Further investigation of crystal hardening inequalities in (110) channel die compression

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A set of geometrically based FCC crystal slip-systems hardening inequalities is analytically investigated in (110) channel die compression for all lateral constraint directions between \( \{112\} \) and \( \{111\} \), following previous analyses of the other two distinct orientation ranges in (110) compression. With all critical slip systems active, it is proved that these inequalities uniquely predict initial lattice stability and finite crystal shearing only in the horizontal channel plane, consistent with experiments for this range of orientations. (The earlier analyses had predicted load-axis stability in both orientation ranges, and lattice stability in one, also commonly found experimentally.) Moreover, it is established that the lateral constraint stress predicted by the hardening inequalities will be less than that given by classic Taylor hardening as this stress evolves with deformation. It is further shown, taking into account experimental stress–strain curves and latent hardening experiments for aluminium and copper, that lattice stability generally can be expected to very large deformations, except perhaps for lateral constraint orientations near the \( \{112\} \) end of the range, which result is consistent with experiment. In appendix A, the possibilities of solutions with a critical slip system inactive are investigated, and predictions of a power law rate-dependent plasticity model are analysed for comparison with the results based on the hardening inequalities.

Keywords: channel die compression; FCC crystals; slip-systems hardening inequalities; finite deformation analysis

1. Introduction

In Havner (2005), an investigation of a diverse set of classic experimental studies (1960–1982) of finite deformations of FCC crystals in high-symmetry axial loading was undertaken. The range of behaviours included axis stability and axisymmetric deformation in \( \{111\} \) and \( \{100\} \) orientations, axis rotation towards a \( \{111\} \) orientation in coplanar double-slip in \( \{110\} \) loading and load-axis rotations towards (from an initial few degrees misalignment) or away from \( \{111\} \) and \( \{100\} \) orientations (with a reduced number of slip planes from the standard three and four). The experimental studies reviewed and analysed were: Kocks (1960, 1964) in aluminium; Nakada et al. (1964) in gold; Ramaswami et al. (1965) in silver and a gold–silver alloy; Keh (1965) in iron (a BCC metal, of course);

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Y. Nakada at Bell Labs in aluminium, briefly reported upon in Wonsiewicz & Chin (1970); and Vorbrugg et al. (1971), Ambrosi et al. (1974), Ambrosi & Schwink (1978) and Franciosi & Zaoui (1982), all in copper.

Use was made in Havner (2005) of experimental information, rigorous kinematic analyses of separate and coincident rotations among lattice, material and loading frames in the various (initial) orientations, and a geometric perspective on relationships among slip-systems hardening moduli. In each experiment analysed, the relative hardening necessary for (or consistent with) the corresponding crystal deformation and lattice rotation (where observed) was determined from the kinematics and the changing resolved shear stresses. A set of hardening inequalities was found (eqn (104) in Havner 2005) that encompassed all the individual responses from multiple-slip experiments and was also consistent with classic latent hardening experiments in single slip (Jackson & Basinski 1967 and Franciosi et al. 1980 among others).

In Havner (2007a), I began an investigation of the applicability of these hardening inequalities to FCC crystals in the very different experimental configuration of channel die compression, specifically (110) loading. As discussed at some length in the Introduction to that paper (which contains an extensive listing of channel die experiments), this family of orientations in the channel die test is of particular importance for investigation because (110) has long been established as a stable normal direction and preferred texture orientation in sheet rolling of polycrystalline metals. (Although my students and I carried out various analyses of channel die compression during the 1980s, a review of which may be found in ch. 5 of Havner (1992; including the theory of ‘minimum plastic spin’, Fuh & Havner 1989), these analyses were primarily focused on the following three specific crystal hardening theories: classic Taylor hardening (Taylor & Elam 1923, 1925); the ‘simple theory’ of Havner & Shalaby (1977, 1978); and the ‘P.A.N. rule’ of Peirce et al. (1982).)

As shown analytically in Havner (2007a), there are three kinematically distinct orientation ranges of the lateral constraint direction (plus three singular orientations) in (110) compression, resulting in different combinations of critical slip systems in each range (apparently first clearly identified by Skalli et al. 1983, fig. 4) and necessitating separate and independent analyses of both initial and subsequent responses. In Havner (2007a,b), all constraint directions between (001) and (112), named ‘range I’, are investigated to large deformations. It is established therein that the hardening inequalities, in combination with experimental stress–strain curves and experimentally based relative values of the moduli, correctly predict the load-axis stability, finite shearing in the horizontal channel plane only and lattice rotation about the (110) load axis that are well-documented experimentally (e.g. Skalli et al. 1983; Skalli 1984; Wrobel et al. 1996).

In Havner (2008), all constraint directions from (111) (commonly called the ‘brass’ orientation) to (110) (the ‘Goss’ orientation) are investigated, again to finite deformations. For this family of lattice orientations, named ‘range III’, it is proved analytically that the hardening inequalities predict the lattice stability that is commonly observed in this range, again with finite shearing only in the horizontal plane (see Chin et al. 1966a; Wonsiewicz & Chin 1970; Driver & Skalli 1982; Kocks & Chandra 1982; Skalli et al. 1983; Skalli 1984; Driver et al. 1994). Moreover, again taking into account experimental data, it is shown that lattice stability should continue to large strains for all orientations in range III.
In this paper, the middle range of constraint directions between \((1\overline{1}2)\) and \((1\overline{1}1)\), named ‘range II’, is investigated using the same hardening inequalities as before (eqn (104) of Havner 2005). In several respects, this orientation range has proven to be the most difficult to fully analyse, partly because, although a range of only slightly more than 19.47° (whereas ranges I and III are each more than 35.26°), it contains two kinematic sub-ranges, which the others do not. The paper is organized as follows.

Section 2 reviews the classic general hardening law, the geometric interactions among slip systems in FCC crystals, and the hardening inequalities. Section 3 contains general kinematics and stress rate equations for (well lubricated) channel die compression, and the equations for the resolved shear stresses in the 12 slip systems of FCC crystals. The range II channel die constraints and their sub-range consequences are given in §4. Section 5 contains the specific hardening rules and consistency conditions for critical slip systems in range II. In §6, a proof of uniqueness of solution with all critical systems active is presented. Its consequences for lattice stability, evolution of the constraint stress and finite crystal shearing are given in §7. The continuation of lattice stability, taking into account experimental stress–strain data from two orientations at the extremes of range II, and including the investigation of hardening of the latent slip systems, is analysed in §8. The main body of the paper ends with a closing discussion in §9, which also takes into account the analyses in appendix A. In appendix Aa, the sub-ranges of range II are investigated without the assumption that all critical systems are active. In appendix Ab, the consequences of the ‘standard’ form of a rate-dependent power law in (110) channel die compression are analysed.

2. Hardening law and inequalities

We adopt the perspective that, for moderate temperatures and strain rates, the \(k\)th slip system is momentarily inactive unless the resolved (Cauchy) shear stress \(\tau_k\) (the ‘Taylor–Schmid stress’) equals a critical value \(\tau^c_k\), the current critical strength of that system. The critical strengths in all \(N\) crystallographic slip systems are considered to evolve with the deformation according to the classic law (Hill 1966, eqn (4))

\[
d\tau^c_k = \sum H_{kj} \, d\gamma_j, \quad j = 1, \ldots, n, \quad k = 1, \ldots, N,
\]

where the \(H_{kj}\) are the non-constant slip-systems hardening moduli (deformation dependent, and at a more fundamental level dislocations dependent); \(d\gamma_j\) is an increment in crystallographic slip in the \(j\)th slip system; and \(n\) is the number of active systems at any stage. Moreover, a critical slip system \((\tau_k = \tau^c_k)\) cannot be active unless the time derivative of its resolved shear stress on a lattice co-rotational frame equals the rate-of-change of its critical strength (a concept first clearly expressed by Hill (1966, p. 96); see also Havner (1972)). This may be stated \(\dot{\tau}_k = \dot{\tau}^c_k\), the consistency condition in an active system (for exact equations with unrestricted lattice straining, see Hill & Havner 1982 or Havner 1992, ch. 3).

Herein the moduli are distinguished only by the five categories of geometric slip-system interactions in FCC crystals. The possible interrelations between systems \(k\) and \(j\) may be defined as follows (Havner 2005):

\[\text{Proc. R. Soc. A} \ (2008)\]
where \( \mathbf{b}_k, \mathbf{n}_k \) denote unit vectors in the slip and slip-plane normal directions, respectively, of the \( k \)th slip system. The specification of these five categories, in some cases with identical names, dates at least to Kocks (1964). The interactions have other names in the context of dislocation theory, in particular with reference to dislocation ‘junctions’ (e.g. Franciosi et al. 1980, p. 279). (Note that in the hardening theory of Bassani & Wu (1991), contributions of slip in other systems to the hardening of a given active system are defined by these same interactions; see their eqn (3.10). However, in comparing with the experiments of Wu et al. (1991), Bassani & Wu adopt a diagonal hardening matrix such that inactive systems do not harden. For a comparative study of Bassani & Wu’s model and four other hardening rules (including the simple theory of Havner & Shalaby 1977) against texture experiments, from the literature, on polycrystal-line copper bars in finite-strain torsion, see Lin & Havner (1996).)

The evolving moduli are labelled \( H_I \) to \( H_V \), corresponding to the above geometric interrelations between pairs of slip systems. The interactions among the 10 possible active systems in (110) channel die compression are given in table 1. No specific equations are proposed for the individual moduli (e.g. their dependence on dislocation density or temperature as well as evolving slips). We adopt only the following hardening inequalities among these moduli (Havner 2005, eqn (104)):

\[
H_V - H_I > H_{IV} - H_{III} \geq 0, \quad H_{II} - H_I > H_{III} - H_{II} > 0.
\]  

These inequalities reflect classic latent hardening experiments on crystals deformed in single slip (Kocks 1964; Kocks & Brown 1966; Jackson & Basinski 1967; Franciosi et al. 1980). As briefly described in §1, they were shown in Havner (2005) to either specifically predict or be consistent with a wide range of
dive experimental behaviours (from the literature) in high-symmetry tensile loading of aluminium, gold, silver, a gold–silver alloy and copper crystals. Moreover, in Havner (2007a, b, 2008), they were applied to the analysis and prediction of finite crystal shearing and (where observed) lattice rotation for two lattice orientation ranges in (110) channel die compression of FCC crystals.

3. Kinematics, stress and stress rate equations in channel die compression

For a crystal in the channel die compression test, we denote loading, lateral constraint and channel axis directions by $X$, $Y$ and $Z$, respectively, and designate unit vectors normal to the positive $X$ and $Y$ crystal faces by $i$, $k$, and a unit vector coincident with the positive channel axis $Z$ by $\mu$. Thus, $\mu k \mu$ constitutes a right-handed orthogonal triad subsequently to be defined relative to the lattice axes [100], [010] and [001].

As in the channel die analyses in Havner (2007a, b, 2008; and earlier ones reviewed in Havner 1992, ch. 5), it is assumed that a crystal is well lubricated, that the lubrication is renewed throughout a test (e.g. Ferry & Humphreys 1996, p. 1294) and that the crystal is initially rectangular with free ends. Correspondingly, the non-zero (Cauchy) stresses are defined by $\sigma_{xx} = -f$ and $\sigma_{yy} = -g$ (initially everywhere, but only away from the free ends of a long crystal after finite deformation), with $f$, $g$ the ‘true’ compressive stresses in the loading and constraint directions, from which the stress tensor is

$$\sigma = -f i \otimes i - g k \otimes k.$$  \hfill (3.1)

The resolved shear stress in the $k$th slip system then may be expressed

$$\tau_k = m_k f + r_k g, \quad m_k = -\nu N_k i, \quad r_k = -\kappa N_k k, \quad N_k = \text{sym} (b \otimes n)_k.$$  \hfill (3.2)

Its rate-of-change in the lattice frame is

$$\dot{\tau}_k = m_k \dot{f} + r_k \dot{g} + \dot{m}_k f + \dot{r}_k g, \quad \dot{m}_k = -2 \nu N_k i, \quad \dot{r}_k = -2 \kappa N_k k,$$

where $\dot{i}$, $\dot{k}$ are the lattice co-rotational derivatives of $i$, $k$, given by (Havner 2007a, eqn (7))

$$\dot{i} = -\dot{e}_L i - \sum (i \cdot b_j) n_j \dot{y}_j, \quad \dot{k} = -\sum (k \cdot b_j) n_j \dot{y}_j,$$

in which use has been made of the channel die constraints

$$d_{xx} = \nu D i = -\dot{e}_L, \quad d_{xy} = \nu D k = 0, \quad d_{yy} = \kappa D k = 0.$$  \hfill (3.5)

Here $e_L = -\ln \lambda$ (figure 1), the logarithmic true compressive strain, and

$$D = \sum N_j \dot{y}_j,$$

the Eulerian strain rate (disregarding the very small lattice strains in comparison with the finite slip deformations that are our focus).

In (110) channel die compression, we have (in Miller index notation)
\[ \{110\}, \{1\bar{1}\bar{b}\}, \{\bar{b}\bar{b}\\}, \sqrt{2}\cot\phi, \infty > b > 0, \]
where \(\phi\) is the clockwise orientation of \(Y\) with respect to [001] (or the anticlockwise orientation of [001] relative to \(Y\); figure 2). We designate the (maximum) 12 slip systems \(a_1, b_2, c_1, c_2, a_3, b_3, d_1, d_2, a_2, b_1, c_3,\) and \(d_3\) by 1–12 in order (with a bar above indicating the opposite sense of slip from that defined in table 2). From equation (3.2), the following are the resolved shear stresses:

\[ \begin{align*}
\tau_1 &= \tau_2 = \left(1/\sqrt{6}\right)\left\{f - gb\left(b - 1\right)/\left(b^2 + 2\right)\right\}, \\
\tau_3 &= \tau_4 = \left(1/\sqrt{6}\right)g\left(b + 2\right)\left(b - 1\right)/\left(b^2 + 2\right), \\
\tau_5 &= \tau_6 = \left(2/\sqrt{6}\right)gb/(b^2 + 2), \\
\tau_7 &= \tau_8 = \left(1/\sqrt{6}\right)g(b + 1)(b - 2)/(b^2 + 2), \\
\tau_9 &= \tau_{10} = \left(1/\sqrt{6}\right)\left\{f - gb\left(b + 1\right)/\left(b^2 + 2\right)\right\}, \\
\tau_{11} &= \tau_{12} = 0.
\end{align*} \]

Lattice orientations \(\infty > b > 2\) \((0 < \phi < 35.26^\circ)\) and \(1 > b > 0\) \((54.74^\circ < \phi < 90^\circ)\), respectively, labelled ranges I and III, were investigated in Havner (2007a,b, 2008). For the remaining range \(2 > b > 1\) \((35.26^\circ < \phi < 54.74^\circ)\), labelled range II, one finds from equation (3.8) that \(\tau_1 > \tau_9, \tau_5 > \tau_3 > 0\) and \(\tau_5 > \tau_7 > 0\). (Opposite sense systems 7, 8 rather than 7, 8 are positively stressed in range II.) Thus, because the three channel constraints, equation (3.5), require that at least three slip systems be active for the onset of finite plastic deformation, the resolved shear stresses in systems 1, 2, 5 and 6 \((a_1, b_2, a_3,\) and \(b_3)\) all must attain the initial critical strength, \(\tau_0\), from which (equation (3.8); also Chidambaram & Havner 1988, eqn (5.1))

\[ g_0b/(b^2 + 2) = f_0/(b + 1), \quad f_0 = (\sqrt{6}/2)\tau_0(b + 1). \]

4. Consequences of the channel die constraints in range II

With only systems 1, 2, 5 and 6 potentially active at the outset, the constraint equations are (from equations (3.5) to (3.7), or reduced from general eqn (6) in Havner 2007a)

\[
\begin{align*}
(d_{xx} = -\dot{e}_L) & \quad \dot{\gamma}_1 + \dot{\gamma}_2 = \sqrt{6}\dot{e}_L, \\
(d_{yy} = 0) & \quad \dot{\gamma}_5 + \dot{\gamma}_6 = (\sqrt{6}/2)\dot{e}_L(b-1), \\
(d_{xy} = 0) & \quad \dot{\gamma}_6 - \dot{\gamma}_5 = (1/4)(2-b)(\dot{\gamma}_1 - \dot{\gamma}_2), & 2 > b > 1.
\end{align*}
\]

Table 2. Designation of slip systems in FCC crystals.

<table>
<thead>
<tr>
<th>plane</th>
<th>(111)</th>
<th>(1\bar{1}1)</th>
<th>(1\bar{1}1)</th>
<th>(1\bar{1}1)</th>
</tr>
</thead>
<tbody>
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<td>direction</td>
<td>[0\bar{1}1]</td>
<td>[10\bar{1}]</td>
<td>[110]</td>
<td>[0\bar{1}1]</td>
</tr>
<tr>
<td>system</td>
<td>a1</td>
<td>a2</td>
<td>a3</td>
<td>b1</td>
</tr>
<tr>
<td>(111)</td>
<td>b2</td>
<td>b3</td>
<td>c1</td>
<td>c2</td>
</tr>
<tr>
<td>(1\bar{1}1)</td>
<td>b\bar{3}</td>
<td>c3</td>
<td>c\bar{1}</td>
<td>b\bar{2}</td>
</tr>
<tr>
<td>(1\bar{1}1)</td>
<td>a\bar{2}</td>
<td>d\bar{1}</td>
<td>d\bar{2}</td>
<td>[110]</td>
</tr>
</tbody>
</table>

4. Consequences of the channel die constraints in range II

With only systems 1, 2, 5 and 6 potentially active at the outset, the constraint equations are (from equations (3.5) to (3.7), or reduced from general eqn (6) in Havner 2007a)
As in previous papers in this series of (110) channel die analyses, we define $x = \dot{\gamma}_1 - \dot{\gamma}_2$. Equation (4.1) then may be rewritten as

$$
\begin{align*}
\dot{\gamma}_1 &= (1/2)(\sqrt{6} \dot{e}_L + x) \geq 0, \\
\dot{\gamma}_2 &= (1/2)(\sqrt{6} \dot{e}_L - x) \geq 0, \\
\dot{\gamma}_5 &= (\sqrt{6}/4) \dot{e}_L (b-1) - (x/8)(2-b) \geq 0, \\
\dot{\gamma}_6 &= (\sqrt{6}/4) \dot{e}_L (b-1) + (x/8)(2-b) \geq 0.
\end{align*}
$$

(4.2)

From the non-negative slip rates, there are two kinematic sub-ranges in range II: $2 > b > 4/3$ and $4/3 > b > 1$, with a singular intermediate orientation $b=4/3$ corresponding to $\phi = 46.69^\circ$. We have the following limits on the magnitude of $x$:

$$
2 > b \geq 4/3, \quad |x| \leq \sqrt{6} \dot{e}_L; \quad 4/3 \geq b > 1, \quad |x| \leq 2\sqrt{6} \dot{e}_L (b-1)/(2-b).
$$

(4.3)

In the first sub-range, both systems 5 and 6 ($a\bar{3}, b\bar{3}$) and at least one of systems 1 and 2 ($a2, b2$) must be active to satisfy the channel die constraints, with the converse true in the second sub-range. At $b=4/3$, only one of each pair necessarily must be active from kinematics alone. Of course, $x=0$ always satisfies the constraints, in which case $\dot{\gamma}_1 = \dot{\gamma}_2$ and $\dot{\gamma}_5 = \dot{\gamma}_6$.

5. Hardening rules and consistency conditions

From table 1 and the general hardening law, equation (2.1), the critical strength rates in the four potentially active systems in range II are

$$
\begin{align*}
(a1) \quad \dot{\tau}_1^c &= H_1 \dot{\gamma}_1 + H_V \dot{\gamma}_2 + H_I \dot{\gamma}_5 + H_{IV} \dot{\gamma}_6, \\
(b2) \quad \dot{\tau}_2^c &= H_V \dot{\gamma}_1 + H_I \dot{\gamma}_2 + H_{IV} \dot{\gamma}_5 + H_I \dot{\gamma}_6, \\
(a3) \quad \dot{\tau}_5^c &= H_I \dot{\gamma}_1 + H_{IV} \dot{\gamma}_2 + H_I \dot{\gamma}_5 + H_{III} \dot{\gamma}_6, \\
(b3) \quad \dot{\tau}_6^c &= H_{IV} \dot{\gamma}_1 + H_I \dot{\gamma}_2 + H_{III} \dot{\gamma}_5 + H_I \dot{\gamma}_6.
\end{align*}
$$

(5.1)

Thus, from the consistency conditions $\dot{\tau}_i^c \geq \dot{\tau}_k$ in critical systems (table 2) and a considerable amount of algebra, we have (with the equality satisfied in an active system)

$$
\begin{align*}
(a1) \quad (H_1 - d_1) \dot{e}_L \geq (h_1 - a_1)x, & \quad (b\bar{2}) \quad (H_1 - d_1) \dot{e}_L \geq -(h_1 - a_1)x, \\
(a\bar{3}) \quad (H_5 - d_5) \dot{e}_L \geq (h_5 - a_5)x, & \quad (b3) \quad (H_5 - d_5) \dot{e}_L \geq -(h_5 - a_5)x,
\end{align*}
$$

(5.2)

where

$$
\begin{align*}
4H_1 &= \sqrt{6}\{2H_V + (b-1)H_{IV} + (b+1)H_I\}, \\
4H_5 &= \sqrt{6}\{2H_{IV} + (b-1)H_{III} + (b+1)H_I\}, \\
4h_1 &= 2H_V - (1/2)(2-b)H_{IV} - (1/2)(b+2)H_I, \\
4h_5 &= 2H_{IV} - (1/2)(2-b)H_{III} - (1/2)(b+2)H_I.
\end{align*}
$$

(5.3)
and

\[
\begin{align*}
12a_1 &= f + gb(2-b)/(b^2 + 2), \\
3a_5 &= gb/(b^2 + 2), \\
\sqrt{6}d_1 &= f' - g'b(b-1)/(b^2 + 2), \\
\sqrt{6}d_5 &= 2g'b/(b^2 + 2), \\
f' &\equiv df/de_L, \\
g' &\equiv dg/de_L.
\end{align*}
\]

This last set of equations is equivalent to eqn (5.5) in Chidambaram & Havner (1988).

Equation (5.2) requires \( H_1 \geq d_1 \) and \( H_5 \geq d_5 \) (at least one of which must be an equality). Therefore, by combining \( 2(H_1 - d_1) + (b-1)(H_5 - d_5) \geq 0 \), we can eliminate the unknown rate-of-change \( g' \) of the constraint stress to obtain

\[
4H_V + 4(b-1)H_{IV} + (b-1)^2H_{III} + (b+1)^2H_I \geq (4/3)f', \tag{5.5}
\]

an inequality we shall make use of throughout the range.

6. A proof of uniqueness with all critical slip systems active

We consider the possibility of a non-unique solution \( x \neq 0 \) in range II with all critical slip systems active. (The latter automatically is the case in all systems for the rate-dependent equations investigated in appendix A.) With systems 1, 2, 5 and 6 active, \( H_1 = d_1 \) and \( H_5 = d_5 \) (from equation (5.2)), whence equation (5.5) is an equality. Moreover, if \( x \neq 0 \) is assumed, we further must have \( h_1 = a_1 \) and \( h_5 = a_5 \), from which (respectively; making use of equation (3.9), \( g_0b/(b^2 + 2) = f_0/(b+1) \), at the onset of the finite deformation)

\[
\begin{align*}
4(H_V - H_I) - (2-b)(H_{IV} - H_I) &= 2f_0/(b+1), \\
3(H_{IV} - H_I) - (3/4)(2-b)(H_{III} - H_I) &= 2f_0/(b+1).
\end{align*}
\]

The difference between these equations can be rearranged as

\[
4(H_V - H_{IV}) + (b-1)H_{IV} + (3/4)(2-b)H_{III} - (1/4)(2+b)H_I = 0.
\]

From the hardening inequality \( H_{IV} \geq H_{III} \), it is seen that the combined second and third terms are not less than \( (1/4)(2+b)H_{III} \). Thus, from \( H_V > H_{IV} \geq H_{III} > H_I \) (equation (2.2)) and the last equation above (a consequence of our initial assumption that \( x \neq 0 \)), we have the ordered sequence

\[
0 = 4(H_V - H_{IV}) + (b-1)H_{IV} + (3/4)(2-b)H_{III} - (1/4)(2+b)H_I \geq 4(H_V - H_{IV}) + (1/4)(2+b)(H_{III} - H_I) > 0,
\]

a contradiction. Thus, from the hardening inequalities, there is no initial solution \( x \neq 0 \) with all four slip systems active. We have only the solution \( x = 0 \), corresponding to equal slip rates within pairs, unless at least one \( \dot{\gamma}_k = 0 \) in critical systems. (Possible solutions with one or more inactive critical slip systems are analysed in appendix A.)

7. Consequences of all critical slip systems active in range II

With only systems 1, 2, 5 and 6 critical in range II, the general equations for \( \mathbf{i} \) and \( \mathbf{k} \) on lattice axes in (110) channel die compression (eqn (8) in Havner 2007a) reduce to

\[
\mathbf{i} = (\sqrt{3}/6)x(0, 0, 1), \quad \mathbf{k} = (b/2)\left\{x/\sqrt{6(b^2 + 2)}\right\}(1, 1, 0),
\]

which obviously satisfy \( \mathbf{t} \cdot \mathbf{i} = 0 \) and \( \mathbf{k} \cdot \mathbf{k} = 0 \) (from \( \mathbf{t} = (110), \mathbf{k} = (11\bar{b}) \)) as required. Thus, the only solution consistent with the hardening inequalities when all critical slip systems are active, \( x = 0 \) (from §6), gives

\[
\mathbf{i} = 0, \quad \mathbf{k} = 0,
\]

corresponding to lattice stability in range II. Henceforth, we shall consider all slip systems to be active and so adopt equation (7.2) and \( x = 0 \), with

\[
\dot{\gamma}_1 = \dot{\gamma}_2 = (\sqrt{6}/2)\dot{\ell}_\mathbf{L}, \quad \dot{\gamma}_5 = \dot{\gamma}_6 = (\sqrt{6}/4)\dot{\ell}_\mathbf{L}(b - 1),
\]

from equation (4.2). Moreover, the previously defined kinematic sub-ranges are no longer relevant, and all equations beginning with equation (7.2) apply throughout range II, \( 2 > b > 1 \). (The solution at the limit \( b = 1 \), the brass orientation, with only systems 1 and 2 active, is precisely that in Havner (2008) for the corresponding limit orientation in range III, wherein slip systems 9 and 10, rather than 5 and 6, otherwise are active.)

(a) Evolution of the constraint stress

With all systems active, \( H_1 = d_1 \) and \( H_5 = d_5 \) from equation (5.2). Consequently,

\[
f' = 3H_V + 3(b - 1)H_{IV} + (3/4)((b - 1)^2H_{III} + (b + 1)^2H_I),
\]

from equation (5.5), and we have the following equation for \( g' \) (from \( H_5 = d_5 \)):

\[
g' = (3/4)(b^2 + 2)(2H_V + (b - 1)H_{III} + (b + 1)H_I)/b. \tag{7.5}
\]

If constraint \( q \), as the loading stress \( f \), were determined experimentally, equations (7.4) and (7.5) would constitute a set of two independent equations for the four evolving moduli \( H_I, H_{III}, H_{IV} \) and \( H_V \). If, further, \( H_{IV}/H_I \) and \( H_V/H_I \) had essentially constant ratios during an interval of the finite deformation, explicit equations for the evolution of \( H_I \) and \( H_{III} \) in terms of \( f, g \) and the fixed lattice orientation could be expressed. Unfortunately, there are apparently no range II orientations for which the constraint stress has been determined experimentally. Consequently, rather than speculating at present about moduli ratios, it seems more useful to obtain a bound on the evolving constraint stress, in particular comparing it with the explicit prediction of classical Taylor hardening (following the precedent in Havner 2007a).

To this end, we first give a less simple equation for \( g' \) than equation (7.5) by subtracting \( H_5 = d_5 \) from \( H_1 = d_1 \) to obtain

\[
g'(b + 1)b/(b^2 + 2) = f' - 3\{(H_V - H_{IV}) + (1/2)(b - 1)(H_{IV} - H_{III})\}. \tag{7.6}
\]
For Taylor hardening, with all hardening moduli equal, the term in wavy brackets is a zero identity. Thus, the constraint stress \( g_T \) for Taylor hardening evolves according to (first given in Chidambaram & Havner 1988, eqn (5.15))

\[
g_T' = f'(b^2 + 2)/(b^2 + b) > f'.
\]

(7.7)

Thus, since \( b \) is constant (from lattice stability),

\[
g_T = f(b^2 + 2)/(b^2 + b) > f.
\]

(7.8)

This may also be deduced from equation (3.9), \( g_0 b/(b^2 + 2) = f_0/(b + 1) \), because the resolved shear stresses in the four active slip systems must increase equally for Taylor hardening, hence the ratio between \( g_T \) and \( f \) remains constant during deformation for a fixed lattice orientation.

We now substitute \( f' \) from equation (7.7) into equation (7.6) to obtain

\[
g' = g_T' - 3(b^2 + 2)\{(H_V - H_{IV}) + (1/2)(b - 1)(H_{IV} - H_{III})\}/(b^2 + b).
\]

(7.9)

Thus, from \( H_V > H_{IV} \geq H_{III} \) of the hardening inequalities,

\[
g < g_T = f(b^2 + 2)/(b^2 + b),
\]

(7.10)

except at the outset, when they both equal \( g_0 \). Thus, we have a comparable result as in range I \((\infty > b > 2)\) and can make an identical statement (Havner 2007a, p. 618): ‘the Taylor hardening constraint stress is an upper bound to the expected constraint stress, and always exceeds it except at the beginning of finite deformation’. (This is in contrast to range III, wherein a (compressive) constraint stress develops according to the hardening inequalities, which stress remains zero for Taylor hardening (Havner 2008, §7).) As also remarked in Havner (2007a), if the constraint stress were determined experimentally, these predictions could be assessed against experiment to either lend support or partially refute the general hardening inequalities as applied to (110) channel die compression.

(b) Finite crystal shearing

We denote angles of possible finite shear in the horizontal \((YZ)\) and vertical \((XZ)\) channel planes (figure 1) by \( \chi_x, \chi_y \) (the anticlockwise rotation about the \( x \)-axis and clockwise rotation about the \( y \)-axis of material lines initially coincident with the \( Y \) and \( X \)-directions, respectively). The correct equations for their evolution are (Havner 2007a, eqn (11); equivalent to eqn (5.15) in Havner & Chidambaram 1987)

\[
\begin{align*}
\tan \chi_x' &= 2d_{yz} + \dot{\varepsilon}_L \tan \chi_x, \\
\tan \chi_y' &= 2d_{xz} + 2\dot{\varepsilon}_L \tan \chi_y.
\end{align*}
\]

(7.11)

We have \( d_{xz} = \iota D\mu \) and \( d_{yz} = \kappa D\mu \), with

\[
\begin{align*}
\iota &= (1/\sqrt{2})(1,1,0), \\
\kappa &= (1, -1, -b)/((b^2 + 2)^{1/2}), \\
\mu &= (1/\sqrt{2})(-b, b, -2)/((b^2 + 2)^{1/2}),
\end{align*}
\]

(7.12)

in full vector form. Thus, from equations (3.6) and (7.12) and table 2, with only slip systems 1, 2, 5 and 6 active

\[ d_{xz} = -(1/4\sqrt{6})(\dot{\gamma}_1 - \dot{\gamma}_2)(b^2 + 2)^{1/2}, \quad d_{yz} = (1/2\sqrt{2})\dot{e}_L b \]  

(7.13)

(in which the equations have been simplified using the three channel die constraints, equation (4.1)).

Remarkably, from equations (7.11) and (7.13), the initial rate of shearing in the horizontal channel plane, \((\tan \chi_x) = (1/\sqrt{2})\dot{e}_L b\), is independent of the solution for \(x\) (i.e. it is independent of whether or not all four slip systems are active in the sub-ranges). Moreover, it is identical with that in range III (Havner 2008, §7).

For all systems active, \(d_{xz} = 0\) from equation (7.3), and there is no finite shearing in the vertical channel plane. Then, as the lattice does not rotate (equation (7.2)), \(b\) is constant. Therefore, in the horizontal plane (from equations (7.11) and (7.13)), \((\tan \chi_x) = (\tan \chi_x + b/\sqrt{2})\dot{e}_L\), whose solution is

\[ \tan \chi_x = (b/\sqrt{2})(\exp e_L - 1). \]  

(7.14)

This is also identical with the range III result in Havner (2008, eqn (27)). Hence, as noted therein (§7), for the common limit point of the two ranges, the brass orientation \(b = 1\) (\(Y = (1\overline{1}1), Z = [\overline{1}1\overline{2}]\)), the equation reduces to an equivalent one first derived by Chin et al. (1966b, eqn (36)), which gave excellent agreement (see their fig. 6) with experimental measurements of Chin et al. (1966a) for a Permalloy crystal. Moreover, this prediction of finite shear in the horizontal channel plane is consistent with measurements of Driver & Skalli (1982) on an aluminium crystal in the brass orientation (see their fig. 4(a)). (In contrast to ranges II and III, in range I the lattice rotates about the (110) load axis, as is well-confirmed experimentally. Hence, \(b\) changes with deformation and the solution is much less simple than equation (7.14), for which see eqns (39) and (42) in Havner 2007a.)

8. Analysis of continued lattice stability

To investigate whether the lattice orientation is predicted to remain stable, with all initially critical systems active, requires two analyses. First, the proof in §6 must be revisited because it made use of the relationship between \(f_0^*\) and \(g_0^*\) at the onset of the finite deformation, and that relationship changes with compressive strain \(e_L\) (equation (7.9)). Second, we must compare the changing resolved shear stresses (equation (3.8)) in the initially non-critical but positively stressed slip systems with their evolving critical strengths (according to table 1) to determine whether they subsequently may become critical.

(a) Investigation of a theoretical critical strain for possible loss of lattice stability

Let us assume that after some finite interval of straining there is a solution \(x \neq 0\). Then, as in §6, we must have both \(h_1 = a_1\) and \(h_5 = a_5\) from equation (5.2), which give

\[
\begin{align*}
6H_V - (3/2)\{(2 - b)H_{IV} + (2 + b)H_I\} &= f_{cr} + g_{cr}(2 - b)b/(b^2 + 2), \\
3H_{IV} - (3/4)\{(2 - b)H_{III} + (2 + b)H_I\} &= 2g_{cr}b/(b^2 + 2),
\end{align*}
\]

(8.1)

Inequalities in channel die compression

with

\[ g_{cr}(b^2 + 2)/(b^2 + b) = f_{cr} - (3/2)\bar{H}_A e_{cr}, \]
\[ \bar{H}_A \equiv 2(H_V - H_{IV}) + (b - 1)(H_{IV} - H_I) \]  

from the integration of equation (7.6) up to the critical strain \( e_{cr} \). (A bar above signifies the mean value of a modulus in the interval, and the subscript ‘A’ is a mnemonic for anisotropic. \( \bar{H}_A \) would be zero for isotropic Taylor hardening.) Upon eliminating \( f_{cr} \) and \( g_{cr} \) between equations (8.1) and (8.2), one obtains (positive from the hardening inequalities)

\[ e_{cr} = \{4(H_V - H_{IV}) + (b - 1)H_{IV} + (3/4)(2 - b)H_{III} - (1/4)(2 + b)H_I\}/H_A > 0. \]  

(8.3)

Thus, there is a theoretical critical strain at which there could be a change from continued lattice stability with all systems active. However, equation (8.3) is not yet the full solution for that strain because \( \bar{H}_A \) contains integrals of the individual moduli up to \( e_{cr} \) (i.e. the mean value of a modulus is given by \( h_a = \{\int_0^{e_{cr}} h_a(e_L) \, de_L\}/e_{cr} \).

Whereas \( h_1 = a_1 \) and \( h_5 = a_5 \) only at the critical strain, \( H_1 = d_1 \) and \( H_5 = d_5 \) from equation (5.2) at all levels of strain up to and including \( e_{cr} \). Thus, equation (7.4) for \( f' \) holds throughout, from which, upon integrating, we have

\[ f_{cr} = f_0 + \bar{h}e_{cr}, \quad \bar{h} = 3\{\bar{H}_V + (b - 1)\bar{H}_{IV} + (1/4)(b - 1)^2\bar{H}_{III} + (1/4)(b + 1)^2\bar{H}_I\}, \]  

(8.4)

where \( h \equiv df/de_L \) and \( \bar{h} = \{\int_0^{e_{cr}} h(e_L) \, de_L\}/e_{cr} \).

Although \( h_a \)'s that must be integrated to obtain \( \bar{H}_A \) and \( \bar{h} \) are, in general, nonlinear functions of the slips and so vary nonlinearly with increasing strain (thus making equations (8.3) and (8.4) complex equations for the critical value), there is experimental support for taking the ratios of the hardening moduli to be essentially constant. This is particularly the case for copper crystals, as discussed in Havner (2008, §8), based upon the classic latent hardening experiments of Jackson & Basinski (1967). Also, there appear to be only relatively small variations with deformation in silver and aluminium, from latent hardening experiments of Kocks (1964) and Kocks & Brown (1966; see discussion in appendix Aa(i) here). Henceforth, in order to obtain an approximate magnitude for \( e_{cr} \), we assume the ratios of the various moduli \( H_a \), where \( a = I, II, III, IV \) and \( V \), to the self-hardening modulus \( H_I \) to be constant and denote (from the hardening inequalities)

\[ r_a = H_a/H_I, \quad r_V > r_{IV} \geq r_{III} > r_{II} > r_I = 1. \]  

(8.5)

with all \( r \)'s constant. We further define orientation-dependent parameters

\[ r_A = 2(r_V - r_{IV}) + (b - 1)(r_{IV} - 1) > 0, \]
\[ r_E = 4(r_V - r_{IV}) + (b - 1)r_{IV} + (3/4)(2 - b)r_{III} - (1/4)(2 + b) > 0. \]  

(8.6)

Then, from equations (8.3)–(8.6), we obtain

\[ f(e_{cr}) - f_0 = (r_E/r_A)h(e_{cr}) \]  

(8.7)
as the governing equation for the critical strain at which non-uniqueness is theoretically possible in range II.

Consider the following analytical form of the compressive stress–strain curve:

\[
\begin{align*}
    f(e_L) &= f_0 + h_T e_L + (h_0 - h_T) e_0 \tanh(e_L/e_0), \\
    h(e_L) &\equiv f'(e_L) = h_T + (h_0 - h_T) \sech^2(e_L/e_0), \\
\end{align*}
\]  

(8.8)

with \( f_0, h_0, h_T \) and \( e_0 \) constants. Although originally intended for representation of uniaxial polycrystalline stress–strain curves when first suggested (without the \( f_0 \) in Havner (1966), it has been shown to well-represent experimental data for various orientations of single crystals of copper (Sue & Havner 1984) and aluminium (Havner & Chidambarrao 1987) in channel die compression and of aluminium in axial tension (Havner & Yu 2005), both with and without the \( f_0 \).

Adopting it here, equation (8.7) for the critical strain becomes

\[
h_T e_{cr} + (h_0 - h_T) e_0 \tanh(e_{cr}/e_0) = (r_E/r_A)\{h_T + (h_0 - h_T) \sech^2(e_{cr}/e_0)\}.
\]

(8.9)

To be specific, the data points of Skalli (1984) to \( e_L=1.0 \), for an aluminium crystal in a nominal (110)(112)[111] orientation in channel die compression (\( b=2 \)), are closely fitted by \( f_0=58.0, h_0=558, h_T=44.5 \) (all MPa) and \( e_0=0.0871 \) (from eqn (5.16) in Havner & Chidambarrao 1987; see also Chidambarrao 1988, fig. 8). Moreover, experimentally based values for the hardening ratios are \( r_V=1.25 \) and \( r_{IV}=r_{III}=1.2 \) (see discussion in the appendix Aa(i)), from which (at \( b=2 \)) \( r_A=0.3 \) and \( r_E=0.4 \) from equation (8.6). Thus, for aluminium at the beginning of range II (\( Y=(112) \)), equation (8.9) gives

\[
44.5 e_{cr} + 44.7 \tanh(11.48 e_{cr}) = (4/3)\{44.5 + 513.5 \sech^2(11.48 e_{cr})\},
\]

from which \( e_{cr}=0.350 \), corresponding to a shear \( \chi_x \) in the horizontal plane of 30.65° (from equation (7.14)). For an aluminium crystal in this orientation, to a strain of approximately 1.0, Skalli et al. (1983, p. 297) indicate (i) finite shear only in the horizontal plane (as here), (ii) a lattice rotation of ‘some degrees’ about the (110) load axis (evidently small from a subsequent statement), and (iii) slip only on lattice planes \( a \) and \( b \). The analysis herein permits no lattice rotation up to the theoretical critical strain of 0.35, but conceivably further analysis might permit a lattice rotation beyond that strain (which would require the \( c \)-plane to become active if \( i=0 \)). However, \( x=0 \) (hence continued lattice stability) is always a theoretically admissible solution consistent with the hardening inequalities. (Many things can contribute to small differences between theory and the finite deformation experiments, of course, including frictional effects along the channel walls not taken into account here.)

At the other limit of range II, the brass orientation \( Y=(111) \), the data points (extending to \( e_L=0.83 \)) and full-range experimental curve B in fig. 2 of Driver et al. (1994) are well represented by \( f=25+60e_L+15.2 \tan h(10.13 e_L) \) MPa (\( f_0=25.0, h_0=214, h_T=60.0 \) (all MPa), and \( e_0=0.0987 \). At \( b=1 \), \( r_A=0.1 \) and \( r_E=0.35 \) from equation (8.6). Thus with the same experimentally based values of the hardening ratios as before, we have (from equation (8.9))

\[
60 e_{cr} + 15.2 \tan h(10.13 e_{cr}) = 3.5\{60 + 154 \sech^2(10.13 e_{cr})\},
\]
which gives \( e_{cr} = 3.25 \)! This corresponds to a thickness reduction of 96%(!) and is far beyond any strain achieved in this orientation range in the experiments of Skalli et al. (1983), Skalli (1984) and Driver et al. (1994). Thus, based on the equations governing the initially active systems, we may conclude that lattice stability should continue (at least in aluminium crystals) through any realistic experimental finite deformation levels for orientations near the end of the range \((b \to 1)\). This is fully consistent with the experiments of Skalli et al. (1983), who found the small tendency towards lattice rotation about \(X = (110)\) at the other end of the range \((b \to 2)\).

For intermediate orientations in range II, it seems reasonable to consider \( e_0 \) in equation (8.9) to be of order-of-magnitude 0.1 (consistent with its empirical values of 0.0871 and 0.0987 at the ends of the range). It would also seem that we may expect \( e_{cr} \) to lie somewhere between the critical strain values calculated above. In such a case, equation (8.9) can be greatly simplified because \( \tanh (e_{cr}/e_0) \) and \( \sinh^2(e_{cr}/e_0) \) then would equal 1 and 0, respectively, to 3 or more significant figures. Thus, we have (to that approximation) the following very simple equation for \( e_{cr} \):

\[
\left( e_{cr} = (r_E/r_A) - (h_0/h_T - 1)e_0 \right). \tag{8.10}
\]

Consider \( b = 4/3 \). Then equations (8.6) and (8.10) give \( e_{cr} = 2.2 - (h_0/h_T - 1)e_0 \) or \( e_{cr} \approx 2.3 - 0.1h_0/h_T \) (with \( e_0 \approx 0.1 \)). From the values above, the empirical ratios \( h_0/h_T \) in aluminium at the beginning and end of the range are 12.5 and 3.57, respectively. Thus, even for a ratio at the high end, say 10 for simplicity, \( e_{cr} \) remains quite large, with an approximate value of 1.3, indicating that the lattice orientation would remain stable through very large deformations.

Now consider copper in \((110)\) channel die compression, again at \( b = 1 \). Sue & Havner (1984) found that \( h_0 = 168 \text{ ksi (1158 MPa)} \), \( h_T = 21.8 \text{ ksi (150 MPa)} \) and \( e_0 = 0.190 \) give extraordinarily close agreement with the experimental stress–strain curve of Wonsiewicz & Chin (1970, fig. 1, curve 4) over an 80% strain range for a copper crystal in this orientation. Also, from the classic latent hardening experiments of Jackson & Basinski (1967) on axially loaded copper crystals, appropriate values of the hardening ratios are (see discussion in Havner 2008, §9) \( r_V = 1.35 \), \( r_{IV} = 1.32 \) and \( r_{II} = 1.28 \), from which equation (8.6) give \( r_E/r_A = 5.5 \) at \( b = 1 \). Thus, from equation (8.10), \( e_{cr} = 4.23 (!) \), again indicating predicted lattice stability to any achievable finite deformation, as found by Wonsiewicz & Chin (1970) in their experiment to a strain of 0.8. (I am not aware of experimental data for copper crystals in other orientations in range II. Thus, the question of lattice stability near the beginning of the range cannot be investigated at present.)

(b) Investigation of the latent slip systems

As noted at the beginning of §8, lattice stability could be affected if initially inactive slip systems subsequently became critical. Thus, we must compare the evolving critical strengths with the changing resolved shear stresses in all positively stressed latent systems in range II.

In the active systems, from equations (5.1) and (7.3),

\[
\begin{align*}
(a1, b2) & \quad \hat{\tau}_1^c = \hat{\tau}_2^c = (\sqrt{6}/4)e_L \{(b + 1)H_I + (b - 1)H_{IV} + 2H_V\}, \\
(a3, b3) & \quad \hat{\tau}_5^c = \hat{\tau}_6^c = (\sqrt{6}/4)e_L \{(b + 1)H_I + (b - 1)H_{III} + 2H_{IV}\},
\end{align*}
\tag{8.11}
\]

and in the positively stressed latent systems, from table 1,

\[
\begin{align*}
(c1, c2) & \quad \dot{\tau}^c_{3} = \dot{\tau}^c_{4} = (\sqrt{6}/4)\dot{\varepsilon}_L\{2H_{III} + (b + 1)H_{IV} + (b - 1)H_{V}\}, \\
(d1, d2) & \quad \dot{\tau}^c_{7} = \dot{\tau}^c_{8} = (\sqrt{6}/4)\dot{\varepsilon}_L\{2H_{II} + (b + 1)H_{IV} + (b - 1)H_{V}\}, \\
(a2, b1) & \quad \dot{\tau}^c_{9} = \dot{\tau}^c_{10} = (\sqrt{6}/4)\dot{\varepsilon}_L\{(b + 1)H_{I} + 2H_{II} + (b - 1)H_{IV}\},
\end{align*}
\]

(8.12)

with the bar above 7 and 8 signifying the opposite sense systems from those defined in table 1. (From equation (3.8), the original systems 7 and 8 are negatively stressed in range II, and slip systems 11 and 12 are unstressed throughout (110) channel die compression.)

Because the lattice has not rotated, the resolved shear stress rates, equation (3.3), reduce to \(\dot{\tau}_k = m_k \dot{f} + r_k \dot{g}\) in general, and to \(\dot{\tau}_k = r_k \dot{g}\) for systems 3 through 8 (the corresponding \(m_k\)'s being zero). Then, recalling from equation (3.8) that \(\tau_5 > \tau_3\), we have \(\dot{\tau}_5 > \dot{\tau}_3\) (as the \(r_k\)'s are constant). Moreover, we see from equations (8.11) and (8.12) that (the first inequality making use of \(H_{III} > H_I\))

\[
\dot{\tau}^c_{3} - \dot{\tau}^c_{5} > (\sqrt{6}/4)\dot{\varepsilon}_L(b - 1)\{(H_{IV} - H_{III}) + (H_{V} - H_{III})\} > 0.
\]

(8.13)

Thus we have \(\dot{\tau}^c_{3} > \dot{\tau}^c_{5} = \dot{\tau}_{5} > \dot{\tau}_{3}\), whence the evolving critical strength in systems 3 and 4 is never attained by their resolved shear stress and those systems remain inactive.

For the remaining positively stressed latent systems, the resolved shear stress rates are (from equations (3.8), (7.4) and (7.5))

\[
\begin{align*}
\dot{\tau}_7 = \dot{\tau}_8 &= (\sqrt{6}/8)\dot{\varepsilon}_L(b + 1)(2 - b)\{(b + 1)H_{I} + (b - 1)H_{III} + 2H_{IV}\}/b, \\
\dot{\tau}_9 = \dot{\tau}_{10} &= (\sqrt{6}/4)\dot{\varepsilon}_L\{(b - 1)(H_{IV} - H_{III}) + 2(H_{V} - H_{IV})\}.
\end{align*}
\]

(8.14)

From the first of these and equation (8.12)\(_2\),

\[
\begin{align*}
\dot{\tau}^c_{7} - \dot{\tau}_7 &= (\sqrt{6}/8)\dot{\varepsilon}_L[2(b - 1)H_{V} + 4(b^2 - 1)H_{IV}/b - (b^2 - 1)(2 - b)H_{III}/b \\
&\quad + 4H_{II} - (b + 1)^2(2 - b)H_{I}/b].
\end{align*}
\]

(8.15)

This can be established as positive for all initial orientations in range II by substituting for \(H_{III}\) and \(H_{II}\) from the hardening inequalities \(H_{III} \leq H_{IV}\) and \(H_{II} > H_{I}\). After some algebra, one obtains

\[
\dot{\tau}^c_{7} - \dot{\tau}_7 > (\sqrt{6}/8)\dot{\varepsilon}_L(b - 1)[2H_{V} + (b + 1)(b + 2)(H_{IV}/b) + (b^2 + b + 2)(H_{I}/b)] > 0,
\]

(Q.E.D.). Thus, as with systems 3 and 4, neither can systems 7 and 8 become critical as deformation proceeds in range III.

Finally, we investigate latent systems 9 and 10. From equations (8.12)\(_3\) and (8.14)\(_2\),

\[
\begin{align*}
\dot{\tau}^c_{9} - \dot{\tau}_9 &= (\sqrt{6}/4)\dot{\varepsilon}_L[(b + 1)H_{I} + 2(H_{II} + H_{IV} - H_{V}) + (b - 1)H_{III}],
\end{align*}
\]

(8.16)

obviously positive for any but an extreme anisotropy (for which there is no experimental support) that would require \(H_{V}\) to be more than three times larger than each of the other moduli. Thus, systems 9 and 10 \((a2, b1)\), as the other latent systems, remain inactive. (For aluminium, adding \(H_{II} = 1.15H_{I}\) consistent with experiments of Kocks & Brown 1966) to the moduli ratios given in §8a,
we find $\tau_0^* - \tau_0 = (\sqrt{6}/2)\epsilon_L(1.1b + 1)H_I > 0$. For copper, adding $H_I = 1.23H_I$ (see Havner 2008, §9, p. 80) to the other ratios in §8a, we have $\tau_0^* - \tau_0 = (\sqrt{6}/2)\epsilon_L(1.14b + 1.06)H_I > 0$.)

In summary, none of the initial latent systems will become active during any finite deformation. Hence equation (8.7) from the analysis in §8a (or the particular analytical equation (8.9)) remains the governing equation for a critical strain and possible loss of lattice stability in range II.

9. Closing discussion

As has been shown, when all critical systems are active, the hardening inequalities from Havner (2005) (equation (2.2) here) uniquely predict initial lattice stability, and a consequent finite shearing only in the horizontal channel plane, for all constraint directions $Y$ between (112) and (111) (range II) in (110) channel die compression, as commonly found experimentally. Also, it is established that the lateral constraint stress predicted by the inequalities will be less than that given by classic Taylor hardening. Moreover, taking into account experimental stress–strain curves for aluminium and copper, and hardening ratios based upon classic latent hardening experiments in those metals, it is found that if there were a subsequent small lattice rotation about the (110) load axis after the finite deformation, it should be expected only near the (112) end of the range for the levels of deformation that have been obtained experimentally, consistent with experiments of Skalli et al. (1983), who reported such a rotation.

In investigating the application of the hardening inequalities in range I (constraint directions between (001) and (112)) in Havner (2007a,b) and range III (constraint directions from (111) to (110)) in Havner (2008), the assumption that all critical slip systems are active was not made. Nevertheless, it was established in those works that the hardening inequalities, together with experimental information on stress–strain curves and experimentally based hardening ratios, uniquely predicted load-axis stability throughout, with lattice rotation about the load axis only in range I, as consistently found experimentally. Consequently, it would seem in order to carry out an analysis for range II without that assumption. This is done in appendix Aa, where it is necessary to perform separate analyses of two sub-ranges, $Y$ between (112) and (334) and between (334) and (111), and of $Y = (334)$ as it is a singular orientation mathematically.

For the first sub-range, making use of the same experimental stress–strain data and hardening ratios for aluminium as before, it is again shown that lattice stability is the unique solution from the hardening inequalities, at least for this metal. However, in the narrow orientation range (only 8.05°) of the second sub-range, including $Y = (334)$ ($\phi_0 = 46.69^\circ$), it is found in appendix Aa(ii),(iii) that other initial solutions are mathematically possible, which would rotate the lattice away from a stable (110) position. However, the hardening inequalities remain consistent with lattice stability in the second sub-range, since they uniquely predict it by considering the four critical slip systems to be active.

Owing to the assumption in the main body of the paper that all critical systems are active, it also seems appropriate to investigate the standard power law form of rate-dependent theory, in which all positively stressed slip systems
automatically are active. This is done in appendix A by for (110) channel die compression. It is shown algebraically that the power law rate-dependent theory never predicts lattice stability exactly for any orientation, but that in the brass orientation \(Y = (111)\), for small values of the ‘rate-sensitivity’ parameter, as typically adopted, lattice rotation would be negligible. This also probably would be the case for other orientations in ranges II and III, but that would require specific selection of parameters and numerical studies for confirmation.

### Appendix A

(a) Investigation of other possible solutions in range II

(i) First sub-range \((2 > b > 4/3)\)

The general requirements in this sub-range are, from §4 and equation (5.2),

\[
H_i \geq d_i, \quad H_5 = d_5, \quad (h_5 - a_5)x = 0, \quad |x| \leq \sqrt{6}e_L. \quad (A\ 1)
\]

Consequently,

\[
g' b/(b^2 + 2) = (3/4)\{2H_{IV} + (b - 1)H_{III} + (b + 1)H_1\}, \quad (A\ 2)
\]

from \(H_5 = d_5\) (and equations (5.3) and (5.4)). Also, from \(H_i \geq d_i\) and equation (A 2), there again follows equation (5.5), \(4H_{IV} + 4(b - 1)H_{IV} + (b - 1)^2H_{III} + (b + 1)^2H_1 \geq (4/3)f'\).

From equation (4.2) and the preceding analysis, the only possible solutions (other than \(x = 0\)) are \(x = \pm \sqrt{6}e_L\), with either system 2 or 1 inactive as \(x\) is positive or negative. (In the remainder of this section, all equations based on speculative values will be designated sequentially by (a1), (b1), ...) For either value of \(x\), \(H_i - d_i = \sqrt{6}(h_1 - a_1) \geq 0\) from the first pair of equation (5.2); and \(h_5 = a_5\) from equation (A 1)3. Because \(x = \pm \sqrt{6}e_L\) corresponds to a rotation of the lattice away from \(i = (110)\) (see equation (7.1)), we must analyse these equations in the initial lattice position at the onset of the finite deformation, where \(g_0 b/(b^2 + 2) = f_0/(b + 1)\). Correspondingly, \(h_5 = a_5\) gives

\[
(2/3)f_0/(b + 1) = H_{IV} - (1/4)(2 - b)H_{III} - (1/4)(2 + b)H_1, \quad (a1)
\]

and after substituting equation (A 2) for \(g'\), \(H_i - d_i = \sqrt{6}(h_1 - a_1)\) (evaluated at the outset) results in (both equations (a1) and (b1) requiring \(x = \pm \sqrt{6}e_L\))

\[
(4/3)f_0' = (3b + 1)H_{IV} + (1/4)(4b^2 - 5b - 2)H_{III} + (1/4)(4b^2 + 9b + 6)H_1. \quad (b1)
\]

To investigate whether these relationships are consistent with the basic range II inequality, \(2(H_i - d_i) + (b - 1)(H_5 - d_5) \geq 0\), we eliminate \(f_0'\) between equation (b1) and equation (5.5), evaluated at the outset, to obtain

\[
4(H_{IV} - H_{IV}) + (b - 1)H_{IV} + (3/4)(2 - b)H_{III} - (1/4)(2 + b) \geq 0. \quad (c1)
\]

Upon substituting \(H_{III} \leq H_{IV}\) in the second term, this gives

\[
4(H_{IV} - H_{IV}) + (1/4)(2 + b)(H_{III} - H_1) > 0,
\]
from the hardening inequalities. Therefore, \( x = \pm \sqrt{6} \varepsilon_L \) does not violate the basic range II inequality. Thus, in contrast to range III \((1 \geq b > 0)\) investigated in Havner (2008), where it is proved that the hardening inequalities alone give \( x = 0 \) as the only admissible solution, we must return to equations (a1) and (b1) and take into account physically probable values of the variables for specific orientations in this sub-range (as also was the case for range I in Havner 2007b).

Although \( b = 2 \), precisely, is a singular position, we shall evaluate equations (a1) and (b1) for that value as a slightly different number will not significantly affect the results, and there is experimental data available for a nominal \( Y = (1\overline{1}2) \) orientation. Thus, at \( b = 2 \), these equations give

\[
(2/3)f_0 = 3(H_{IV} - H_I), \quad (4/3)f_0' = 7H_{IV} + H_{III} + 10H_I,
\]

from which we obtain the requirement (using \( H_{III} > H_I \) from the hardening inequalities)

\[
6f_0' - 7f_0 > 81H_I. \tag{e1}
\]

As noted in §8a, a very close analytical fit to the stress–strain data of Skalli (1984) for an aluminium crystal in a nominal \((110)(1\overline{1}2)[\overline{1}11]\) orientation \((b \approx 2)\), from the onset of plastic deformation to a logarithmic strain of 1, is the equation (Havner & Chidambrao 1987) \( f(\varepsilon_L) = 58.0 + 44.5\varepsilon_L + 44.7 \tanh (11.48\varepsilon_L) \) (MPa), from which \( f_0 = 58 \) and \( f_0' = 558 \) (MPa). There remains to define an experimentally based value for \( H_{IV} \) relative to \( H_I \) in order to confirm or refute inequality (e1) (required if \( x = \pm \sqrt{6} \varepsilon_L \)).

For uniaxial compression of aluminium crystals, with the load axis within the \( a2 \) triangle (as defined here) of a stereographic projection (so that slip system 1 was the only active one), Kocks (1964) gave 11 critical strength values (apparently from three crystal specimens) for cross-slip system \( b3 \) relative to system \( a2 \) as the deformation evolved. The range of the hardening ratio \( H_{IV}/H_I \) for the three groups of points (also displayed in fig. 19 of Khedro & Havner 1991) is 1.14–1.30 (the largest group), with an overall mean slope of approximately 1.25. For another cross-slip system, \( d3 \), Kocks (1964) gave 12 critical strength values relative to system \( a2 \) (again from three different crystal specimens), all of which lie close to a slope of 1.10. (Also see separate displays of all points for systems \( b3 \) and \( d3 \) in Khedro & Havner 1991; fig. 19 and 20.) Based on these values, it seems reasonable to consider \( H_{IV}/H_I \approx 1.2 \). Moreover, this is consistent with the statement of Kocks & Brown (1966, p. 87), from extensive latent hardening experiments on aluminium crystals, that all slip systems not coplanar with the primary system had ‘a flow stress raised by 10–30%’. Thus, setting \( H_{IV} = 1.2H_I \), we have \( H_I = (10/9)f_0 = 64.4 \) MPa from the first of equation (d1), using the data of Skalli (1984). Consequently, inequality (e1) gives \( 654 > 1159 \), a strong contradiction! Or, we may evaluate \( H_{III} \) from the second of equation (d1) to obtain \( H_{III} = -442 \) MPa, another impossible result. (The apparent experimental extremes, 1.1–1.3, of the relative hardening range \( H_{IV}/H_I \) in aluminium also result in negative values for \( H_{III} \) and a strong violation of inequality (e1), based on the experimental data.) Thus, specifically for aluminium, we may conclude that \( x = 0 \) is the only physically possible solution near \( b = 2 \) in range II.

Consider now the other end of this sub-range, corresponding to \( b = 4/3 \). Again, this is a singular orientation, but we shall evaluate equations (a1) and (b1) using this number because, as before, a slightly different number will not significantly
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affect the results. Thus, for $b=4/3$, equations (a1) and (b1) give

$$12f_0 = 42H_{IV} - 7H_{III} - 35H_1, \quad 24f'_0 = 90H_{IV} - 7H_{III} + 113H_1,$$

from which (again using the hardening inequality $H_{III} > H_1$)

$$6f'_0 - (45/7)f_0 > 49H_1,$$

an inequality at $b=4/3$ that must hold if $x = \pm \sqrt[3]{6} \varepsilon_L$, comparable with inequality (e1) at $b=2$ that we have shown does not hold (at least for aluminium). Then, once more adopting $H_{IV} = 1.2H_1$, we find $H_1 > (10/7)f_0$ and $f'_0 > 12.74f_0$. (At $b=2$, equations (d1) and (e1) require $H_1 > (10/9)f_0$ and $f'_0 > 16.17f_0$ for $H_{IV} = 1.2H_1$.)

I have not found experimental stress–strain results for aluminium, or any other metal, in (110) channel die compression for a constraint direction at or near $b=4/3$ (approx. the centre of orientation range II). However, in the brass orientation $Y = (111)$ at the far end of range II ($b=1$), $f_0$ and $f'_0$ can be determined from the aluminium stress–strain curve of Driver et al. (1994, fig. 2, curve ‘B’), as discussed in §8a. The results are $f_0 = 25$ and $f'_0 = 214$ (MPa), whence $f'_0/f_0 = 8.56$ at $b=1$. Recall that, at the beginning of the range ($b=2$), $f'_0/f_0 = 558/58 = 9.62$ from the aluminium data of Skalli (1984). Thus, it seems reasonable to expect that the $f'_0/f_0$ ratio is 10 or less throughout this 19.47° range. Consequently, because inequality (g1) requires $f'_0/f_0 > 12.74$ at $b=4/3$ for $H_{IV} = 1.2H_1$, we conclude for aluminium that $x=0$ is the only physically probable solution near $b=4/3$ as well.

All other orientations within the first sub-range give minimum values that $f'_0/f_0$ must exceed (for $x = \pm \sqrt[3]{6} \varepsilon_L$) between 16.17 and 12.74, each a significantly greater number than a value of 10 or less that we may expect from experiments on aluminium. (It can be shown analytically that, for $H_{IV} \geq H_{III} > H_1$ and $H_{IV} = 1.2H_1$, $f'_0/f_0$ is continuously decreasing with $b$ in the interval.) Thus, it may be concluded, based on the hardening inequalities and experimental stress–strain data, that $x=0$ is the only physically possible (initial) solution throughout the first sub-range in aluminium crystals.

For other metals, such an extreme case as $x = \pm \sqrt[3]{6} \varepsilon_L$, requiring one of the equally stressed systems $a1$ and $b2$ to be inactive, also may seem unlikely. (As established in §6, the hardening inequalities preclude any solution, other than $x=0$, with all critical systems active.) However, as with the foregoing analysis for aluminium, experimental data would be needed for any other metal to demonstrate that the hardening inequalities guarantee the experimentally expected result of equal pairwise slip, corresponding to $x=0$. (In copper crystals, e.g. the $H_{IV}/H_1$ ratio may be taken as approx. 1.3, based upon the latent hardening experiments of Jackson & Basinski (1967). However, experimental stress–strain data for copper in this sub-range is not readily found in the literature.)

(ii) Second sub-range ($4/3 > b > 1$)

In this sub-range, with systems 1 and 2 and at least one of systems 5 and 6 necessarily active, the governing equations are (again from §4 and equation (5.2))

$$H_1 = d_1, \quad H_5 \geq d_5, \quad (h_1 - a_1)x = 0, \quad |x| \leq 2\sqrt[3]{6} \varepsilon_L (b-1)/(2-b).$$

Inequalities in channel die compression

\[ H_1 = d_1 \] gives (from equations (5.3) and (5.4))

\[ g' b (b - 1)/(b^2 + 2) = f' - (3/2)\{2H_V + (b - 1)H_{IV} + (b + 1)H_I\}, \tag{A 4} \]

and equation (5.5), which we rewrite as

\[ 3H_V + 3(b - 1)H_{IV} + (3/4)\{(b - 1)^2H_{III} + (b + 1)^2H_I\} \geq f', \tag{A 5} \]

is re-established from equations (A 3)\(_2\) and (A 4).

As in the first sub-range, we investigate possible solutions other than \(x = 0\) at the outset of the finite deformation. From §4 and the analysis and proof in §6, the only non-zero solutions are \(x = \pm 2\sqrt{6\varepsilon_L}(b - 1)/(2 - b)\), with either system 5 or 6 inactive as \(x\) is positive or negative. (In this section, equations based on these speculative values will be designated (a2), (b2), etc. to distinguish them from similar equations in appendix A\(a(i)\).) Either value of \(x\) gives \((2 - b)(H_5 - d_5) = 2\sqrt{6(b - 1)(a_5 - h_5)} \geq 0\) from the second pair of equation (5.2) (note the reversal of the \(h_5, a_5\) order from that of \(h_1, a_1\) in the first sub-range). Moreover, \(h_1 = a_1\) from equation (A 3)\(_3\) if \(x \neq 0\). From this second condition and \(g_0 b/(b^2 + 2) = f_0/(b + 1)\), we have

\[ 2f_0/(b + 1) = 4H_V - (2 - b)H_{IV} - (2 + b)H_I. \tag{a2} \]

Then, substituting \(g'\) from equation (A 4), \(g_0\) in terms of \(f_0\), and equation (a2) into the first condition above, all evaluated at the outset, we obtain

\[ (2 - b)f_0' = (2 + 5b - 4b^2)H_V + (b - 1)(1 + 3b - b^2)H_{IV} + (1/2)(4 + 3b - b^3)H_I, \tag{b2} \]

with each modulus multiplier positive in the sub-range \((4/3 > b > 1)\). Both equations (a2) and (b2) are required for a solution \(x \neq 0\) in this sub-range (with either system 5 or 6 inactive).

Upon eliminating \(f_0'\) between equations (A 5) (at the outset) and (b2), one obtains

\[ (b - 1)^2\{16H_V - 4(5 - b)H_{IV} + 3(2 - b)H_{III} + (b + 2)H_I\} \geq 0, \tag{c2} \]

which must be satisfied if \(x \neq 0\). This is a zero identity in the limit \(b = 1\) (at which \(x\) goes to zero in any case); and the l.h.s. is otherwise positive in the range \(4/3 > b > 1\), based on \(H_V > H_{IV} \geq H_{III} > H_I > 0\). Therefore, the solutions \(x = \pm 2\sqrt{6\varepsilon_L}(b - 1)/(2 - b)\) in the second sub-range, with either system 5 or 6 inactive, do not violate the basic range II inequality, equation (5.5) (as similarly was the case for \(x = \pm 2\sqrt{6\varepsilon_L}\) in the first sub-range). To further investigate possible non-uniqueness in this sub-range, we again consider experimentally based values of the parameters.

As before, consider an aluminium crystal with \(H_{IV} = 1.2H_I\) and \(H_{V} = 1.25H_I\) (consistent with the experiments of Kocks & Brown (1966), already quoted). In the limiting position \(b = 4/3\), equations (a2) and (b2) give \(f_0 = 1.011H_I\) and \(f_0' = 9.07H_I\), whence \(f_0'/f_0 = 8.97\). In the second limiting position, \(b = 1\), they give \(f_0 = 0.87H_I\) and \(f_0' = 6.75H_I\), whence \(f_0'/f_0 = 8.44\), with the ratio lying between these values elsewhere in the sub-range (e.g. \(f_0'/f_0 = 8.62\) at \(b = 1.1\)). In terms of the range II experimental values discussed in appendix A\(a(i)\), these \(f_0'/f_0\) ratios would appear possible for the extreme case of either system 5 or 6 inactive at the outset.
To further assess this possibility, consider the argument adopted in Havner (2007b, p. 893) that initial values of the critical resolved shear stress \( \tau_0 \) and the hardening moduli ‘ideally should be properties of the crystal and so independent of its orientation in a channel die’, with the proviso that ‘At most they may be expected to vary only by relatively small amounts with change in initial orientation’. From equation (3.9), \( 2f_0/(b+1) = \sqrt{6}\tau_0 \) in range II. Thus, from \( h_i = a_i \) (required for \( x \neq 0 \) in the sub-range) and the resulting equation (a2), we have

\[
4H_V - (2 - b)H_{IV} - (2 + b)H_I = \sqrt{6}\tau_0.
\]

With the same relative values of \( H_V, H_{IV} \) and \( H_I \) as above, this gives a range for \( H_I/(\sqrt{6}\tau_0) \) from 1.154 (\( b=4/3 \)) to 1.25 (\( b=1 \)), a difference of 10.4% in this sub-interval of 8.05°. Perhaps this may be viewed as an acceptable ‘relatively small’ variation in what essentially should be an experimental constant for a given crystal. If so, there appears to be no argument available from the preceding analyses that, based on the hardening inequalities and experimentally based relative values of the moduli, the solution must be \( x=0 \), uniquely, in the second sub-range. There is only what seems the physical unlikelihood that one of two (initially) equally stressed slip systems, 5 and 6 (\( a_3 \) and \( b_3 \)), would be inactive in this sub-range, as required by the hardening inequalities if \( x \neq 0 \) (§6).

(iii) Orientation \( b=4/3 \) (\( \phi_0 = 46.69^\circ \))

For this orientation, from the kinematic constraints, equation (4.1), \( \dot{\gamma}_6 = (1/6)\dot{\gamma}_1 \) and \( \dot{\gamma}_5 = (1/6)\dot{\gamma}_2 \). Then, from §6, the only possible non-zero solutions are \( x = \pm \sqrt{6}\ell, \) requiring systems 2 and 5 or systems 1 and 6 to be inactive as \( x \) is positive or negative. In either case, there follow from equation (5.2): \( H_1 - d_1 = \sqrt{6}(a_1 - h_1) \geq 0 \) and \( H_5 - d_5 = \sqrt{6}(a_5 - h_5) \geq 0 \), with \( g_0 = (17/14)f_0 \) from equation (3.9). Eliminating \( g'_0 \) between these two equations, we obtain

\[
2f'_0 = f_0 + 4H_{IV} + (37/3)H_I.
\]

The inequalities \( h_1 \geq a_1 \) and \( a_5 \geq h_5 \) provide upper and lower bounds on \( f_0 \) (which is satisfied by the general hardening inequalities)

\[
(7/2)H_{IV} - (7/12)H_{III} - (35/12)H_I \leq f_0 \leq (14/3)H_V - (7/9)H_{IV} - (35/9)H_I.
\]

Thus, we have the following bounds on \( f'_0 \) for \( x \neq 0 \):

\[
22.5H_{IV} - 1.75H_{III} + 28.25H_I \leq 6f'_0 \leq 14H_V + (29/3)H_{IV} + (76/3)H_I.
\]

We again evaluate for \( H_V = 1.25H_I \) and \( H_{IV} = 1.2H_I \), representative values for aluminium as we have seen. Thus, with \( H_{IV} \geq H_{III} \), we obtain \( 8.86 \leq f'_0/H_I \leq 9.07 \), whence \( f'_0/H_I = 8.97 \) within 1.1%. The bounds on \( f_0 \) are 0.583 \( \leq f_0/H_I \leq 1.011 \), considerably less close.

With the lower found on \( f_0 \), we have \( f'_0/f_0 = 15.2 \) and with the upper bound, \( f'_0/f_0 = 8.97 \). The upper bound is within the limits 9.62–8.56 of the ratio for range II determined in appendix Aa(i) (based on experimental data of Skalli (1984) and Driver et al. (1994)). The lower bound far exceeds these limits. Thus, it cannot be unambiguously determined whether or not there can be a solution other than \( x=0 \) in the singular orientation \( b=4/3 \). Similarly as in the second sub-range, we can only assert the physical unlikelihood that two of the four equally stressed systems at the outset would be inactive, which (from the proof in §6) the hardening inequalities require if \( x \neq 0 \).

(b) Predictions in channel die compression

Although the hardening inequalities are fully consistent with lattice stability in all range II orientations, it was necessary to take all critical systems as active (§§6 and 7) for that stability to be a strict consequence of the inequalities, as seen in appendix Aa. Consequently, it seems appropriate to investigate, for comparison in (110) channel die compression, the ‘standard’ power law-type rate-dependent theory in which every positively stressed slip system is active automatically

\[ \dot{\gamma}_k = \dot{\gamma}_0 (\tau_k / \tau_k^R)^{n-1}, \]

which was introduced by Hutchinson (1976), with \( \dot{\gamma}_0 > 0 \) a ‘convenient reference creep-rate’, \( \tau_k^R \) a positive ‘reference stress’, and \( 1/n \) the positive rate-sensitivity parameter. Although Hutchinson intended his model for crystalline materials at relatively high homologous temperatures, it has been adopted by many other investigators for polycrystal analyses at ordinary temperatures. The early applications, in both crystal and polycrystal calculations were by Peirce et al. (1983), Asaro & Needleman (1985), Nemat-Nasser & Obata (1986), Molinari et al. (1987), Toth et al. (1988), Harren & Asaro (1989), Harren et al. (1989), McHugh et al. (1989), Wenk et al. (1989) and Neale et al. (1990) (for a concise review of those works, see Havner 1992, pp. 201–204). Since 1990 there have been literally hundreds more calculations using the power law form of equation (A 6). One may note, for example, (i) Wu et al. (1997), in which the hardening matrix for the \( \tau_k^R \) in Asaro & Needleman (1985) is applied to the calculation of ‘forming limit diagrams’ of aluminium alloy sheets, (ii) the review article (with results from extensive finite-element calculations) by Cuitiño & Ortiz (1992), and (iii) §6.5 of Nemat-Nasser (2004). The last two references provide further treatment of rate effects in crystals, including consideration of dislocation interactions (see also Gurtin (2006) for a continuum theory of the flow of edge and screw dislocations).

Hutchinson (1976) took the \( \tau_k^R \) to be constant at high temperatures and the same in all slip systems. Peirce et al. (1983), followed by many others (in applications at room temperature), adopted equal values at the outset but introduced hardening rules (both isotropic and anisotropic) for the ‘reference stresses’. Here, as we shall compare predictions only at the onset of the finite deformation, we also adopt an initial reference value \( \tau_0 \) that is the same in all systems. (This of course need not equal the initial ‘critical strength’ \( \tau_0 \) used throughout the rest of this paper.) Thus, we may write

\[ \dot{\gamma}_k = A_0 \tau_k |\tau_k|^{n-1}, \quad A_0 = \dot{\gamma}_0 / (\tau_0^R)^n, \]

whence from equation (3.8), for the biaxial stress state of (110) channel die compression,

\[ \begin{aligned}
\dot{\gamma}_1 &= \dot{\gamma}_2 = A_0 (1/\sqrt{6})^n \{ f - gb(b-1)/(b^2 + 2) \}^n, \\
\dot{\gamma}_3 &= \dot{\gamma}_4 = A_0 (1/\sqrt{6})^n \{ g(b+2)(b-1)/(b^2 + 2) \}^n, \\
\dot{\gamma}_5 &= \dot{\gamma}_6 = A_0 (2/\sqrt{6})^n \{ g(b/(b^2 + 2) \}^n, \\
\dot{\gamma}_7 &= \dot{\gamma}_8 = A_0 (1/\sqrt{6})^n \{ g(b+1)(b-2)/(b^2 + 2) \}^n, \\
\dot{\gamma}_9 &= \dot{\gamma}_{10} = A_0 (1/\sqrt{6})^n \{ f - gb(b+1)/(b^2 + 2) \}^n, \\
\dot{\gamma}_{11} &= \dot{\gamma}_{12} = 0.
\end{aligned} \]

with the following provisos (reflecting the absolute value expression in equation (A 7)). In ranges II and III (2 > b > 0), because systems 7, 8 are active rather than 7, 8 (as noted in §8b), (b−2) is replaced by the positive (2−b) (as also was done in equation (8.14)) and the negative \( \gamma_7, \gamma_8 \) that equation (A 8) would give are replaced by positive \( \gamma_7, \gamma_8 \). Similarly, in range III (1 ≥ b > 0), systems 3, 4 are active rather than 3, 4; (b−1) is replaced by the positive (1−b) and the negative \( \gamma_3, \gamma_4 \) from equation (A 8) are replaced by positive \( \gamma_3, \gamma_4 \).

Now consider the general form of the channel die constraints and the general equations for \( \dot{i}, \dot{k} \) when all 10 slip systems are included. These are (Havner 2007a, eqns (6) and (8)) as follows:

\[
\begin{align*}
d_{xx} &= -\dot{e}_L : \quad \dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_9 + \dot{\gamma}_{10} = \sqrt{6}\dot{e}_L, \\
d_{xy} &= 0 : \quad (b-2)(\dot{\gamma}_1 - \dot{\gamma}_2 - \dot{\gamma}_7 + \dot{\gamma}_8) + (b + 2)(\dot{\gamma}_3 - \dot{\gamma}_4 + \dot{\gamma}_9 - \dot{\gamma}_{10}) \\
&\quad - 4(\dot{\gamma}_5 - \dot{\gamma}_6) = 0, \\
d_{yy} &= 0 : \quad b(b-1)(\dot{\gamma}_1 + \dot{\gamma}_2) - (b + 2)(b-1)(\dot{\gamma}_3 + \dot{\gamma}_4) - 2b(\dot{\gamma}_5 + \dot{\gamma}_6) \\
&\quad - (b + 1)(b-2)(\dot{\gamma}_7 + \dot{\gamma}_8) + b(b + 1)(\dot{\gamma}_9 + \dot{\gamma}_{10}) = 0,
\end{align*}
\]

(A 9)

and

\[
\begin{align*}
\dot{i} &= (\sqrt{3}/6)(\dot{\gamma}_3 - \dot{\gamma}_4 + \dot{\gamma}_7 - \dot{\gamma}_8, -\dot{\gamma}_3 + \dot{\gamma}_4 - \dot{\gamma}_7 + \dot{\gamma}_8, \\
&\quad \dot{\gamma}_1 - \dot{\gamma}_2 - \dot{\gamma}_3 + \dot{\gamma}_4 + \dot{\gamma}_7 - \dot{\gamma}_8 + \dot{\gamma}_9 - \dot{\gamma}_{10}), \\
\dot{k} &= (1/\sqrt{6})[(b-1)(\dot{\gamma}_1 - \dot{\gamma}_2 + \dot{\gamma}_3 + \dot{\gamma}_4) - 2(\dot{\gamma}_5 - \dot{\gamma}_6) - (b + 1) \\
&\quad \times (\dot{\gamma}_7 + \dot{\gamma}_8 - \dot{\gamma}_9 + \dot{\gamma}_{10}), (b-1)(\dot{\gamma}_1 - \dot{\gamma}_2 - \dot{\gamma}_3 - \dot{\gamma}_4) - 2(\dot{\gamma}_5 - \dot{\gamma}_6) \\
&\quad + (b + 1)(\dot{\gamma}_7 + \dot{\gamma}_8 + \dot{\gamma}_9 - \dot{\gamma}_{10}), (b-1)(\dot{\gamma}_1 + \dot{\gamma}_2 - \dot{\gamma}_3 - \dot{\gamma}_4) \\
&\quad - 2(\dot{\gamma}_5 + \dot{\gamma}_6) - (b + 1)(\dot{\gamma}_7 + \dot{\gamma}_8 - \dot{\gamma}_9 - \dot{\gamma}_{10})]/(b^2 + 2)^{1/2}.
\end{align*}
\]

(A 10)

Thus, from the pairwise equality of slip rates in the power law form,

\[
\begin{align*}
d_{xx} &= -\dot{e}_L : \quad \dot{\gamma}_1 + \dot{\gamma}_9 = (\sqrt{6}/2)\dot{e}_L, \\
d_{xy} &= 0 : \quad 0 = 0, \\
d_{yy} &= 0 : \quad b(b-1)\dot{\gamma}_1 - (b + 2)(b-1)\dot{\gamma}_3 - 2b\dot{\gamma}_5 - (b + 1)(b-2)\dot{\gamma}_7 \\
&\quad + b(b + 1)\dot{\gamma}_9 = 0,
\end{align*}
\]

(A 11)

and

\[
\begin{align*}
\dot{i} &= 0, \quad \dot{k} = \left(2/3\right)(1/b)\{(b-1)\dot{\gamma}_3 - (b + 1)\dot{\gamma}_7\}(b, -b, 2)/(b^2 + 2)^{1/2},
\end{align*}
\]

(A 12)

in which we have made use of equation (A 11)3 to greatly simplify the \( \dot{k} \) equation. (Again, negative \( \dot{\gamma}_7 \) is replaced by positive \( \dot{\gamma}_7 \) in ranges II and III, making the second term positive; negatives \( \dot{\gamma}_3 \) and \( -b-1 \) are replaced by \( \dot{\gamma}_3 \) and \( (1-b) \) in range III; and \( b=0 \) is excluded.) Thus, the power law form predicts load-axis stability throughout (110) channel die compression (as do the hardening inequalities), which stability is well established experimentally. However, in contrast to experiments and stable lattice predictions from the hardening
inequalities in ranges II and III (for the latter, see Havner 2008, §6), the power law rate-dependent form cannot predict lattice stability algebraically in any orientation. Rather, the general form of equation (A 6) always results in a rotation of the lattice about the (110) load axis. Nevertheless, this can be extremely small, as we shall see below.

Denoting the components of the axial vector of the lattice spin tensor on the channel frame by $\omega_x$, $\omega_y$, $\omega_z$, we have (from Havner 1992, eqn (5.28))

$$
\begin{align*}
\omega_x &= -\mu \cdot \hat{k} = (2/\sqrt{3})(1/b)\{(b-1)\hat{\gamma}_3 - (b+1)\hat{\gamma}_7\} \equiv \phi, \\
\omega_y &= \mu \cdot \hat{i} = 0, \\
\omega_z &= \nu \cdot \hat{k} = -\kappa \cdot \hat{i} = 0,
\end{align*}
$$

(A 13)

with $\nu$, $\kappa$, $\mu$ given by equation (7.12). Because $\hat{\gamma}_3$ and $\hat{\gamma}_7$ cannot be zero simultaneously from equation (A 8), $\phi$ is never zero algebraically. To demonstrate that it nevertheless may be negligible, consider the transition orientation $b=1$ (the brass orientation) from range II to range III, with $\gamma_3 = 0$ and $\gamma_7 = -\gamma_7$ (as of course it is throughout both ranges). We then have $\phi = 4/3\hat{\gamma}_7 > 0$ from equation (A 13)1, $\hat{\gamma}_5 = \hat{\gamma}_7 = (2g/3f)^n\hat{\gamma}_1$ from equation (A 8) and $\hat{\gamma}_0 = \hat{\gamma}_5 + \hat{\gamma}_7$ from equation (A 11)3. Upon substituting into equation (A 11)1, we obtain

$$
\hat{\gamma}_1 = (\sqrt{6}/2)\hat{e}_L/\{1 + (2g/3f)^n\}, \quad \phi = 2\sqrt{2}\hat{e}_L/\{1 + (3f/2g)^n\}.
$$

The ratio of $f$ to $g$ is determined by substituting equation (A 8) into $\hat{\gamma}_9 = \hat{\gamma}_5 + \hat{\gamma}_7$. We find $3f = 2g(1 + 2^{1/n})$, which is approximately $4g$ for the relatively large $n$ that commonly is chosen in calculations. Therefore, in the brass orientation

$$
\phi = 2\sqrt{2}\hat{e}_L/\{1 + (1 + 2^{1/n})^n\},
$$

which approaches $2\sqrt{2}\hat{e}_L/(1 + 2^n)$ as $n$ becomes large and so becomes negligible in comparison with $\hat{e}_L$. Thus, the power law form can be numerically consistent with both experiment (see Chin et al. (1966a), for a Permalloy crystal; Wonsiewicz & Chin (1970), for copper; and Driver & Skalli (1982), Skalli et al. (1983) and Driver et al. (1994), among others, for aluminium crystals) and the hardening inequalities' unique algebraic prediction of lattice stability in this orientation.

References


Skalli, A. 1984 Etude theorique et experimentale de la deformation plastique en compression plane de cristaux d’aluminium. These d’Etat, Ecole des Mines de Saint-Etienne, France.


