An asymptotic model for the formation and evolution of air gaps in vertical continuous casting

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The formation of an air gap at the mould–metal interface in vertical continuous casting has long been known to have a detrimental effect on the efficiency of the process, and has therefore attracted attempts at mathematical modelling. While almost all current efforts consist of complex three-dimensional numerical simulations of the phenomenon, this paper considers instead an asymptotic model that captures the essential characteristics. The model is thermomechanical and is derived for a geometry, where the generalized plane strain approximation is appropriate. Although two-way coupling between the thermal and mechanical problems is accounted for, it is found that the problems decouple at leading order anyway, and that the thickness of the air gap does not depend on the constitutive relation used for describing the inelastic strains. Furthermore, a criterion for the onset of air-gap formation is derived in terms of the process operating parameters. Mathematically, we obtain a moving boundary problem for a parabolic partial differential equation with a degenerate initial condition and a non-standard Neumann-type boundary condition. Sample computations are performed using parameters for the continuous casting of the copper, and the results, qualitative trends and possible extensions are discussed.

Keywords: continuous casting; air-gap formation; asymptotics

1. Introduction

Air-gap formation in the industrial continuous casting of metals and metal alloys has long been recognized as having an adverse effect on process efficiency. A two-dimensional schematic of the situation is given in figure 1, which shows molten metal, typically copper, aluminium or steel alloys, passing vertically downwards through a cooled mould, solidifying and being withdrawn at casting speed, $V_{\text{cast}}$. At each mould wall, there is typically first a region where liquid metal is in contact with the mould wall, followed by a region where the solidified shell is in contact; after this, at $z=z_{\text{gap}}$, an air gap begins to form between the solidified shell and the mould wall. Eventually, at some location $z=z_{\text{mid}}$, complete solidification occurs at the centreline. In particular, the formation of the air gap

Received 15 November 2008
Accepted 29 January 2009

prohibits effective heat transfer between the mould and shell, leading to longer solidification lengths and requiring supplementary process design considerations, such as mould tapering.

As previously stated by several authors (Schwerdtfeger et al. 1998; Kron et al. 2004, 2005; Fredriksson & Åkerlind 2006), the air gap is thought to form when the solid metal shell, in going from a plastic to elastic state, is strong enough to withstand the metallostatic pressure of the adjacent molten metal, thereby receding from the mould as a result of contraction; a contributing factor is also thought to be the expansion of the mould itself (Kim 2003). Prior to this, when the applied pressure dominates, no air gap can form. In view of the detrimental effect that the air gap has on process efficiency, mathematical models of varying degrees of complexity have been derived to describe the phenomenon. Early analytical models for predicting the onset of air-gap formation, although not its evolution, were due to Savage (1962), Richmond & Tien (1971), Kristiansson (1982) and Tien & Richmond (1982). Most subsequent models have been numerical and have adopted different approaches for handling the thermomechanical interaction at the air gap.

(i) A heat transfer coefficient is assumed for the thermal calculation, and the temperature field that is computed is then used to determine the displacement of the solid strand from the mould (Schwerdtfeger et al. 1998); there therefore no back-coupling from the mechanical to the thermal problem.

(ii) Experiments are carried out for a particular solidification process in order to correlate the heat transfer coefficient with the air gap width (Trovant & Argyropoulos 1998, 2000; Kron et al. 2004); this heat transfer coefficient is then used for the thermal calculation.

(iii) Complete two-way coupling between thermal and mechanical calculations is allowed for, so that the thermal boundary conditions are dependent on the surface displacements and stresses of the solidifying shell, and vice versa.

Figure 1. Two-dimensional schematic of air-gap formation.
While approach (iii) is no doubt most representative of the actual thermomechanical interaction that occurs in the course of air-gap formation, it is computationally expensive and unwieldy: for example, none of the models under (iii) has given the qualitative understanding of the air gap’s dependence on different operating parameters, or indeed whether it is possible to avoid air-gap formation completely. The alternative approach that we suggest here is to use asymptotic methods to reduce the complexity of the problem, although without sacrificing any physical features; the reason why this appears to be possible is because the thickness of the air gap is much smaller than all other length scales in the problem. There are several benefits to adopting the asymptotic route: for initial design purposes, to gain insight prior to more general and expensive simulations; for use as a benchmark for testing more general formulations; for use in online control systems. Furthermore, the model presented here can serve as the basis for input to other asymptotic models for casting processes (Bland 1984; Hill & Wu 1994; DiLellio & Young 1995; Smelser & Johnson 1995; Johnson & Cherukuri 1999; Cherukuri & Johnson 2001). There is also an experimental perspective to this model, and approach (iii) in general, which is worth highlighting. While there exist numerous experimental methods for determining the appropriate boundary conditions that are necessary for modelling via approach (i), e.g. thermocouples embedded into the mould wall (Åberg et al. 2005b; Aberg & Fredriksson 2007), these require two measurements at any particular z-location in order to determine a suitable heat transfer coefficient. However, having a model for air-gap formation and evolution means that we require only one measurement at each z-location; furthermore, there is then the possibility to compare approaches (i), (ii) and (iii).

The layout of the paper is as follows. In §2, we formulate the appropriate thermomechanical problem for a geometry in which the generalized plane strain approximation holds. In particular, we discuss the appropriate condition for describing the onset of air-gap formation, as well as how to incorporate air-gap evolution. In §3, we non-dimensionalize the governing equations, perform asymptotic reduction and derive a criterion for air-gap formation in terms of the operating parameters. From the asymptotics, we see that the thermal and mechanical problems decouple, and that the thickness of the air gap is independent of the constitutive relation chosen to describe the inelastic strains. In §4, we describe briefly some of the numerical aspects of the problem, and in §5 a finite-element method is used to solve the governing equations; by this stage, these have been reduced to a moving boundary problem for a parabolic partial differential equation (PDE) with degenerate initial conditions and a non-standard Neumann-like boundary condition. Conclusions are drawn in §6.

2. Mathematical formulation

We consider a steady-state, two-dimensional problem, as already shown in figure 1, in which pure liquid metal at its melting temperature, $T_{\text{melt}}$, enters a mould region at $z=0$, solidifies and is withdrawn at a casting speed $V_{\text{cast}}$; the
extension to the case where the liquid metal is at a temperature greater than its melting temperature will be considered elsewhere, but the working assumption that we use allows us to take \( y_m(0) = 0 \) and thereby to avoid extraneous details that are not essential here. Subsequently, an air gap starts to form at the inner mould surface at \( z = z_{\text{gap}} \). For \( 0 < z < z_{\text{gap}} \), solidification occurs in the region \( 0 < y < y_m(z) \), whereas for \( z > z_{\text{gap}} \), air occupies \( 0 < y < y_a(z) \) and solid occupies \( y_a(z) < y < y_m(z) \). Eventually, after complete solidification has occurred at \( z = z_{\text{mid}} \), the solid region occupies \( y_a(z) < y < W \); due to symmetry, we need to consider only the left-hand side of figure 1.

(a) Heat transfer

The heat transfer aspects are as follows. For \( 0 < z < z_{\text{gap}} \) and \( 0 < y < y_m(z) \), and then \( z > z_{\text{gap}} \) and \( y_a(z) < y < y_m(z) \), we have

\[
\rho c_{ps} V_{\text{cast}} \frac{\partial T_s}{\partial z} = k_s \frac{\partial^2 T_s}{\partial y^2}, \tag{2.1}
\]

where \( k_s \) is the thermal conductivity of the solid metal; \( c_{ps} \) is its specific heat capacity; and \( \rho \) its density. In equation (2.1), we use the fact that casting geometries are often slender, which motivates us to assume that \( \partial^2 / \partial z^2 \ll \partial^2 / \partial y^2 \).

For boundary conditions at \( y = y_m(z) \), we have

\[
T_s = T_{\text{melt}}, \tag{2.2}
\]

and the Stefan condition,

\[
k_s \frac{\partial T_s}{\partial y} = \rho \Delta H_l V_{\text{cast}} \frac{dy_m}{dz}, \tag{2.3}
\]

where \( \Delta H_l \) is the latent heat of fusion; this form for (2.3) also makes use of the fact that the geometry is slender and that there is no heat flux in the liquid phase. However, once solidification is complete, at \( z = z_{\text{mid}} \), we treat \( y = W \) as a symmetry axis, so that (2.2) and (2.3) are replaced by just

\[
\frac{\partial T_s}{\partial y} = 0. \tag{2.4}
\]

At the left-hand surface of the solid phase, separate considerations are necessary for \( 0 < z < z_{\text{gap}} \) and \( z > z_{\text{gap}} \). For \( 0 < z < z_{\text{gap}} \), where the solid shell is in contact with the mould, it is reasonable to assume continuity of temperature and heat flux, i.e. at \( y = 0 \),

\[
T_s = T_{M}, \quad k_s \frac{\partial T_s}{\partial y} = k_M \frac{\partial T_M}{\partial y}, \tag{2.5}
\]

where \( k_M \) is the thermal conductivity of the mould. In addition, assuming only heat conduction in the slender mould, we have

\[
\frac{\partial^2 T_M}{\partial y^2} = 0 \quad \text{for} \quad -H_M \leq y \leq 0, \tag{2.6}
\]
subject to
\[ T_M = T_o(z) \quad \text{at} \quad y = -H_M, \quad (2.7) \]
where \( T_o(z) \) is the experimentally measurable temperature at the outer surface of the mould; in practice, measurements may also be made by means of thermocouples located within the mould itself (Åberg et al. 2005b; Åberg & Fredriksson 2007). So,
\[ T_M(y, z) = T_o(z) + (y + H_M)A_M(z), \quad (2.8) \]
where \( A_M(z) \) is to be determined; substituting into (2.5), we arrive at
\[ k_a \frac{\partial T_a}{\partial y} = \frac{k_M}{H_M} (T_s - T_o(z)) \quad \text{at} \quad y = 0. \quad (2.9) \]
For \( z > z_{\text{gap}} \), on the other hand, if we assume heat conduction only in the air gap and continuity of temperature and heat flux at the mould–air and air–shell interfaces, we have
\[ \frac{\partial^2 T_a}{\partial y^2} = 0 \quad \text{for} \quad 0 \leq y \leq y_a(z), \quad (2.10) \]
with additional interface conditions
\[ T_a = T_M, \quad k_a \frac{\partial T_a}{\partial y} = k_M \frac{\partial T_M}{\partial y} \quad \text{at} \quad y = 0 \quad (2.11) \]
and
\[ T_a = T_s, \quad k_a \frac{\partial T_a}{\partial y} = k_s \frac{\partial T_s}{\partial y} \quad \text{at} \quad y = y_a(z), \quad (2.12) \]
where \( k_a \) is the thermal conductivity of air. Now, using (2.8) and
\[ T_a(y, z) = B_a(z) + yA_a(z), \quad (2.13) \]
as the general form of the solution to (2.10), boundary conditions (2.11) and (2.12) lead to
\[ T_o(z) + H_M A_M(z) = B_a(z), \quad k_a A_a(z) = k_M A_M(z), \]
\[ B_a(z) + y_a(z)A_a(z) = (T_s)_{y = y_a}, \]
giving
\[ A_M(z) = \frac{k_a \{(T_s)_{y = y_a} - T_o(z)\}}{k_a H_M + y_a(z)k_M}, \quad A_a(z) = \frac{k_M \{(T_s)_{y = y_a} - T_o(z)\}}{k_a H_M + y_a(z)k_M}, \]
\[ B_a(z) = T_o(z) + \frac{H_M k_a \{(T_s)_{y = y_a} - T_o(z)\}}{k_a H_M + y_a(z)k_M}. \]
Following straightforward manipulation to eliminate $T_a$, we obtain

$$k_s \frac{\partial T_s}{\partial y} = \frac{1}{\frac{H_M}{k_M} + \frac{y_a(z)}{k_a}}(T_s - T_o(z)) \quad \text{at} \quad y = y_a(z). \quad (2.14)$$

Whether conduction is the only heat transfer mechanism in the air gap has been the subject of some debate (Ho & Pehlke 1984; Nishida et al. 1986); if it is not the dominating mechanism, then we would need to replace (2.10) with something else and the subsequent manipulation leading to (2.14) would not hold. We assume here that conduction is the dominating mechanism, since this gives a closed form for the required boundary condition, although we note that other possibilities, e.g. convection and radiation, can be introduced in systematic fashion. Of use to process engineers is the heat transfer coefficient, $h$, between the solidified shell and the mould, defined in dimensional variables from the equation

$$k_s \left( \frac{\partial T_s}{\partial y} \right)_{y=y_a} = h((T_s)_{y=y_a} - T_M(0, z)); \quad (2.15)$$

retracing the manipulation that led to (2.14), we arrive at

$$h = \frac{k_s}{y_a(z)}. \quad (2.16)$$

Since governing equation (2.1) is a parabolic PDE, an initial condition for $T_s$ is necessary at $z=0$. As this is where solidification starts, most appropriate is

$$T_s = T_{\text{melt}}. \quad (2.17)$$

In addition, we must have

$$y_m(0) = 0, \quad (2.18)$$

i.e. the solid phase is initially of zero thickness. Consequently, the problem is initially degenerate; we demonstrate later how to deal with this potential difficulty.

A final comment concerns the form of the boundary conditions (2.9) and (2.14). Since this study adopts the simplest possible approach in order to ensure analytical tractability, using the temperature at $y = -H_M$ appears at first sight not to be the simplest; this would instead be a constant temperature condition at $y=0$, as used for the thermomechanical analysis of Weiner & Boley (1963), for example. There turn out to be at least two reasons for our approach here: the first is simply that the inner mould surface is not isothermal in practice; the second, and somewhat deeper, reason is that such a boundary condition would not give rise to air-gap formation, i.e. the criterion for air-gap formation, which we introduce in §2b, would never be satisfied. More details of this are given in appendix A in the electronic supplementary material; for a more sophisticated treatment of the appropriate boundary condition, although without consideration of air-gap formation, see the models for the vertical continuous casting of steel by Bland (1984), Hill & Wu (1994) and DiLellio & Young (1995).
In Cartesian coordinates in three dimensions, the equilibrium of forces is given by

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{2.19}
\]

\[
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{2.20}
\]

and

\[
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \sigma_z}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0, \tag{2.21}
\]

where \(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}\) are the stress components, with \(\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}\). In continuous casting, the generalized plane strain approximation is often invoked. Although there are different definitions for this in literature (Babuška & Szabó 2006), here, we take it to mean that the length scale in the \(z\)-direction is much greater than that in the \(x\)- and \(y\)-directions, which enables us to reduce (2.19)–(2.21) to

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0; \tag{2.22}
\]

however, we still allow for the possibility for expansion in the \(z\)-direction and take \(\varepsilon_z = \varepsilon_z(z)\), where \(\varepsilon_z\) is the longitudinal strain. In addition, if we exclude bending, we can set \(\tau_{xy} = \tau_{yz} = 0\), as in Weiner & Boley (1963) and Kristiansson (1982), leaving just

\[
\frac{\partial \sigma_x}{\partial x} = 0, \quad \frac{\partial \sigma_y}{\partial y} = 0; \tag{2.23}
\]

there is, however, axial stress, i.e. \(\sigma_z \neq 0\). However, as discussed in detail by Jablonka (1995) and Schwerdtfeger et al. (1998), a more convenient form for these when considering a body translating in the \(z\)-direction is

\[
\frac{\partial \sigma_x}{\partial x} = 0, \quad \frac{\partial \sigma_y}{\partial y} = 0, \tag{2.24}
\]

where the dots denote differentiation with respect to \(z\).

The stress components are then related to the elastic strain rates—\(\varepsilon_x^{\text{el}}, \varepsilon_y^{\text{el}}, \varepsilon_z^{\text{el}}\)—through

\[
\begin{bmatrix}
\dot{\sigma}_x \\
\dot{\sigma}_y \\
\dot{\sigma}_z
\end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & \nu \\
\nu & 1 - \nu & \nu \\
\nu & \nu & 1 - \nu
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^{\text{el}} \\
\varepsilon_y^{\text{el}} \\
\varepsilon_z^{\text{el}}
\end{bmatrix}. \tag{2.25}
\]

We will assume that thermal (\(\varepsilon_x^{\text{th}}, \varepsilon_y^{\text{th}}, \varepsilon_z^{\text{th}}\)), elastic and inelastic strains (\(\varepsilon_x^{\text{i}}, \varepsilon_y^{\text{i}}, \varepsilon_z^{\text{i}}\)) occur simultaneously and that they are additive, so that

\[
\varepsilon_j = \varepsilon_j^{\text{el}} + \varepsilon_j^{\text{th}} + \varepsilon_j^{\text{i}}, \quad j = x, y, z, \tag{2.26}
\]
where \( \varepsilon_x \) and \( \varepsilon_y \) are strain components in the \( x \)- and \( y \)-direction, respectively. With \( \dot{\varepsilon}_x = \dot{\varepsilon}_y = \dot{\varepsilon}_z = \alpha \dot{T} \), equations (2.25) and (2.26) are then summarized by

\[
\begin{bmatrix}
\dot{\sigma}_x \\
\dot{\sigma}_y \\
\dot{\sigma}_z
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu \\
\nu & 1-\nu & \nu \\
\nu & \nu & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\dot{\varepsilon}_x - \dot{\varepsilon}_x \\
\dot{\varepsilon}_y - \dot{\varepsilon}_y \\
\dot{\varepsilon}_z - \dot{\varepsilon}_z
\end{bmatrix} - \frac{E\alpha \dot{T}}{(1-2\nu)}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

(2.27)

where \( E \) is the Young modulus; \( \nu \) is the Poisson ratio; and \( \alpha \) is the thermal expansion coefficient, and the strain components are in turn related to the displacements, \( u, v, w \), by

\[
\dot{\varepsilon}_x = \frac{\partial \dot{u}}{\partial x}, \quad \dot{\varepsilon}_y = \frac{\partial \dot{v}}{\partial y}, \quad \dot{\varepsilon}_z = \frac{\partial \dot{w}}{\partial z}.
\]

(2.28)

Note that, in writing (2.27), we have implicitly assumed that \( E \) and \( \alpha \) are independent of temperature. Using the expressions for copper given by Freed (1988),

\[
E = 127 \, 000 - 41 \, T_s - 0.027 \, T_s^2 \, \text{MPa},
\]

(2.29)

\[
\alpha = 1.6 \times 10^{-5} + 5 \times 10^{-9} \, T_s \, \text{K}^{-1},
\]

(2.30)

over the temperature interval suggested by the computed results of Mahmoudi (2006), \( \sim 1250 \, \text{K} \leq T_s \leq 1356 \, \text{K} \), implies that \( E \) will decrease from \( 3.5 \times 10^{10} \, \text{Pa} \) to \( 2 \times 10^{10} \, \text{Pa} \), whereas \( \alpha \) will increase from \( 2.2 \times 10^{-5} \, \text{K}^{-1} \) to \( 2.3 \times 10^{-5} \, \text{K}^{-1} \); for simplicity, and to ensure analytical tractability, we keep \( E \) and \( \alpha \) constant in this paper and use the values given in Table 1, which are themselves taken from earlier computations for the vertical continuous casting of copper (Mahmoudi et al. 2001, 2003; Mahmoudi 2006). Furthermore, strain compatibility conditions give that, for this geometry (Weiner & Boley 1963; Kristiansson 1982)

\[
\dot{\varepsilon}_x = \dot{\varepsilon}_x = \dot{\varepsilon}(z), \quad \dot{\varepsilon}_y = \dot{\varepsilon}_y(y, z),
\]

(2.31)

\[
\delta_x = \delta_x = \delta(y, z), \quad \delta_y = \delta_y(z);
\]

(2.32)

for a detailed discussion of the derivation of these, see Boley & Weiner (1997), pp. 87 and 527. In addition, if we use the empirical strain hardening equation for viscoplastic strains in the form suggested by Schwerdtfeger et al. (1998) for \( \dot{\varepsilon}_y \), we find that

\[
\dot{\varepsilon}_x = \dot{\varepsilon}_y = \dot{\varepsilon}(z), \quad \dot{\varepsilon}_y = \dot{\varepsilon}_y(y, z),
\]

(2.33)

since \( \dot{\varepsilon}_x \) and \( \dot{\varepsilon}_z \) are both the same function of \( \sigma_y(z) \) and \( \dot{\varepsilon}_y \) is a function of \( \sigma(y, z) \). A similar conclusion results if we instead use Norton’s law for secondary creep (Kristiansson 1982), and so we choose to proceed on the assumption that (2.33) holds; for our purposes, it turns out that we do not need to specify them, as far as computing the width of the air gap is concerned. Thence, equation (2.27), combined with (2.31) and (2.32), can be reduced to
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Table 1. Parameters for computations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{ps}$</td>
<td>485</td>
<td>J kg$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$E$</td>
<td>$3 \times 10^{10}$</td>
<td>N m$^{-2}$</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81</td>
<td>m s$^{-2}$</td>
</tr>
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<td>$H_M$</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>$k_a$</td>
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<td>W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$k_M$</td>
<td>160</td>
<td>W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$k_s$</td>
<td>335</td>
<td>W m$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$L$</td>
<td>0.38</td>
<td>m</td>
</tr>
<tr>
<td>$T_{melt}$</td>
<td>1356</td>
<td>K</td>
</tr>
<tr>
<td>$T_{\min}^o$</td>
<td>323</td>
<td>K</td>
</tr>
<tr>
<td>$V_{cast}$</td>
<td>0.1</td>
<td>m s$^{-1}$</td>
</tr>
<tr>
<td>$W$</td>
<td>0.0135</td>
<td>m</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$2.25 \times 10^{-5}$</td>
<td>K$^{-1}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>8000</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>$\Delta H_t$</td>
<td>205 000</td>
<td>J kg$^{-1}$</td>
</tr>
</tbody>
</table>

\[
\dot{\sigma} = \frac{E}{(1 + \nu)(1 - 2\nu)} \left( \dot{\varepsilon} - \dot{\varepsilon}^i + \nu(\dot{\varepsilon}_y - \dot{\varepsilon}_y^i) \right) - \frac{E\alpha \dot{T}_s}{(1 - 2\nu)} \tag{2.34}
\]

and

\[
\dot{\sigma}_y = \frac{E}{(1 + \nu)(1 - 2\nu)} \left( 2\nu(\dot{\varepsilon} - \dot{\varepsilon}) + (1 - \nu)(\dot{\varepsilon}_y - \dot{\varepsilon}_y^i) \right) - \frac{E\alpha \dot{T}_s}{(1 - 2\nu)}. \tag{2.35}
\]

The structure mechanical problem is then fully specified by imposing three conditions. Normally, these consist of boundary conditions at $y=0$ and $y=y_{m}(z)$, as well as a global constraint (Weiner & Boley 1963; Kristiansson 1982; Schwerdtfeger et al. 1998; Åberg et al. 2005a). Considering first $y=y_{m}(z)$, we will have

\[
\sigma_y = -p(z), \tag{2.36}
\]

where $p(z)$ is the hydrostatic pressure, which we write as $p=p_0 + \rho g z$, with $p_0$ as the pressure at $z=0$, and then

\[
\dot{\sigma}_y + y_{m} \frac{\partial \sigma_y}{\partial y} = -\dot{p}(z). \tag{2.37}
\]

Since also $\partial \sigma_y / \partial y = 0$ from (2.23), we conclude that $\dot{\sigma}_y = -\dot{p}(z)$ everywhere, which then gives

\[
\sigma_y(y, z) \equiv -p(z), \tag{2.38}
\]

everywhere. We note also that in general $\sigma_x = \sigma_z = -p(z)$ at $y=y_{m}(z)$, giving

\[
\sigma = -p(z) \quad \text{at } y = y_{m}(z). \tag{2.39}
\]
However, (2.38) will only hold as far as $z = z_{\text{mid}}$; for $z > z_{\text{mid}}$, we will have $\sigma_y = \sigma_y(z)$, but now we set

$$v = 0, \quad \frac{\partial u}{\partial y} = 0,$$

(2.40)

at $y = W$, i.e. symmetry conditions, which leads to

$$\dot{v} = 0, \quad \frac{\partial \dot{u}}{\partial y} = 0.$$

(2.41)

The other boundary lies at $y = 0$ for $z \leq z_{\text{gap}}$, where we set zero normal displacement,

$$v = 0,$$

(2.42)

which then gives

$$\dot{v} = 0.$$

(2.43)

For $z > z_{\text{gap}}$, we set

$$\sigma = 0 \quad \text{at} \quad y = y_a(z),$$

(2.44)

an alternative candidate would have been $\sigma_y = -p_a$, where $p_a$ is the pressure of the surrounding atmosphere. Equation (2.44) is motivated by the condition, used by Richmond & Tien (1971), that the lateral stress, i.e. $\sigma_x$, should vanish when the air gap forms; we continue to use that condition thereafter also. On the other hand, if we use $\sigma_y = -p_a$, we find that it is not possible to satisfy (2.38) without presumably having $\tau_{xy} \neq 0$, thereby losing analytical tractability; hence, we settle for (2.44). Note also, however, that (2.44) holds only if $y_a(z) > 0$ and, as we find later, for parameter ranges for which $y_a$ returns to zero, it is plausible that (2.42) should be reinstated.

Boundary conditions (2.38) and (2.43), or (2.44), will not suffice, however, to determine $\dot{\varepsilon}(z)$, and therefore additional considerations are required. A further condition results from the fact that $\sigma_z$ has to be compensated by the total external force, $F_z$, acting on the cross-sectional area, i.e.

$$F_z = \begin{cases} \int_0^{y_a(z)} \sigma_z \, dy & \text{if} \; z \leq z_{\text{gap}}, \\ \int_{y_a(z)}^{y_a(z)} \sigma_z \, dy & \text{if} \; z_{\text{gap}} < z < z_{\text{mid}}, \\ \int_{y_a(z)}^{W} \sigma_z \, dy & \text{if} \; z_{\text{mid}} < z < L. \end{cases}$$

(2.45)

Different assumptions can be made to obtain $F_z$; this is discussed briefly for the continuous casting of steel by Schwerdtfeger et al. (1998). In the mould region, $F_z$ will equal the weight of the liquid metal column above the cross-sectional area of the shell if the friction within the mould is negligible and the strand is held by the guiding system below the mould, i.e. the slice of the shell is subjected to an overall compression. In principle, there can also be overall longitudinal tension in

the shell with positive $F_z$, exerted by the withdrawal machine, if the friction in the mould is large; also, if friction and weight force compensate each other, $F_z = 0$. Here, the most relevant case is that for the mould region, so we set

$$
F_z = \begin{cases} 
-p_0 y_m(z) - \rho g \left\{ z y_m(z) - \int_0^z y_m(z') \, dz' \right\} & \text{if } z \leq z_{\text{gap}}, \\
-p_0 (y_m(z) - y_a(z)) - \rho g \left\{ z y_m(z) - y_a(z) z_m(y_a(z)) - \int_{z_m(y_a(z))}^z y_m(z') \, dz' \right\} & \text{if } z_{\text{gap}} < z < z_{\text{mid}}, \\
-p_0 (W - y_a(z)) - \rho g \left\{ z_{\text{mid}} W - y_a(z) z_m(y_a(z)) - \int_{z_m(y_a(z))}^{z_{\text{mid}}} y_m(z') \, dz' \right\} & \text{if } z_{\text{mid}} < z < L; 
\end{cases}
$$

in each case, the first term is due to the pressure $p_0$ at the meniscus level, whereas the second term is due to weight of the liquid metal column. Differentiating (2.45) and (2.46) with respect to $z$, replacing $s_z$ by $s$, and equating then gives

$$
\left\{ \begin{array}{l}
\int_0^{y_m(z)} \dot{\sigma} \, dy' = 0 & \text{if } z \leq z_{\text{gap}}, \\
\int_{y_a(z)}^{y_m(z)} \dot{\sigma} \, dy' = (p_0 + \rho g z_m(y_a(z))) \dot{y}_a(z) & \text{if } z_{\text{gap}} < z < z_{\text{mid}}, \\
\int_{y_a(z)}^W \dot{\sigma} \, dy' = (p_0 + \rho g z_m(y_a(z))) \dot{y}_a(z) & \text{if } z_{\text{mid}} < z < L.
\end{array} \right.
$$

3. Analysis

(a) Non-dimensionalization

To non-dimensionalize, we write

$$
Y = \frac{y}{W}, \quad Y_m = \frac{y_m}{W}, \quad Y_a = \frac{y_a}{k_a H_M / k_M}, \quad Z = \frac{z}{L}, \quad P = \frac{p}{E \alpha \Delta T},
$$

$$
\theta = \frac{T_{\text{melt}} - T_s}{\Delta T}, \quad \theta_o = \frac{T_{\text{melt}} - T_0(z)}{\Delta T}, \quad \Sigma = \frac{\sigma}{E \alpha \Delta T}, \quad \Sigma_Y = \frac{\sigma_Y}{E \alpha \Delta T},
$$

$$
\epsilon = \frac{\epsilon}{\alpha \Delta T}, \quad \epsilon_Y = \frac{\epsilon_Y}{\alpha \Delta T}, \quad \epsilon^i = \frac{\epsilon^i}{\alpha \Delta T}, \quad \epsilon^i_Y = \frac{\epsilon^i_Y}{\alpha \Delta T},
$$

where $\Delta T$ is a temperature scale to be determined. Equation (2.1) then becomes

$$
\widetilde{Pe} \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Y^2},
$$

where $\widetilde{Pe}$ is the reduced Peclet number, given by

$$
\widetilde{Pe} = \frac{\rho c_{pm} V_{\text{cast}} L}{k_s} \left( \frac{W}{L} \right)^2.
$$
The boundary conditions for $\theta$ are then

$$\theta = 0, \quad \frac{\partial \theta}{\partial Y} = -\frac{\bar{P}_e}{St} \frac{d Y_m}{d Z} \quad \text{at} \quad Y = Y_m(Z), \quad (3.3)$$

$$\frac{\partial \theta}{\partial Y} = \kappa (\theta - \theta_o(Z)) \quad \text{at} \quad Y = 0, \quad \text{for} \quad Z < Z_{\text{gap}} \quad (3.4)$$

and

$$\frac{\partial \theta}{\partial Y} = \frac{\kappa}{(1 + Y_a(Z))} (\theta - \theta_o(Z)) \quad \text{at} \quad Y = \delta Y_a(Z), \quad \text{for} \quad Z > Z_{\text{gap}}, \quad (3.5)$$

where $Z_{\text{gap}} = z_{\text{gap}}/L$ and $St$ is the Stefan number, given by

$$St = \frac{c_{\text{ps}} \Delta T}{\Delta H_f}, \quad (3.6)$$

with

$$\kappa = \frac{k_M W}{k_s H_M}, \quad \delta = \frac{k_{\text{air}} H_M}{k_M W}. \quad (3.7)$$

Also, (3.5) reveals why $y_a$ was rescaled with $k_a H_M/k_M$ rather than $W$: this is the thickness it needs to be to make a leading-order contribution to the heat transfer in the problem. The initial conditions (2.17) and (2.18) are, respectively,

$$\theta = 0 \quad \text{at} \quad Z = 0 \quad (3.8)$$

and

$$Y_m(0) = 0. \quad (3.9)$$

For the structural mechanics part of the problem, we have from (2.34) and (2.35)

$$\dot{\Sigma} = \frac{1}{(1 + \nu)(1 - 2\nu)} (\ddot{\epsilon} - \ddot{\epsilon}^i + \nu (\ddot{\epsilon}_Y - \ddot{\epsilon}_Y^i)) + \frac{\dot{\theta}}{(1 - 2\nu)} \quad (3.10)$$

and

$$\dot{\Sigma}_Y = \frac{1}{(1 + \nu)(1 - 2\nu)} (2\nu(\ddot{\epsilon} - \ddot{\epsilon}^i) + (1 - \nu)(\ddot{\epsilon}_Y - \ddot{\epsilon}_Y^i)) + \frac{\dot{\theta}}{(1 - 2\nu)}. \quad (3.11)$$

From the discussion leading to (2.38), we have

$$\Sigma_Y(Y, Z) = -P(Z), \quad (3.12)$$

everywhere, where $P(Z) = P_0 + P_1 Z$, with

$$P_0 = \frac{p_0}{E\alpha \Delta T}, \quad P_1 = \frac{\rho g L}{E\alpha \Delta T}; \quad (3.13)$$

also, from (2.39),

$$\Sigma = -P(Z) \quad \text{at} \quad Y = Y_m(Z). \quad (3.14)$$

Equation (3.12) will only hold for $Z \leq Z_{\text{mid}}$; for $Z > Z_{\text{mid}}$, we will have $\Sigma_Y = \Sigma_Y(Z)$, but now we have symmetry conditions at $Y = 1$, i.e.
\[ \dot{V} = 0, \quad \frac{\partial \dot{U}}{\partial Y} = 0, \]  
(3.15)

where dots now denote differentiation with respect to \( Z \). For the boundary facing the mould, we have

\[ \dot{V} = 0 \quad \text{at} \quad Y = 0 \quad \text{for} \quad Z \leq Z_{\text{gap}} \]  
(3.16)

and

\[ \Sigma = 0 \quad \text{at} \quad Y = \delta Y_a(Z) \quad \text{for} \quad Z > Z_{\text{gap}}, \]  
(3.17)

Finally, (2.47) becomes

\[
\begin{cases}
\int_0^{Y_a(Z)} \dot{\Sigma} \, dY' = 0 & \text{if} \ Z \leq Z_{\text{gap}}, \\
\int_{\delta Y_a(Z)}^{Y_a(Z)} \dot{\Sigma} \, dY' = (P_0 + P_1 Z_m(\delta Y_a(Z))) \delta Y_a(Z) & \text{if} \ Z_{\text{gap}} < Z \leq Z_{\text{mid}}, \\
\int_{\delta Y_a(Z)}^{Y_a(Z)} \dot{\Sigma} \, dY' = (P_0 + P_1 Z_m(\delta Y_a(Z))) \delta Y_a(Z) & \text{if} \ Z_{\text{mid}} < Z \leq 1.
\end{cases}
\]  
(3.18)

To make further analytical headway, we need to consider the orders of magnitude of the various dimensionless parameters that are present: \( Pe, St, \kappa, \delta, P_0, P_1 \). Using the data in table 1, we have

\[ \kappa \sim 2, \quad \delta \sim 10^{-4}, \quad \widetilde{Pe} \sim 0.5; \]  
(3.19)

for the other three parameters, we need to determine \( \Delta T \). This quantity ought to characterize the temperature difference across the solidified shell; furthermore, it should be chosen so that \( \theta \sim O(1) \). However, it is not straightforward to know the appropriate scale \emph{a priori}, because the only temperature associated with the boundary condition at the outer surface of the shell is actually that associated with the mould, \( T_0(z) \), although this temperature is likely to be much lower than that at the outer surface of the shell. Nevertheless, using this value does set an upper bound on \( \Delta T \); hence, we take \( \Delta T = T_{\text{melt}} - T_{0\text{min}} \), where \( T_{0\text{min}} = \min(T_0(z) \mid z \geq 0) \). This gives

\[ St \sim 2, \quad P_0 \sim 10^{-4}, \quad P_1 \sim 10^{-5}; \]

while these orders of magnitude depend on the fact that we have taken \( \Delta T \sim 10^3 \text{K} \), we note that even if we had taken \( \Delta T \sim 10^2 \text{K} \), leading to

\[ St \sim 0.2, \quad P_0 \sim 10^{-3}, \quad P_1 \sim 10^{-4}, \]

there will be no quantitative differences in what follows.

More significant, however, is the fact that \( \delta \ll 1 \), which permits still further simplification.

\((b)\ \delta \ll 1\)

For the thermal model, we now have (3.1), subject to initial conditions (3.8) and (3.9) and the following boundary conditions:
for $0 < Z \leq Z_{\text{gap}}$, boundary conditions (3.3) and (3.4),

for $Z_{\text{gap}} < Z \leq Z_{\text{mid}}$, (3.4) and

$$\frac{\partial \theta}{\partial Y} = \frac{\kappa}{(1 + Y_a(Z))} (\theta - \theta_o(Z)) \quad \text{at } Y = 0,$$  \hspace{1cm} (3.20)

and

for $Z_{\text{mid}} < Z \leq 1$, (3.20) and

$$\frac{\partial \theta}{\partial Y} = 0 \quad \text{at } Y = 1.$$  \hspace{1cm} (3.21)

To determine $Y_a(Z)$ and $Z_{\text{gap}}$, we return to the structure mechanics. Equation (3.18) now becomes, at leading order in $d$,

$$J \left[ \int_0^{Y_m(Z)} \dot{\Sigma} \, dY' = 0 \quad \text{if } Z \leq Z_{\text{gap}}, \right.$$

$$J \left[ \int_0^{Y_m(Z)} \dot{\Sigma} \, dY' = 0 \quad \text{if } Z_{\text{gap}} < Z \leq Z_{\text{mid}}, \right.$$

$$J \left[ \int_0^{Y_m(Z)} \dot{\Sigma} \, dY' = 0 \quad \text{if } Z_{\text{mid}} < Z \leq 1. \right.$$  \hspace{1cm} (3.22)

Now, in order to treat $Z_{\text{mid}} < Z \leq Z_1$ simultaneously, we set $\Sigma_Y \equiv -\phi(Z)$. From the foregoing analysis, $\phi(Z) = P(Z)$ for $Z \leq Z_{\text{mid}}$, but its functional form for $Z > Z_{\text{mid}}$ is unknown \textit{a priori}. Combining (3.10) and (3.11) gives

$$\dot{\Sigma} = \left( \frac{1}{1 - \nu} \right) (\dot{\epsilon} - \dot{\epsilon}^\ddagger + \dot{\theta}) - \frac{\nu}{1 - \nu} \phi(Z).$$  \hspace{1cm} (3.23)

Using (3.22), we have

$$\int_0^{Y_m(Z)} \{ \dot{\epsilon} - \dot{\epsilon}^\ddagger + \dot{\theta} - \nu \phi(Z) \} \, dY' = 0,$$  \hspace{1cm} (3.24)

where, for simplicity, we have implicitly taken $Y_m(Z) = 1$ for $Z_{\text{mid}} < Z < 1$. On rearranging,

$$\dot{\epsilon} - \dot{\epsilon}^\ddagger = \frac{-1}{Y_m(Z)} \int_0^{Y_m(Z)} \dot{\theta} \, dY' + \nu \phi(Z).$$  \hspace{1cm} (3.25)

Furthermore, we obtain

$$\dot{\Sigma} = \left( \frac{1}{1 - \nu} \right) \left( \frac{-1}{Y_m(Z)} \int_0^{Y_m(Z)} \dot{\theta} \, dY' + \dot{\theta} \right),$$  \hspace{1cm} (3.26)

which can also be written as

$$\dot{\Sigma} = \left( \frac{1}{1 - \nu} \right) \left( \frac{1}{Pc Y_m(Z)} \left( \frac{\dot{\Sigma}}{St} \frac{dY_m}{dZ} + \kappa (\theta_{\text{wall}} - \theta_o(Z)) \right) + \dot{\theta} \right).$$  \hspace{1cm} (3.27)

Also, we have

$$\dot{\epsilon}_Y - \dot{\epsilon}^\ddagger_Y = \frac{1}{1 - \nu} \left\{ -(1 + \nu) \dot{\theta} + \frac{2\nu}{Y_m(Z)} \int_0^{Y_m(Z)} \dot{\theta} \, dY' \right\} \left[ \frac{2\nu + \nu^2 - 1}{1 - \nu} \right] \phi(Z).$$  \hspace{1cm} (3.28)
from which we can obtain \( \epsilon - \epsilon_i, \epsilon_Y - \dot{\epsilon}_Y \) and \( \Sigma \) as

\[
\epsilon - \epsilon_i = - \int_Z Z_m(Y) (\dot{\epsilon} - \dot{\epsilon}_i) \, dZ',
\]

\[
\epsilon_Y - \dot{\epsilon}_Y = - \int_Z Z_m(Y) (\dot{\epsilon}_Y - \dot{\epsilon}_Y) \, dZ',
\]

\[
\Sigma = - \varphi(Z_m(Y)) - \int_Z Z_m(Y) \Sigma \, dZ',
\]

where \( Z_m(Y) \) is the inverse function of \( Y_m(Z) \); thus, \( \epsilon - \epsilon_i, \epsilon_Y - \dot{\epsilon}_Y \) and \( \Sigma \) can be calculated from (3.29)–(3.31), once \( \theta \) has been determined. Of particular note here is the fact that \( \Sigma \) does not depend on the functional forms used for \( \epsilon_i \) and \( \dot{\epsilon}_Y \). Most significantly, this will mean that the thickness of the air gap will not depend, at leading order, on the constitutive relation chosen for the inelastic strains; this conclusion, although initially somewhat surprising, is consistent with the computed results of Schwerdtfeger et al. (1998).

Now, we focus attention on \( Z \leq Z_{\text{mid}} \), where \( \dot{\varphi}(Z) = \dot{P}(Z) \). First, we observe that \( Z_{\text{gap}} \) is given by

\[
-P_0 + \int_{0}^{Z_{\text{gap}}} \dot{\Sigma}(0, Z') \, dZ' = 0,
\]

which indicates that if we neglect the hydrostatic pressure distribution by setting \( P_0 = 0 \), we would obtain \( Z_{\text{gap}} = 0 \), implying an overprediction in the subsequent air-gap thickness; from (3.32), we then obtain

\[
(\theta)_{Y=0} = (1 - \nu) P_0 - \frac{1}{P_e} \int_{0}^{Z_{\text{gap}}} \left[ \frac{\dot{P}eSt^{-1}Y_m(Z')}{Y_m(Z')} + \frac{\kappa((\theta)_{Y=0} - \theta_o(Z'))}{1 + Y_a(Z')} \right] \, dZ'.
\]

For \( Z_{\text{gap}} < Z \leq Z_{\text{mid}} \), we have

\[
(\theta)_{Y=0} = (1 - \nu) P_0 - \frac{1}{P_e} \int_{0}^{Z_{\text{mid}}} \left[ \frac{\dot{P}eSt^{-1}Y_m(Z')}{Y_m(Z')} + \frac{\kappa((\theta)_{Y=0} - \theta_o(Z'))}{1 + Y_a(Z')} \right] \, dZ' + \int_{Z_{\text{mid}}}^{Z} \frac{\kappa((\theta)_{Y=0} - \theta_o(Z'))}{1 + Y_a(Z')} \, dZ';
\]

but for \( Z > Z_{\text{mid}} \),

\[
(\theta)_{Y=0} = (1 - \nu) P_0 - \frac{1}{P_e} \left\{ \int_{0}^{Z_{\text{mid}}} \left[ \frac{\dot{P}eSt^{-1}Y_m(Z')}{Y_m(Z')} + \frac{\kappa((\theta)_{Y=0} - \theta_o(Z'))}{1 + Y_a(Z')} \right] \, dZ' \right\}.
\]

thus, (3.34) and (3.35) constitute the extra conditions at \( Y=0 \), subsidiary to (3.20), that are necessary in order to find \( Y_a(Z) \). As boundary conditions, (3.34) and (3.35) are rather unwieldy, particularly with regard to later numerical implementation, and it proves more convenient to manipulate them to a simpler
form. Differentiating (3.34) with respect to $Z$ gives
\[
\left( \frac{\partial \theta}{\partial Z} \right)_{Y=0} = -\frac{1}{\text{Pe} Y_m(Z)} \left[ \text{Pe} St^{-1} \dot{Y}_m(Z) + \frac{\kappa((\theta)_{Y=0} - \theta_0(Z))}{1 + Y_a(Z)} \right]
\]
(3.36)
and $Y_a(Z)$ can be eliminated completely by using (3.20) to give
\[
\frac{\partial \theta}{\partial Y} = -\text{Pe} Y_m(Z) \frac{\partial \theta}{\partial Z} - \text{Pe} St^{-1} \dot{Y}_m(Z) \quad \text{at } Y = 0 \quad \text{for } Z_{\text{gap}} < Z \leq Z_{\text{mid}}.
\]
(3.37)
A similar procedure using (3.20) and (3.35) gives
\[
\frac{\partial \theta}{\partial Y} = -\text{Pe} \frac{\partial \theta}{\partial Z} \quad \text{at } Y = 0 \quad \text{for } Z_{\text{mid}} < Z \leq 1.
\]
(3.38)

(c) Onset of air-gap formation (analysis for $Z \ll 1$)

While there is no doubt that $\Sigma < 0$ when solidification begins, there is no guarantee that there is a solution for $Z_{\text{gap}}$ for all possible combinations of operating parameters; such a situation was found by Richmond & Tien (1971). For example, if $\Sigma(0, Z) < 0$, the force on the solidified shell becomes more compressive, meaning a lower likelihood that the shell will separate away from the cooling mould. Since the air gap is thought to form just a short distance below the meniscus, it is therefore instructive to consider the analysis for $Z \ll 1$, when regular series expansions for $\theta$ and $Y_m$ in terms of $Z$ ought still to be valid. Setting
\[
\theta = Y_m(Z) F(Z, Y/Y_m(Z)), \quad \eta = Y/Y_m(z).
\]
(3.39)
(3.1) becomes
\[
\text{Pe} Y_m(\dot{Y}_m F + Y_m F_z - \eta \dot{Y}_m F_\eta) = F_\eta,
\]
(3.40)
with boundary conditions (3.3) and (3.4) becoming
\[
F_\eta = \kappa(Y_m F - \theta_0(Z)) \quad \text{at } \eta = 0
\]
(3.41)
and
\[
F = 0, \quad (F_\eta)_{\eta=1} = -\text{Pe} St^{-1} \dot{Y}_m \quad \text{at } \eta = 1,
\]
(3.42)
respectively. Now, in the transformed coordinates, (3.26) becomes
\[
\dot{\Sigma} = \frac{1}{1 - \nu} \left( -\int_0^1 \dot{\theta} \, d\eta' + \dot{\theta} \right),
\]
(3.43)
where $\dot{\theta} = \dot{Y}_m F + Y_m F_z - \eta \dot{Y}_m F_\eta$.

As $Z \to 0$, a self-consistent boundary-value problem is obtained if $Y_m(Z) = \lambda Z + o(Z)$, where $\lambda$ is a strictly positive constant whose value is to be determined. However, it turns out to be more convenient to write
\[
F = F_0(\eta) + ZF_1(\eta) + \ldots
\]
\[
Y_m(Z) = \lambda_1 Z + \lambda_2 Z^2 + \ldots
\]
in addition, we expect a similar expansion for $\hat{\Sigma}(\eta, Z)$, and we come to the form of this shortly. At $Z^0$, equation (3.40) reduces to

$$F_{0\eta\eta} = 0,$$

subject to

$$F_{0\eta} = -\kappa \theta_o(0) \quad \text{at} \ \eta = 0,$$  \hspace{1cm} (3.44)

$$F_0 = 0, \quad F_{0\eta} = -\widetilde{Pe} St^{-1} \lambda_1 \quad \text{at} \ \eta = 1,$$  \hspace{1cm} (3.45)

respectively. So,

$$\lambda_1 = \kappa \widetilde{Pe} St\theta_o(0),$$  \hspace{1cm} (3.46)

$$F_0(\eta) = \kappa \theta_o(0)(1 - \eta).$$  \hspace{1cm} (3.47)

As regards air-gap formation, we need to look in particular at $\hat{\Sigma}(0, Z)$. At leading order in $Z$, which is $Z^0$, we obtain

$$(1 - \nu)\hat{\Sigma}(0, Z) = -\lambda_1 \int_0^1 (F_0 - \eta F_0') \, d\eta + \lambda_1 F_0(0);$$  \hspace{1cm} (3.48)

however, substituting (3.47) and (3.48), we find that this vanishes, and so we turn instead to $Z^1$. Hence, we expect instead that

$$(1 - \nu)\hat{\Sigma}(0, Z) \sim \lambda_1 \left\{ 2 F_1(0) - \int_0^1 (2F_1 - \eta F_1') \, d\eta' \right\} Z;$$  \hspace{1cm} (3.49)

this requires additional analysis to find $F_1$.

Equation (3.40) at $Z^1$ gives

$$\widetilde{Pe} \lambda_1^2 (F_0 - \eta F_0') = F_{1\eta\eta},$$  \hspace{1cm} (3.50)

subject to

$$F_{1\eta} = \kappa (\lambda_1 F_0 - \dot{\theta}_o(0)) \quad \text{at} \ \eta = 0,$$  \hspace{1cm} (3.51)

and

$$F_1 = 0, \quad F_{1\eta} = -2 \lambda_2 \widetilde{Pe} St^{-1} \quad \text{at} \ \eta = 1.$$  \hspace{1cm} (3.52)

Using (3.48), we have

$$\widetilde{Pe} \lambda_1^2 \kappa \theta_o(0) = F_{1\eta\eta}.$$

So,

$$F_1 = \frac{1}{2} \widetilde{Pe} \lambda_1^2 \kappa \theta_o(0) \eta^2 + A_1 \eta + B_1,$$  \hspace{1cm} (3.53)
where $A_1$ and $B_1$ are constants of integration. Applying the boundary conditions, we obtain

$$A_1 = \kappa \dot{\theta}_o(0) - \left(1 + \frac{1}{2} St\theta_o(0)\right) \kappa^3 \bar{P}e^{-1} St\theta_o^2(0) \left(1 + \frac{1}{2} St\theta_o(0)\right),$$

(3.56)

$$B_1 = \kappa \left(\kappa^2 \bar{P}e^{-1} St\theta_o^2(0) - \dot{\theta}_o(0)\right)$$

(3.57)

and

$$\lambda_2 = \frac{1}{2 Pe St^{-1}} \left\{ \kappa \lambda_1 \kappa\theta_o(0) - \dot{\theta}_o(0) - \frac{1}{2} \bar{P}e \lambda_1 \kappa\theta_o(0) \right\}. \quad (3.58)$$

This time, we observe that

$$\dot{\Sigma}(0, Z) \sim \frac{\kappa^2 St\theta_o(0)}{2(1 - \nu) \bar{P}e} \left(\bar{P}e \dot{\theta}_o(0) - \kappa^2 St\theta_o^2(0)\right) Z.$$  

(3.59)

Now, if

$$\bar{P}e \theta_o(0) > \kappa^2 St\theta_o^2(0),$$

(3.60)

we will have $\dot{\Sigma}(0, Z) > 0$, so that $\Sigma(0, Z)$ is initially increasing; in dimensional terms, (3.60) is written as

$$\dot{T}_o(0) < -\left(\frac{k^2 M}{\Delta Hr \rho V_{cast} k_s \lambda M^2}\right) (T_{melt} - T_o(0))^2.$$  

(3.61)

Although, as we shall see later, (3.61) is not a hard and fast criterion to predict whether an air gap will form or not, it does help to determine combinations of $V_{cast}$, $T_o(0)$ and $\dot{T}_o(0)$ for which an air gap is more likely to form. Setting

$$\chi_1 = \frac{\kappa^2 St\theta_o(0)}{2(1 - \nu) \bar{P}e} \left(\bar{P}e \dot{\theta}_o(0) - \kappa^2 St\theta_o^2(0)\right),$$

(3.62)

an air gap is more likely to form for combinations lying below the surface $\chi_1 = 0$ given in figure 2. Furthermore, a promising feature of inequality (3.61) is that even though the non-dimensional analysis was carried out using the width and length of the casting geometry, these parameters are not present in the final dimensional expression, which is in terms of quantities local to the onset. Also, since

$$\Sigma(0, Z) \sim -P_o + \frac{1}{2} \chi_1 Z^2,$$

the gap should start to form at $Z_{gap} \approx (2P_o/\chi_1)^{1/2}$.

The reason for the caution exercised above is that $\dot{\Sigma}(0, Z)$ may be dominated by terms of order higher than $Z$ before the theoretical value of $Z_{gap}$ is reached; consequently, $\dot{\Sigma}(0, 0) < 0$ might still lead to air-gap formation, and $\dot{\Sigma}(0, 0) > 0$ might still lead to the absence of an air gap. To clarify further, we need to consider an additional term in the Taylor series expansion for $\dot{\Sigma}(0, Z)$, i.e.

$$\dot{\Sigma}(0, Z) \sim \chi_1 Z + \chi_2 Z^2.$$  

(3.63)
In particular, if \( c_1 > 0 \), an air gap will form if \( c_2 > 0 \); on the other hand, even if \( c_1 < 0 \), a gap may still form if \( c_2 < 0 \). Hence, the sign of \( c_2 \), which we can obtain by considering (3.40) at \( Z^2 \) as

\[
\chi_2 = \frac{k^2 St}{12 Pe^3} \left( k^4 St^2 \theta_0^4(0)(20 St \theta_0(0) + 3 St \theta_0^2(0) + 21) - 8 Pe (St \theta_0(0) + 3 St \theta_0^2(0) + 3 Pe^2 \left( 2 \theta_0(0) \ddot{\theta}_0(0) + \ddot{\theta}_0^2(0) \right) \right)
\]

(3.64)
is of importance; for the derivation of (3.64), see appendix B in the electronic supplementary material. We make use of (3.64) for interpreting the numerical results in §5.

### 4. Numerical implementation

The numerical task at hand is the solution of (3.1), subject to boundary conditions (3.3), (3.4), (3.21), (3.37) and (3.38), and initial conditions (3.8) and (3.9). Since the problem is degenerate at \( Z=0 \), it is convenient to use (3.39), although now not just for small \( Z \). Hence, we have (3.40), subject to initial conditions

\[
F = \kappa \theta_0(0)(1 - \eta) \quad \text{at} \quad Z = 0,
\]

(4.1)

\[
Y_m(0) = 0
\]

(4.2)

and boundary conditions:

- for \( 0 < Z \leq Z_{\text{gap}} \), (3.41) and (3.42),
- for \( Z_{\text{gap}} < Z \leq Z_{\text{mid}} \), (3.42) and

\[
F_\eta = -Pe \dot{Y}_m \left( \ddot{Y}_m F + \dot{Y}_m F_Z \right) - Pe St^{-1} \dot{Y}_m \quad \text{at} \quad \eta = 0
\]

(4.3)

and
for $Z_{\text{mid}} < Z \leq 1$,
\begin{align}
F_{\eta} &= -\overline{Pe} F Z \quad \text{at } \eta = 0, \\
F_{\eta} &= 0 \quad \text{at } \eta = 1.
\end{align}

In the transformed variables, $Z_{\text{gap}}$ will be given by the solution to
\begin{equation}
Y_m(Z_{\text{gap}})(F)_{\eta=0} = (1 - \nu) P_0 - \frac{1}{P_e} \int_0^{Z_{\text{gap}}} \left[ \overline{Pe} St^{-1} Y_m(Z') + \kappa(Y_m(Z')(F)_{\eta=0} - \theta_o(Z')) \right] \frac{dZ'}{Y_m(Z')},
\end{equation}
whereas $Z_{\text{mid}} := \{ \min Z | Y_m(Z) = 1 \}$; after $F$ and $Y_m$ have been determined, $Y_a$ is given by
\begin{equation}
Y_a(Z) = \frac{\kappa}{(F_{\eta})_{\eta=0}} (Y_m(Z)(F)_{\eta=0} - \theta_o(Z)) - 1.
\end{equation}

Some aspects of the numerical method used to solve a system of equations similar to the above—in particular, order of accuracy and correct initialization—were considered recently by Mitchell & Vynnycky (submitted). Whereas they used a variety of finite-difference methods, here we use the finite-element software COMSOL MULTIPHYSICS.

5. Preliminary model results

Although non-dimensional variables were useful for the analysis in §§3 and 4 for easily identifying orders of magnitudes of various terms, as well as formulating the leading-order problem, it turns out to be more useful to return to dimensional variables in order to consider actual model predictions. To demonstrate the use of the model, it is clear that one should be guided by (3.61); otherwise, one might choose combinations of $T_o(0), \dot{T}_o(0)$ and $V_{\text{cast}}$ for which an air gap does not form at all.

As a basis for computations, we use the geometry and operating conditions given in table 1, and choose $T_o(z)$ of the form
\begin{equation}
T_o(z) = T_0 + (T_o(0) - T_0) \exp(-z/z_c),
\end{equation}
where $z_c = -(T_o(0) - T_0)/\dot{T}_o(0)$. We choose (5.1) because it is a comparatively simple profile that contains the qualitative features we desire: $\dot{T}_o(0) < 0$, and $T_o(z)$ approximately constant away from $z=0$, as is suggested by the computed results of Mahmoudi (2006).

Air-gap formation and evolution

Figure 3 shows the curve

\[ \dot{T}_0(0) = - \left( \frac{k_M^2}{\Delta H_f \rho V_{\text{cast}} k_S H_M^2} \right) (T_{\text{melt}} - T_0(0))^2, \]  

(5.2)

for \( V_{\text{cast}} = 0.1 \text{ m s}^{-1} \). For \( \dot{T}_0(0) = -3 \times 10^3 \text{ K m}^{-1} \), (5.2) predicts that \( \dot{\sigma}(0, z) \) will be positive at \( z=0 \) if \( T_0(0) \geq T_{\text{crit}} \approx 553.6 \text{ K} \); subsequently, figure 4 shows \( \sigma(0, z) \) versus \( z \) for several values of \( T_0(0) \) around this value. When \( T_0(0) > T_{\text{crit}} \), \( \dot{\sigma}(0, z) > 0 \) at \( z=0 \), so that \( \sigma(0, z) = 0 \) soon after \( z=0 \). As \( T_0(0) \) decreases, so does \( \dot{\sigma}(0, z) \) at \( z=0 \); when \( T_0(0) < T_{\text{crit}} \), \( \dot{\sigma}(0, z) < 0 \) at \( z=0 \). Note, however, that this does not necessarily mean that an air gap will not form for all values of \( T_0(0) < T_{\text{crit}} \); although \( \dot{\sigma}(0, 0) = 0 \) when \( T_0(0) < T_{\text{crit}} \), further analysis indicates that \( \chi_2 > 0 \) for \( T_0(0) < T_{\text{crit}} \), giving the possibility that the second term in (3.63) may dominate the first when \( |\chi_1| \) is small. As \( T_0(0) \) decreases further, and \( |\chi_1| \) becomes larger, the first term in (3.63) dominates once again. In summary, the actual value of \( T_0(0) \) for which an air gap will subsequently form is slightly lower than that given by (5.2); this result is consistent with the profiles in figure 4, which indicate that the actual value of \( T_0(0) \), below which an air gap does not form, lies between 525 and 545 K.

(b) Air-gap evolution

Fixing now

\[ \dot{T}_0(0) = -3 \times 10^4 \text{ K m}^{-1}, \quad T_0(0) = 1300 \text{ K}, \]

which is a combination that is known from figure 2 to lead to air-gap formation, we vary \( V_{\text{cast}} \) to investigate its effect on quantities such as \( y_m(z) \), \( y_a(z) \) and the heat transfer coefficient, \( h \). First, however, we compare in figure 5 the profiles for
Figure 4. $\sigma(0, z)$ versus $z$ in the vicinity of $z=0$ for $T_o(0) = -3 \times 10^3$ K m$^{-1}$, $V_{\text{cast}} = 0.1$ m s$^{-1}$ for three values of $T_o(0)$. The curves for $T_o(0)=525$ (solid line) and 545 K (dashed line) have been truncated after the point where $\sigma(0, z)=0$ (dot-dashed line, $T_o(0)=565$ K).

Figure 5. Comparison of $y_m$ versus $z$, obtained with (dotted line) and without (solid line) the air-gap model for $T_o(0) = -3 \times 10^3$ K m$^{-1}$, $V_{\text{cast}} = 0.1$ m s$^{-1}$.

$y_m$ for $V_{\text{cast}} = 0.1$ m s$^{-1}$ obtained when the air-gap model is included and omitted; for the case where the air-gap model is omitted, we simply use (3.41) for all $Z$, rather than switching to (4.3) when $\sigma(0, z)=0$. From the figure, it is evident that omission of the model predicts complete solidification within the mould region, whereas inclusion predicts the opposite; hence, inclusion of such a model is essential to any prediction of heat transfer within the mould region.

Figure 6 serves to identify the extent of solidification for three casting speeds: $V_{\text{cast}} = 0.01$, 0.03 and 0.1 m s$^{-1}$. For the two lower speeds, solidification is complete; this figure will subsequently help to interpret the later figures.
Figure 7 shows $y_a(z)$ for these three casting speeds. For each of these, the air-gap thickness is of the order of magnitude computed in models and inferred from experiment by other authors, albeit for the continuous casting of steel (Kelly et al. 1988; Huespe et al. 2000). When solidification is incomplete, we see that $y_a(z)$ increases rapidly to begin with and then decreases slowly; qualitatively, this resembles distinctly the full numerical results shown by Huespe et al. (2000). For $V_{\text{cast}} = 0.01$ (solid line), 0.03 (dot-dashed line) and 0.1 (dotted line) m s$^{-1}$, decreasing the casting speed further, we find that $y_a$ ultimately returns to zero. Note also that using (3.61) for this combination of $\dot{T}_o(0)$ and $T_o(0)$ indicates that an air gap will not form for $V_{\text{cast}} < 0.005$ m s$^{-1}$.

Figure 6. $y_m$ versus $z$ for $\dot{T}_o(0) = -3 \times 10^4$ K m$^{-1}$, $V_{\text{cast}} = 0.01$ (solid line), 0.03 (dot-dashed line) and 0.1 (dotted line) m s$^{-1}$.
Figure 8 shows the temperature at the outer surface of the solidified shell, whereas figure 9 shows the temperature at the mould wall at \( y=0 \). Naturally, these coincide for regions where there is no air gap, although this occurs only for very small values of \( z \) for all values of \( V_{\text{cast}} \) and for \( z \geq 0.3 \text{ m} \) for \( V_{\text{cast}} = 0.01 \text{ m s}^{-1} \). An interesting feature of figure 9 is that \((T_M)_{y=0}\) increases as \( y_a \) decreases, whereas it decreases once \( y_a = 0 \). For reference, the profile for \( T_o(z) \), which was taken to be the same for all values of \( V_{\text{cast}} \), has been included in figure 9.

Figure 10 shows the heat flux, \( Q \), given by

\[
Q = \left( -k_M \frac{\partial T_M}{\partial y} \right)_{y=0} ;
\]
this of the same order of magnitude as in Mahmoudi (2006). In all cases, we note that, following an initial steep decrease for small $z$ before the air gap forms, $Q$ is more or less constant for intervals where an air gap is present. Finally, figure 11 shows the heat transfer coefficient, and this is also of the same order of magnitude as in other available works on the continuous casting of copper; note, however, that $h$ is by definition unbounded when $y_a=0$, which explains its behaviour in figure 11 at locations when $y_a=0$ in figure 7.

6. Conclusions

In this paper, we have developed an asymptotic model to describe the formation and evolution of the air gap that is present in vertical continuous casting. Starting with a coupled thermomechanical model, we demonstrated that the
thermal and mechanical models could be decoupled, and that the thickness of the air gap was independent of the thermoplastic constitutive relation used—while somewhat surprising, this is in line with observations made elsewhere (Schwerdtfeger et al. 1998). In addition, the analysis has given a criterion, equation (3.61), which serves as a guideline, in terms of operating parameters, as to whether an air gap will form at all. Preliminary calculations were carried out using parameters for the continuous casting of copper.

The analytical framework presented here can now be extended in many ways. The simplest would be application to an actual casting process; for this, continuous casting in a cylindrical mould would be the most appropriate (Schwerdtfeger et al. 1998; Johnson & Cherukuri 1999; Cherukuri & Johnson 2001). Other extensions would be to include the effect of mould taper, which can also be treated asymptotically, and mould contraction, which would require solving for the stresses and displacements in the mould itself. A further step would be to consider the effect of superheat, i.e. with the temperature of the incoming molten metal being greater than the melting temperature. By comparison with the case of zero superheat, we would expect solidification to start lower down the mould wall and the onset of air-gap formation to occur even further down, in view of the compound effect of the superheat and the extra liquid head that this leads to.

Finally, we note that air-gap formation also occurs in a variety of other metal casting processes and the current asymptotic methodology, which uses the fact that the length scale associated with the air-gap thickness is much smaller than all other length scales in the problem, has the potential to be of use there too.

The author acknowledges the support of the Mathematics Applications Consortium for Science and Industry (www.macsi.ie) funded by the Science Foundation Ireland Mathematics Initiative grant 06/MI/005.

References


