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A note on the unsteady motion under gravity of a corner point on a free surface: a generalization of Stokes' theory

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In this paper, we develop a theory based on local asymptotic coordinate expansions for the unsteady propagation of a corner point on the constant-pressure free surface bounding an incompressible inviscid fluid in irrotational motion under the action of gravity. This generalizes the result of Stokes and Michell relating to the horizontal propagation of a corner at constant speed.

Keywords: Stokes wave; inviscid irrotational free surface flow; Stokes' 120° corner flow

1. Introduction

In this paper, we analyse the two-dimensional, unsteady, irrotational flow close to a propagating corner on a constant-pressure free surface bounding an incompressible inviscid fluid under the action of gravity (with surface tension neglected) via local asymptotic coordinate expansions. With the location of the corner at time $t \geq 0$ given by $\boldsymbol{r}_{\mathrm{c}}(t)$ (relative to an origin that is fixed in space) and the angle of the corner measured through the fluid given by $0 < \beta(t) < \pi$, we establish that $\beta(t) = 2\pi/3 \equiv 120°$, independent of t, while the orientation of the corner is such that the vector $\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + g\boldsymbol{j}$ *bisects* the exterior corner angle of $4\pi/3$ when drawn at the corner point, $\boldsymbol{r}_{\mathrm{c}}(t)$. As a consequence of this, the motion of the corner relative to its tip is a rigid body rotation with angular speed $\dot{\theta}_{\mathrm{c}}(t)$, where

$$\cos\theta_{\mathrm{c}}(t)\boldsymbol{i} + \sin\theta_{\mathrm{c}}(t)\boldsymbol{j} = \frac{\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + g\boldsymbol{j}}{|\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + g\boldsymbol{j}|}.$$

Here, $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors pointing horizontally and vertically upwards, respectively; g is the acceleration due to gravity; and a dot denotes $\mathrm{d}/\mathrm{d}t$. The corner point is convected with the fluid, so that the fluid velocity vector at the corner point is $\dot{\boldsymbol{r}}_{\mathrm{c}}(t)$, and we show that the fluid flow *relative to the corner point* is non-trivial if and only if $\ddot{\boldsymbol{r}}_{\mathrm{c}} \neq -g\boldsymbol{j}$. This theory generalizes the classical result of Stokes (1880) and Michell (1893; see also Whitham 1974, p. 475) relating to a corner propagating horizontally with constant speed. Rigorous

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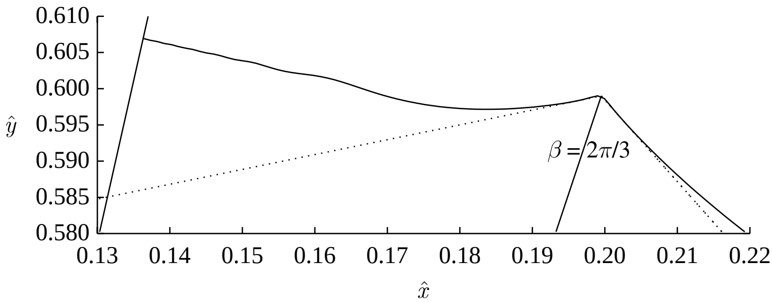

Figure 1. The straight solid line shows the direction of the local acceleration of the corner point, which bisects the corner angle (gravity is negligible in this example). The two dotted lines each meet the straight solid line at 60°. The free surface meets a flat solid wall that makes an angle of approximately 102.6° with the horizontal. Note that $\hat{x} = x/t^{\gamma}$ and $\hat{y} = y/t^{\gamma}$, with $\gamma > 1$ being a positive constant, so that, in terms of the fixed coordinates (x, y), this flow is unsteady and accelerating (for further details, see Needham *et al.* 2008).

theory for the uniform horizontal propagation of a free surface corner has since been established by, for example, Grant (1973), McLeod (1996), Toland (1996) and Fraenkel (2007). The theory we develop here is in the spirit of Stokes' approach; we construct formal, but rational, local asymptotic coordinate expansions. A fully rigorous theory for the results developed here would be the next (extremely difficult) step.

The motivation for this work comes from our recent discovery that accelerating free surface corners can be generated by the motion of an inclined wavemaker into a strip of initially stationary inviscid fluid (Needham *et al.* 2008). Fig. 12 of that paper shows a numerical solution of the small-time, inner asymptotic problem, and is reproduced here as figure 1. It shows the corner point that is formed on the free surface and accelerates away from the wavemaker. The motion of this corner point is consistent with the results of the present paper, in that the corner angle is 120° and the local acceleration vector at the corner point bisects the corner. Note that the oscillations on the free surface are a result of the upward acceleration of the fluid. A local asymptotic expansion shows that both the amplitude and the wavelength of these oscillations decrease algebraically with distance from the point of contact with the solid wall (for further details, see Needham *et al.* 2008). The solution shown in figure 1 is the generic inner solution for flows driven from an initial state of rest by an accelerating, surface-piercing, solid body. However, there are other situations, such as flows close to a sloping beach or flows driven by an upwelling of fluid, where it is possible that local accelerations could lead to the formation and propagation of a free surface corner.

2. Equations of motion

We consider the irrotational motion of an incompressible inviscid fluid under the action of gravity, bounded by a constant-pressure free surface. As for Stokes' highest wave, the local effect of surface tension at the free surface is to smooth off the corner that can form in the absence of surface tension. In the asymptotic

limit of small surface tension, this smoothing occurs in a passive inner region. At leading order, the inner solution matches with an outer solution identical to that in the absence of surface tension. We therefore neglect surface tension here.

In terms of the fixed Cartesian coordinates (x, y), with y measuring distance vertically upwards, the location of the free surface can be written parametrically as

$$\boldsymbol{r} = (x, y) = \boldsymbol{R}(s, t) = (X(s, t), Y(s, t)), \quad \text{for } -\infty < s < \infty, \ t \geq 0,$$

where s is the arc length measured along the free surface and t is the time. Our intention is to analyse the motion close to a propagating corner on the free surface, which we take to have interior angle through the fluid $\beta(t)$, with

$$0 < \beta(t) < \pi, \quad \text{for } t \geq 0. \tag{2.1}$$

The location of the corner on the free surface is

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{c}}(t) = (x_{\mathrm{c}}(t), y_{\mathrm{c}}(t)), \quad \text{for } t \geq 0,$$

so that its velocity and acceleration are, respectively,

$$\left.\begin{aligned} &\boldsymbol{v}_{\mathrm{c}}(t) = \dot{\boldsymbol{r}}_{\mathrm{c}}(t) = (\dot{x}_{\mathrm{c}}(t), \dot{y}_{\mathrm{c}}(t)), \quad \text{for } t \geq 0, \\ \text{and} \quad &\boldsymbol{a}_{\mathrm{c}}(t) = \ddot{\boldsymbol{r}}_{\mathrm{c}}(t) = (\ddot{x}_{\mathrm{c}}(t), \ddot{y}_{\mathrm{c}}(t)), \quad \text{for } t \geq 0. \end{aligned}\right\} \tag{2.2}$$

It is convenient to measure the arc length along the free surface from the corner point, so that $s=0$ at the corner point for all times $t \geq 0$. Thus,

$$\boldsymbol{R}(0, t) = (X(0, t), Y(0, t)) = (x_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) = \boldsymbol{r}_{\mathrm{c}}(t), \quad \text{for } t \geq 0$$

and

$$\boldsymbol{R}_t(0, t) = (X_t(0, t), Y_t(0, t)) = (\dot{x}_{\mathrm{c}}(t), \dot{y}_{\mathrm{c}}(t)) = \dot{\boldsymbol{r}}_{\mathrm{c}}(t), \quad \text{for } t \geq 0.$$

The unit vector normal to the free surface, pointing *out* of the fluid is

$$\hat{\boldsymbol{n}}(s, t) = (-Y_s(s, t), X_s(s, t)), \quad \text{for } -\infty < s < \infty \text{ and } t \geq 0,$$

and the unit tangent vector in the direction of increasing s is

$$\hat{\boldsymbol{t}}(s, t) = R_s(s, t) = (X_s(s, t), Y_s(s, t)), \quad \text{for } -\infty < s < \infty \text{ and } t \geq 0.$$

With this convention, the region occupied by the fluid is always to the right of the direction of increasing s on the free surface. The situation is illustrated in figure 2. Since the motion of the fluid is irrotational, the flow can be described by a velocity potential $\phi(x, y, t)$, such that the fluid velocity vector is $\nabla\phi$, and

$$\nabla^2\phi = 0, \quad \text{for } (x, y) \in D(t) \text{ and } t \geq 0, \tag{2.3}$$

where $D(t) \subset \mathbb{R}^2$ is the region occupied by the fluid for $t \geq 0$. The usual kinematic and Bernoulli boundary conditions on the free surface take the form

$$(\boldsymbol{R}_t(s, t) - \nabla\phi) \cdot \hat{\boldsymbol{n}} = 0, \quad \text{at } \boldsymbol{r} = \boldsymbol{R}(s, t), \quad \text{for } -\infty < s < \infty \text{ and } t \geq 0, \tag{2.4}$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\boldsymbol{R}(s, t) \cdot \boldsymbol{j} = 0, \quad \text{at } \boldsymbol{r} = \boldsymbol{R}(s, t), \quad \text{for } -\infty < s < \infty \text{ and } t \geq 0. \tag{2.5}$$

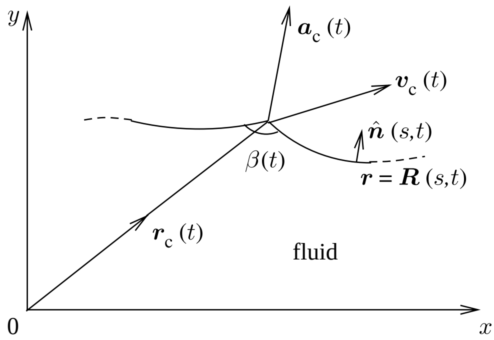

Figure 2. Definitions of the corner angle, $\beta(t)$, the corner position, velocity and acceleration, $\boldsymbol{r}_c(t)$, $\boldsymbol{v}_c(t)$ and $\boldsymbol{a}_c(t)$, respectively, the position of the free surface, $\boldsymbol{r}=\boldsymbol{R}(s,t)$, and the outward unit normal, $\hat{\boldsymbol{n}}(s,t)$.

If the flow in the region containing the corner has length scale l_c, we can define dimensionless variables as

$$x = l_c x', \quad y = l_c y', \quad t = \sqrt{\frac{l_c}{g}}t, \quad \boldsymbol{R} = l_c \boldsymbol{R}', \quad s = l_c s', \quad \phi = \sqrt{g l_c^3}\phi'.$$

Of course, l_c depends on the nature of the global flow away from the corner, but we note that the corner studied in Needham *et al.* (2008) actually develops on a length scale that grows with time. This does not affect the details of the local analysis, and is probably a typical situation for corners driven by accelerating flows.

In terms of these dimensionless variables, all definitions and equations remain the same, *except* that g must be set to unity in (2.5). We therefore do this and drop the primes for notational convenience.

3. Local analysis at the corner

At the corner point, we introduce the local polar coordinates (ρ, θ), so that

$$x - x_c(t) = \rho \cos\theta, \qquad y - y_c(t) = \rho \sin\theta,$$

with the usual unit vectors $\hat{\boldsymbol{\rho}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$. We take the edges of the corner to lie along the lines $\theta = -\delta(t)$ and $\theta = -\delta(t) - \beta(t)$, with

$$-\pi < \delta(t) \le \pi, \quad \text{for } t \ge 0. \tag{3.1}$$

As we approach the corner on the free surface, $s \to 0$, and we have

$$\hat{\boldsymbol{t}}(s,t) \sim \begin{cases} \hat{\boldsymbol{\rho}}(-\delta(t)) & \text{as } s \to 0^+, \\ -\hat{\boldsymbol{\rho}}(-\delta(t) - \beta(t)) & \text{as } s \to 0^-, \end{cases} \tag{3.2}$$

and

$$\hat{\boldsymbol{n}}(s,t) \sim \begin{cases} \hat{\boldsymbol{\theta}}(-\delta(t)) & \text{as } s \to 0^+, \\ -\hat{\boldsymbol{\theta}}(-\delta(t) - \beta(t)) & \text{as } s \to 0^-, \end{cases} \tag{3.3}$$

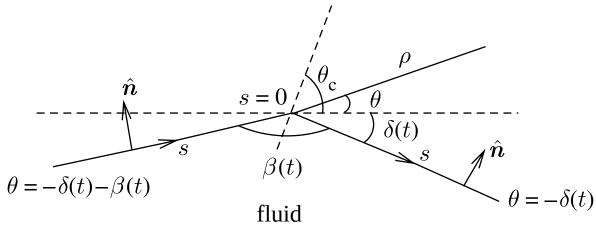

Figure 3. A sketch showing definitions of angles at the corner. Gravity acts vertically downwards, so the line $\theta=0$ is horizontal. The line $\theta=\theta_c$ bisects the corner.

together with

$$\boldsymbol{R}(s,t)=\boldsymbol{r}_c(t)+\begin{cases} s\hat{\boldsymbol{\rho}}(-\delta(t))+O(s^\alpha), & \text{for } s>0,\\ -s\hat{\boldsymbol{\rho}}(-\delta(t)-\beta(t))+O(s^\alpha), & \text{for } s<0,\end{cases} \tag{3.4}$$

as $s\to 0$ for $t\geq 0$, with $\alpha>1$ being a constant to be determined. In addition, with the requirement that the fluid velocity should be finite at the corner point,

$$\phi(\rho,\theta,t)=\phi_c(t)+\rho\hat{\boldsymbol{\rho}}(\theta)\cdot\boldsymbol{q}(t)+\rho^{\mu(t)}F(\theta,t)+O(\rho^{\gamma(t)}), \tag{3.5}$$

as $\rho\to 0$, with $-\delta(t)-\beta(t)\leq\theta\leq-\delta(t)$ for $t\geq 0$. Here, $\phi_c(t)$, the potential at the corner point, $\gamma(t)>\mu(t)>1$, and the velocity vector $\boldsymbol{q}(t)$ are to be determined. The situation is illustrated in figure 3. It now follows from (3.4) and (3.5) that:

$$\boldsymbol{R}_t(s,t)=\dot{\boldsymbol{r}}_c(t)+\begin{cases} -s\dot{\delta}(t)\hat{\boldsymbol{\theta}}(-\delta(t))+O(s^\alpha), & \text{for } s>0,\\ s(\dot{\delta}(t)+\dot{\beta}(t))\hat{\boldsymbol{\theta}}(-\delta(t)-\beta(t))+O(s^\alpha), & \text{for } s<0,\end{cases} \tag{3.6}$$

as $s\to 0$ for $t\geq 0$, and,

$$\nabla\phi(\rho,\theta,t)=\boldsymbol{q}(t)+\rho^{\mu(t)-1}\{\mu(t)F(\theta,t)\hat{\boldsymbol{\rho}}(\theta)+F_\theta(\theta,t)\hat{\boldsymbol{\theta}}(\theta)\}+O(\rho^{\gamma(t)-1}), \tag{3.7}$$

as $\rho\to 0$ with $-\delta(t)-\beta(t)\leq\theta\leq-\delta(t)$ for $t\geq 0$. It now remains to substitute from (3.3) to (3.7) into (2.3) to (2.5) (with g replaced by unity), noting that

$$\phi_t(x,y,t)=\phi_t(\rho,\theta,t)-\dot{\boldsymbol{r}}_c(t)\cdot\nabla\phi(\rho,\theta,t)$$

in (2.5).

At leading order in (2.4) and (2.5), we obtain, via (3.6) and (3.7),

$$(\dot{\boldsymbol{r}}_c(t)-\boldsymbol{q}(t))\cdot\hat{\boldsymbol{\theta}}(-\delta(t))=0, \tag{3.8}$$

$$(\dot{\boldsymbol{r}}_c(t)-\boldsymbol{q}(t))\cdot\hat{\boldsymbol{\theta}}(-\delta(t)-\beta(t))=0, \tag{3.9}$$

$$\dot{\phi}_c(t)-\boldsymbol{q}(t)\cdot\dot{\boldsymbol{r}}_c(t)+\frac{1}{2}|\boldsymbol{q}(t)|^2+\boldsymbol{r}_c(t)\cdot\boldsymbol{j}=0, \tag{3.10}$$

for $t\geq 0$. Since the unit vectors $\hat{\boldsymbol{\theta}}(-\delta(t))$ and $\hat{\boldsymbol{\theta}}(-\delta(t)-\beta(t))$ are linearly independent (via (2.1)), (3.8) and (3.9) show that

$$\boldsymbol{q}(t)=\dot{\boldsymbol{r}}_c(t),\quad \text{for } t\geq 0, \tag{3.11}$$

after which (3.10) gives

$$\dot{\phi}_{\mathrm{c}}(t) = \frac{1}{2}|\dot{\boldsymbol{r}}_{\mathrm{c}}(t)|^2 - \boldsymbol{r}_{\mathrm{c}}(t)\cdot\boldsymbol{j}, \quad \text{for } t \geq 0. \tag{3.12}$$

On using (3.6), (3.7), (3.11) and (3.12), conditions (2.4) and (2.5) become, for $t \geq 0$,

$$s^{\mu(t)-1}F_\theta(-\delta(t), t) + s\dot{\delta}(t) + O(|s|^\alpha, |s|^{\gamma(t)-1}) = 0, \quad \text{as } s \to 0^+, \tag{3.13}$$

$$(-s)^{\mu(t)-1}F_\theta(-\delta(t)-\beta(t), t) - s(\dot{\delta}(t) + \dot{\beta}(t)) + O(|s|^\alpha, |s|^{\gamma(t)-1}) = 0, \quad \text{as } s \to 0^-, \tag{3.14}$$

$$\begin{aligned} &s^{\mu(t)}F_t(-\delta(t), t) + \dot{\mu}(t)F(-\delta(t), t)s^{\mu(t)} \log s \\ &+\frac{1}{2}s^{2(\mu(t)-1)}\{\mu^2(t)F^2(-\delta(t), t) + F_\theta^2(-\delta(t), t)\} + s\hat{\boldsymbol{\rho}}(-\delta(t))\cdot(\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + \boldsymbol{j}) \\ &+O(|s|^{\gamma(t)}, \dot{\gamma}(t)|s|^{\gamma(t)} \log|s|, |s|^{\mu(t)+\gamma(t)-2}, |s|^\alpha) = 0 \quad \text{as } s \to 0^+, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &(-s)^{\mu(t)}F_t(-\delta(t)-\beta(t), t) + \dot{\mu}(t)F(-\delta(t)-\beta(t), t)(-s)^{\mu(t)} \log(-s) \\ &+\frac{1}{2}(-s)^{2(\mu(t)-1)}\{\mu^2(t)F^2(-\delta(t)-\beta(t), t) + F_\theta^2(-\delta(t)-\beta(t), t)\} \\ &-s\hat{\boldsymbol{\rho}}(-\delta(t)-\beta(t))\cdot(\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + \boldsymbol{j}) + O(|s|^{\gamma(t)}, \dot{\gamma}(t)|s|^{\gamma(t)} \log|s|, |s|^{\mu(t)+\gamma(t)-2}, |s|^\alpha) \\ &= 0, \quad \text{as } s \to 0^-. \end{aligned} \tag{3.16}$$

Recalling that $\gamma(t) > \mu(t) > 1$ and $\alpha > 1$, a balancing of dominant terms as $s \to 0$ in (3.15) and (3.16) requires that $2(\mu(t)-1) = 1$, so that

$$\mu(t) = \frac{3}{2}, \quad \text{for } t \geq 0. \tag{3.17}$$

In addition, using (3.17), a balancing of the correction terms in (3.13) and (3.14) requires that $\gamma(t) = 2$ for $t \geq 0$. Similarly, (3.15) and (3.16) show that $\alpha = 3/2$ for consistency at the orders we are considering in detail.

With (3.17), then (3.13), (3.14) (at $O(s^{1/2})$) and (3.15), (3.16) (at $O(s)$) become

$$F_\theta(-\delta(t), t) = F_\theta(-\delta(t)-\beta(t), t) = 0, \quad \text{for } t \geq 0, \tag{3.18}$$

together with

$$\hat{\boldsymbol{\rho}}(-\delta(t))\cdot(\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + \boldsymbol{j}) = -\frac{9}{8}F^2(-\delta(t), t), \tag{3.19}$$

$$\hat{\boldsymbol{\rho}}(-\delta(t)-\beta(t))\cdot(\ddot{\boldsymbol{r}}_{\mathrm{c}}(t) + \boldsymbol{j}) = -\frac{9}{8}F^2(-\delta(t)-\beta(t), t), \tag{3.20}$$

for $t \geq 0$. Now, substitution from (3.5) into (2.3) requires that

$$F_{\theta\theta} + \frac{9}{4}F = 0 \quad \text{in } -\delta(t) - \beta(t) < \theta < -\delta, \quad \text{for } t \geq 0. \tag{3.21}$$

The solution of (3.21) that satisfies $(3.18)_1$ is

$$F(\theta, t) = A(t)\cos\frac{3}{2}(\theta + \delta(t)), \quad \text{for } t \geq 0, \tag{3.22}$$

with $A(t)$ a function that is to be determined. Applying condition $(3.18)_2$, then requires that

$$\sin\frac{3}{2}\beta(t) = 0, \quad \text{for } t \geq 0,$$

which, via (2.1), means that we must have

$$\beta(t) = \frac{2}{3}\pi \equiv 120°, \quad \text{for } t \geq 0. \tag{3.23}$$

Finally, conditions (3.19) and (3.20), with (3.22) and (3.23), become

$$\hat{\boldsymbol{\rho}}(-\delta(t))\cdot(\boldsymbol{a}_{\mathrm{c}}(t) + \boldsymbol{j}) = \hat{\boldsymbol{\rho}}\left(-\delta(t) - \frac{2}{3}\pi\right)\cdot(\boldsymbol{a}_{\mathrm{c}}(t) + \boldsymbol{j}) = -\frac{9}{8}A^2(t), \quad \text{for } t \geq 0, \tag{3.24}$$

with $\boldsymbol{a}_{\mathrm{c}}(t)$ the vector acceleration of the corner point, defined in (2.2). The conditions (3.24) establish geometrically that the orientation of the corner on the free surface is such that when the vector $\boldsymbol{a}_{\mathrm{c}}(t)+\boldsymbol{j}$ is drawn at the corner point, it points out of the fluid and bisects its exterior angle. In terms of the local polar coordinates (ρ, θ), we can write, for $t \geq 0$,

$$\boldsymbol{a}_{\mathrm{c}}(t) + \boldsymbol{j} = \rho_{\mathrm{c}}(t)\{\cos\theta_{\mathrm{c}}(t)\boldsymbol{i} + \sin\theta_{\mathrm{c}}(t)\boldsymbol{j}\}, \tag{3.25}$$

with

$$\rho_{\mathrm{c}}(t) = |\boldsymbol{a}_{\mathrm{c}}(t) + \boldsymbol{j}| \geq 0. \tag{3.26}$$

It then follows from (3.24) and (3.25) that

$$\delta(t) = \frac{2}{3}\pi - \theta_{\mathrm{c}}(t), \quad \text{for } t \geq 0 \tag{3.27}$$

and

$$A(t) = \pm\frac{2}{3}\sqrt{\rho_{\mathrm{c}}(t)}, \quad \text{for } t \geq 0. \tag{3.28}$$

Thus, as the vector $\boldsymbol{a}_{\mathrm{c}}(t)+\boldsymbol{j}$ will, in general, change with time, the corner performs a rigid body rotation relative to its tip, with angular speed $\dot{\theta}_{\mathrm{c}}(t)$. It should also be noted via (3.5), (3.11) and (3.17) that

$$\nabla\phi(x_{\mathrm{c}}(t), y_{\mathrm{c}}(t), t) = \dot{\boldsymbol{r}}_{\mathrm{c}}(t), \quad \text{for } t \geq 0, \tag{3.29}$$

so that the corner point in the free surface is convected with the fluid. In addition, via (3.5), (3.11), (3.17), (3.22) and (3.28), it follows that the flow *relative to the corner point* is non-trivial if and only if $\boldsymbol{a}_{\mathrm{c}}(t) \neq -\boldsymbol{j}$.

4. Summary

We have considered the dynamical motion of a corner point of interior angle measured through the fluid $0<\beta(t)<\pi$, located at $\boldsymbol{r}_{\mathrm{c}}(t)$ relative to a fixed origin, and constructed local asymptotic coordinate expansions which demonstrate that, for all $t\geq 0$,

(i) the interior angle of the corner point measured through the fluid is $\beta(t)=2\pi/3\equiv 120°$,
(ii) the orientation of the corner is such that the vector $\ddot{\boldsymbol{r}}_{\mathrm{c}}(t)+\boldsymbol{j}$ bisects the *exterior* corner angle of $4\pi/3\equiv 240°$,
(iii) the motion of the corner, relative to its tip, is a rigid body rotation with angular speed $\dot{\theta}_{\mathrm{c}}(t)$, where

$$\cos\theta_{\mathrm{c}}(t)\boldsymbol{i}+\sin\theta_{\mathrm{c}}(t)\boldsymbol{j}=\frac{\ddot{\boldsymbol{r}}_{\mathrm{c}}(t)+g\boldsymbol{j}}{|\ddot{\boldsymbol{r}}_{\mathrm{c}}(t)+g\boldsymbol{j}|}.$$

(iv) the corner point is convected with the fluid, so that the fluid velocity vector at the corner point is $\dot{\boldsymbol{r}}_{\mathrm{c}}(t)$, and
(v) the fluid flow in the corner *relative to the corner point* is non-trivial if and only if $\ddot{\boldsymbol{r}}_{\mathrm{c}}(t)\neq -g\boldsymbol{j}$.

Finally, we observe that when the corner point moves with uniform velocity, $\ddot{\boldsymbol{r}}_{\mathrm{c}}(t)=\boldsymbol{0}$, and so via (ii), the orientation of the corner is such that $\boldsymbol{j}$ bisects its exterior angle. Thus, the corner must be symmetrically disposed about the upward vertical, which reproduces Stokes' result for steadily propagating waves with corners.

To conclude, we remark that, having demonstrated that an unsteady corner is able to propagate on a free surface under gravity, and having examined the structure of such a propagating corner, it is natural to ask whether such a corner can be realized in practice; in other words, are unsteady corner structures temporally stable to small spatial perturbations? It has been demonstrated in detail by Longuet-Higgins & Cleaver (1994) and Longuet-Higgins & Tanaka (1997) that a Stokes corner propagating horizontally with constant speed is temporally unstable, and we might expect that any constant velocity corner will also be temporally unstable. However, it is unclear at present whether or not the effect of particular accelerations may stabilize the corner structure, much in the way that particular accelerations of the pivot point of a rigid pendulum under gravity can stabilize the usually unstable, vertically upward equilibrium. This aspect of the motion of free surface corners is currently under consideration. Naturally, another major and imperative step forward would involve putting the formal, but rational, theory presented here on a fully rigorous basis. That is, can this local structure be realized as part of the solution of an appropriate initial boundary-value problem in a rigorous framework? In this context, we believe that the local structure exhibited here provides valuable evidence for a conjecture in this direction.

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