. Then the value of the option at time $s \leq t$ is

$$C_{st} = P_{st} \mathbb{E} \left[ (S(t, \xi_t) - K)^+ | \xi_s \right].$$

Once again we find it convenient to obtain first the price process for the Arrow–Debreu security. This is on account of the relation

$$C_{st} = P_{st} \mathbb{E} \left[ \int_0^\infty \delta(\xi_t - y) (S(t, y) - K)^+ \, dy \bigg| \xi_s \right]$$

$$= P_{st} \int_0^\infty \mathbb{E} \left[ \delta(\xi_t - y) | \xi_s \right] (S(t, y) - K)^+ \, dy = \int_0^\infty A_{st}(y) (S(t, y) - K)^+ \, dy,$$
where $S(t, y)$ is defined as in (5.2) and $\{A_{st}\}_{0 \leq s \leq t \leq T}$ is given by $P_{st}E[\delta(\xi_t - y)|\xi_s]$. By taking the conditional expectation, we obtain the following result.

**Proposition 7.1.** The price process $\{A_{st}(y)\}_{0 \leq s \leq t \leq T}$ of the Arrow–Debreu security that pays out $\delta(\xi_t - y)$ at $t$ is

$$A_{st}(y) = P_{st} \frac{1_{\{y > \xi_s\}}(y - \xi_s)^{m(t-s)-1}}{B[m(t-s), m(T-t)]} \int_{\xi_s}^{y} p(x)x^{1-mT}(x-y)^{m(T-t)-1} \, dx,$$

where $p(x)$ is the density of $X_T$.

This result is established later in this section. By substitution of (7.3) in (7.2) we see that the price process of the option is given by

$$C_{st} = \frac{P_{st}}{B[m(t-s), m(T-t)]} \int_{\xi_s}^{y} p(x)x^{1-mT}(x-\xi_s)^{m(T-s)-1} \, dx$$

$$\times \int_{y=\xi_s}^{\infty} (y - \xi_s)^{m(t-s)-1} \left[ \int_{y}^{\infty} p(x)x^{1-mT}(x-y)^{m(T-t)-1} (P_{tT}x - K) \, dx \right]^+ \, dy.$$

Assuming that there is only one critical value $y^*$ that solves (5.9), we find that the integration over $y$ in (7.4) vanishes for $y < y^*$. In this case, we can lift the max function, and by interchanging the order of integration we obtain

$$C_{st} = \frac{P_{st}}{B[m(t-s), m(T-t)]} \int_{\xi_s}^{y} p(x)x^{1-mT}(P_{tT}x - K) \int_{y=\xi_s}^{y^*} (y - \xi_s)^{m(t-s)-1}(x-y)^{m(T-t)-1} \, dy \, dx.$$

Making the substitutions $y = z + \xi_s$ and $z = w(x - \xi_s)$ we get

$$\int_{y=\xi_s}^{y^*} (y - \xi_s)^{m(t-s)-1}(x-y)^{m(T-t)-1} \, dy = \int_{z=\xi_s}^{y-x} \int_{z=\xi_s}^{y-x} \frac{m(t-s)-1}{m(T-t)-1} \, dz \, dw.$$

We see that together with the beta function in the denominator of (7.5) the integral term on the right side of (7.6) gives rise to a complementary beta distribution function. Therefore, the option price can be written in the form

$$C_{st} = P_{st} \int_{y=\xi_s}^{\infty} \frac{p(x)x^{1-mT}(x-\xi_s)^{m(T-s)-1}}{\int_{\xi_s}^{\infty} p(x)x^{1-mT}(x-\xi_s)^{m(T-s)-1} \, dx} (P_{tT}x - K)B\left(\frac{y^* - \xi_s}{x - \xi_s}\right) \, dx.$$

Finally, we observe that the quotient in the integrand is the conditional density $\pi_s(x)$. The price at time $s \leq t$ thus reduces to the following expression:

$$C_{st} = P_{st} \int_{y=\xi_s}^{\infty} \pi_s(x)(xP_{tT} - K)B\left(\frac{y^* - \xi_s}{x - \xi_s}\right) \, dx.$$

As in the case of the initial price, the range of integration in (7.8) must be modified if there is more than one critical value for which (5.9) is satisfied. We now proceed to derive the expression for the Arrow–Debreu price process.

**Proof of proposition 7.1.** Using the Fourier representation for \( \delta(x) \) we have

\[
E[\delta(\xi_t - y)|\xi_s] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iy\lambda} \mathbb{E}[e^{i\lambda\xi_t} | \xi_s] \, d\lambda.
\]

To determine the conditional expectation \( \mathbb{E}[e^{i\lambda\xi_t} | \xi_s] \), the following result is handy.

**Lemma 7.2.** Let \( \{\xi_t\}_{0 \leq t \leq T} \) be given by \( \xi_t = X_T \gamma_T \), where \( \{\gamma_T\} \) is a gamma bridge and \( X_T \) is an independent positive random variable. Then for fixed \( s \) such that \( 0 \leq s \leq t \leq T \) we have \( \xi_t = \xi_s + Z_T \delta_{tT} \), where \( Z_T = (1 - \gamma_s)X_T \) and where the process \( \delta_{tT} = (\gamma_T - \gamma_s)/(1 - \gamma_s) \) is a gamma bridge over the interval \( t \in [s, T] \) and is independent of \( \xi_s \) and \( Z_T \).

By use of \( \xi_t = \xi_s + Z_T \delta_{tT} \) and the tower property we find that

\[
E[\exp(i\lambda\xi_t)|\xi_s] = E[E[\exp(i\lambda(\xi_s + Z_T \delta_{tT}))]|\xi_s] = \exp(i\lambda\xi_s) E[E[\exp(i\lambda Z_T \delta_{tT})|\xi_s, \delta_{tT}]|\xi_s].
\]

Since \( Z_T = X_T - \xi_s \) and \( \{\delta_{tT}\} \) is independent of \( \xi_s \) and \( X_T \), the inner expectation can be carried out explicitly by use of the conditional density for \( X_T \):

\[
E\left[ \exp(i\lambda\xi_t) \bigg| \xi_s \right] = \exp(i\lambda\xi_s) E\left[ \int_{x=\xi_s}^{\infty} \exp(i\lambda(x-\xi_s)\delta_{tT})\pi_s(x) \, dx \bigg| \xi_s \right].
\]

By substituting (7.11) in (7.9) we deduce that

\[
E[\delta(\xi_t - y)|\xi_s] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\lambda(y-\xi_s)) \int_{x=\xi_s}^{\infty} \Phi_\delta[\lambda(x-\xi_s)]\pi_s(x) \, dx \, d\lambda
\]

\[
= \int_{x=\xi_s}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\lambda(y-x))\Phi_\delta[\lambda(x-x_s)] \, d\lambda \right) \pi_s(x) \, dx
\]

\[
= \int_{x=\xi_s}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-i\frac{y-x}{x-\xi_s}z\right)\Phi_\delta(z) \, dz \right) \frac{1}{x-\xi_s} \pi_s(x) \, dx
\]

\[
= \int_{x=\xi_s}^{\infty} \frac{\pi_s(x)}{x-\xi_s} f_\delta \left( \frac{y-x}{x-\xi_s} \right) \, dx,
\]

where \( \Phi_\delta \) is the characteristic function for \( \delta_{tT} \) and \( f_\delta \) is the density of \( \delta_{tT} \). Since \( \delta_{tT} \) is beta distributed with parameters \( m(t-s) \) and \( m(T-t) \), we deduce expression (7.3) for the Arrow–Debreu price.

**Proof of lemma 7.2.** The decomposition \( \xi_t = \xi_s + Z_T \delta_{tT} \) can be verified if one sets \( \{\gamma_{tT}\} = \{\gamma_t/\gamma_T\} \), where \( \{\gamma_t\} \) is a standard gamma process. To see that \( \{\delta_{tT}\}_{s \leq t \leq T} \) is, for fixed \( s \), a gamma bridge over \([s, T]\), it suffices to note that \( \delta_{tT} = (\gamma_T - \gamma_s)/(\gamma_T - \gamma_t) \) and that \( \{\gamma_t - \gamma_s\}_{s \leq t < \infty} \) is a gamma process. In particular, we observe that the independent increments property holds, and that \( \gamma_t - \gamma_s \) is gamma distributed with mean \( m(t-s) \). Finally, to see that \( \{\delta_{tT}\} \) is independent.
of $\xi_s$ and $Z_T$, it suffices to show that $\delta_{t,T}$, $\gamma_s$ and $\gamma_T-\gamma_s$ are independent. We have

$$\mathbb{Q}(\{\delta_{t,T} < a\} \cap \{\gamma_T - \gamma_s < b\} \cap \{\gamma_s < c\}) = \mathbb{E}[\mathbb{1}_{\{\delta_{t,T} < a\}} \mathbb{1}_{\{\gamma_T - \gamma_s < b\}} \mathbb{1}_{\{\gamma_s < c\}}]$$

$$= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\delta_{t,T} < a\}} \mathbb{1}_{\{\gamma_T - \gamma_s < b\}} \mathbb{1}_{\{\gamma_s < c\}}|\gamma_s]] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\delta_{t,T} < a\}} \mathbb{1}_{\{\gamma_T - \gamma_s < b\}}|\gamma_T - \gamma_s] \mathbb{1}_{\{\gamma_s < c\}}]$$

$$= \mathbb{E}[\mathbb{1}_{\{\delta_{t,T} < a\}} \mathbb{1}_{\{\gamma_T - \gamma_s < b\}} \mathbb{1}_{\{\gamma_s < c\}}]$$

(7.13)

Here, we have used the fact that $\gamma_s$ is independent of $\delta_{t,T}$ and $\gamma_T-\gamma_s$, which follows from the independent increments property of $\{\gamma_t\}$. We have also used lemma 3.3, together with the fact that we can write $\delta_{t,T}=B/(A+B)$ and $\gamma_T-\gamma_s=A+B$, with $A=\gamma_T-\gamma_t$ and $B=\gamma_t-\gamma_s$, from which it follows that $\delta_{t,T}$ and $\gamma_T-\gamma_s$ are independent.

The result of lemma 7.2 leads to a remark concerning model calibration. Suppose that the aggregate claims process is given, and we reinitialize the model at a specified intermediate time. We would like the dynamics moving forward from the intermediate time to be represented by an aggregate claims process of the same type. It follows from lemma 7.2 that the process $\{\eta_t\}_{s \leq t \leq T}$ defined by $\eta_t = Z_T \delta_{t,T}$, is an aggregate claims process spanning the interval $[s,T]$. The random variable $Z_T$ represents the information about $X_T$ that is ‘not yet revealed’ at time $s$. The idea is that at time $s$ the value of $\xi_s$ is known, and the ‘new’ gains process $\{\eta_t\}_{s \leq t \leq T}$ begins to reveal the value of $Z_T$ in such a way that $\eta_s=0$ and $\eta_T=Z_T$.

Alternatively, at time $s$ we can use the knowledge of $\xi_s$ to compute the new a priori density for $X_T$. Thus, at $s$ the a priori density $p(x)$ is replaced by the a posteriori density $\pi_s(x)$. On account of the relation $Z_T = X_T - \xi_s$, we have

$$\mathbb{Q}[Z_T < z|\xi_s] = \mathbb{Q}[X_T < \xi_s + z|\xi_s],$$

(7.14)

from which it follows that the conditional density of $Z_T$ is given at time $s$ by $\pi_s(\xi_s + z)$. We can think of $\pi_s(\xi_s + z)$ as a new a priori density, now for the random variable $Z_T$. Given this density we calculate the conditional probability $\mathbb{Q}[Z_T < z|\eta_t]$ for $t \in [s,T]$. By the method used to establish proposition 4.1 and the probability law for the gamma bridge $\{\delta_{t,T}\}$ we deduce that

$$\frac{d}{dz} \mathbb{Q}[Z_T < z|\eta_t] = \mathbb{1}_{z>\eta_t} \frac{\pi_s(\xi_s + z)z^{1-m(T-s)}(z-\eta_t)^{m(T-t)-1}}{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z)z^{1-m(T-s)}(z-\eta_t)^{m(T-t)-1} dz},$$

(7.15)

from which we see that the value process can be represented in the following form:

$$S_t = P_t \left[ \xi_s + \frac{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z)z^{2-m(T-s)}(z-y)^{m(T-t)-1} dz}{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z)z^{1-m(T-s)}(z-\eta_t)^{m(T-t)-1} dz} \right].$$

(7.16)

Making the substitution $z=x-\xi_s$ and also substituting $\eta_t=\xi_t-\xi_s$, we see that this expression reduces to the value process obtained in (4.4).
8. Gamma-distributed terminal gains

When the terminal payout $X_T$ of the cumulative gains process (1.1) is gamma distributed with mean $\kappa m T$ and variance $\kappa^2 m T$ for some $\kappa$, the value process $\{S_t\}$ has an especially simple structure, and we are led back to the $\mathbb{Q}$-gamma model discussed in §1. This can be seen as follows. Let $\{\gamma_t\}$ be a standard gamma process with rate $\gamma$ and let $\{\gamma_{it}\}$ be the associated bridge. Then $X_T$ and $\kappa \gamma_T$ have the same distribution; but since $\gamma$ and $\gamma_{it}$ are independent, $\{X_T \gamma_{it}\}$ and $\{\kappa \gamma_T \gamma_{it}\}$ have the same probability law; therefore, $\{\xi_t\}$ has the same law as $\{\kappa \gamma_t\}$ and hence is a $\mathbb{Q}$-gamma process with scale $\kappa$ and standard growth rate $m$. The fact that $\xi_t$ is gamma distributed can be verified directly as follows. The characteristic function of $X_T$ is $\phi_X(\lambda) = (1 - i \kappa \lambda)^{-m T}$. Substituting this into (5.22) and setting $z = (1 - u)/(1 - i \kappa \lambda u)$, we deduce that $\phi_\xi(\lambda) = (1 - i \kappa m)^{-m t}$, which is the characteristic function of a gamma variable with mean $\kappa m T$ and variance $\kappa^2 m T$.

It is interesting to note that although the $\mathbb{Q}$-gamma process has independent increments the general cumulative gains process (1.1) has dependent increments. In particular, for the covariance of $\xi_s$ and $\xi_t - \xi_s$ in the general case we have

$$\text{cov} [\xi_s, \xi_t - \xi_s] = \frac{ms(t-s)}{T(mT+1)} E[X_T^2] - \frac{s(t-s)}{T^2} (E[X_T])^2. \quad (8.1)$$

Hence a necessary condition for independent increments is given by $(E[X_T])^2 = m T \text{var}[X_T]$.

The value processes for various claims in the $\mathbb{Q}$-gamma model can be worked out explicitly. For the density of $X_T$ we have the expression defined by (2.8). Substituting this into (4.4) and carrying out the integration, we are led to the following linear expression for the reserve process:

$$S_t = P_{it}(\xi_t + \kappa m (T-t)). \quad (8.2)$$

We observe that $S_0 = P_{0it} \kappa m T$ and that $S_T = X_T$, as required. An alternative derivation of (8.2) is as follows. Since $\{\xi_t\}$ is a gamma process with scale parameter $\kappa$ and standardized growth rate $m$, by the Markov property we have $S_t = P_{it} E[\xi_T | \xi_t]$, and (8.2) follows from the independent increments property of the gamma process.

These relations lead to simplifications in the valuation of contingent claims. Let us work out the value $C_{it}$ at time $t$ of a stop-loss reinsurance contract that pays $\max(X_T - K, 0)$ at $T$ for some fixed threshold $K$. In the $\mathbb{Q}$-gamma model we have $C_{it} = P_{it} E[(\xi_T - K)^+ | \xi_t]$, and hence by use of the independent increments property we deduce that

$$C_{it} = P_{it} \int_{(K-\xi_t)/\kappa}^{\infty} (\kappa z + \xi_t - K) \frac{z^{m(T-t)-1} e^{-z}}{\Gamma[m(T-t)]} \, dz$$

$$= P_{it} \left[ \kappa \frac{\Gamma[m(T-t)+1,(K-\xi_t)/\kappa]}{\Gamma[m(T-t)]} - (K - \xi_t) \frac{\Gamma[m(T-t), (K-\xi_t)/\kappa]}{\Gamma[m(T-t)]} \right], \quad (8.3)$$

where $\Gamma[a,z] = \int_z^{\infty} x^{a-1} e^{-x} \, dx$ denotes the incomplete gamma integral.
We proceed to calculate the associated Arrow–Debreu price $A_{st}$ in this model. By substituting (8.2) in (7.3), we deduce that

$$A_{st}(y) = P_{st} \frac{\kappa^{-(m(t-s))}}{\Gamma(m(t-s))} (y - \xi_s)^{(m(t-s)-1)} \exp \left( -\frac{1}{\kappa} (y - \xi_s) \right).$$ (8.4)

Writing $R_s = P_{T^{-1}}(K - (S_s + \kappa m(t-s)))$ we see by use of (8.2) that the price at time $s$ of a reinsurance contract with payout $(S_t - K)^+$ at $t$ is given by

$$C_{st} = P_{st} \mathbb{E}_s[(S_t - K)^+] = \int_0^\infty A_{st}(y)[P_{st}(y + \kappa m(T-t)) - K]^+ \, dy$$

$$= P_{st} \left[ \frac{\Gamma(m(t-s) + 1, \kappa^{-1} R_s)}{\Gamma(m(t-s))} - \kappa^{-1} R_s \frac{\Gamma(m(t-s), \kappa^{-1} R_s)}{\Gamma(m(t-s))} \right].$$ (8.5)

It may be surprising that the linear martingale $\{\gamma_t - mt\}$ can be used as the basis of a model for a limited liability asset, but this is consistent with the finite time horizon over which the asset is defined. In fact, for each value of $n$, the associated Laguerre martingale can be ‘positivized’ over the time interval $[0,T]$ by the addition of suitable lower order counter terms. In particular, for each $n \in \mathbb{N}$ we consider the martingale $L_{n}^{m(T-t) - n}(\gamma_t)$, $t \geq 0$, where $L_{n}^{k}(z)$ is defined as in (2.6). In this way we obtain the positive martingales (i) $\{\gamma_t + m(T-t)\}$, (ii) $\{(\gamma_t + m(T-t))^2 + m(T-t)\}$, (iii) $\{m(T-t)(\gamma_t + m(T-t)) + 2m(T-t)\}$, and so on. These processes arise naturally if we take the gamma process to be the information process associated with an $X$-factor (as defined in the sense of Macrina 2006 and Brody et al. 2007, 2008) given by the terminal value of the gamma process, and let the terminal pay-off of the asset be a polynomial in the $X$-factor.

This example also points towards the way in which the theory put forward here can be developed further to take into account multiple cash flows, families of interdependent assets, and filtrations of greater complexity. Each asset is defined by a series of one or more cash flows, each such cash flow being dependent on a set of one or more independent market factors ($X$-factors). Each $X$-factor has an associated information process, which may be of the Brownian bridge type considered in the references cited above, or the gamma bridge type considered here, or possibly some generalization thereof; and the market filtration is taken to be generated collectively by this set of information processes.

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