Electrically and magnetically charged vortices in the Chern–Simons–Higgs theory

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In this paper, we prove the existence of finite-energy electrically and magnetically charged vortex solutions in the full Chern–Simons–Higgs theory, for which both the Maxwell term and the Chern–Simons term are present in the Lagrangian density. We consider both Abelian and non-Abelian cases. The solutions are smooth and satisfy natural boundary conditions. Existence is established via a constrained minimization procedure applied on indefinite action functionals. This work settles a long-standing open problem concerning the existence of dually charged vortices in the classical gauge field Higgs model minimally extended to contain a Chern–Simons term.

Keywords: Chern–Simons–Higgs vortices; constraint minimization; Abelian and non-Abelian gauge theory

1. Introduction

Dirac (1931), in his celebrated work, showed that the existence of a magnetic monopole solution to the Maxwell equations has the profound implication that electric charges in the universe are all quantized. Later, Schwinger (1969) further explored the idea of Dirac and proposed the existence of both electrically and magnetically charged particle-like solutions, called dyons, and used them to model quarks. In particular, Schwinger (1969) generalized the electric charge quantization condition of Dirac (1931) to a quantization condition relating electric and magnetic charges of a dyon. In modern theoretical physics, dyons are considered as excited states of magnetic monopoles. Both magnetic monopoles and dyons and their abundance are predicted by grand unified theories (Lykken & Strominger 1980; l’Yi et al. 1982; Grossman 1983; Nelson 1983; Preskill 1984; Affleck 1986; Vachaspati 1996). The well-known finite-energy singularity-free magnetic monopole and dyon solutions in the Yang–Mills–Higgs theory include the monopole solutions due to Polyakov (1974), 't Hooft (1974), Bogomol’nyi (1976), Prasad & Sommerfeld (1975), Jaffe & Taubes (1980) and Taubes (1982) and the dyon solutions due to Julia & Zee (1975), Bogomol’nyi (1976) and Prasad & Sommerfeld (1975). See also Cho & Maison (1997) and Yang (1998).

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for the construction of dyon solutions in the Weinberg–Salam electroweak theory. These are all static solutions of the governing gauge field equations in three-space dimensions.

Vortices arise as static solutions to gauge field equations in two-space dimensions. Unlike monopoles, magnetic vortices not only arise as theoretical constructs but also play important roles in areas such as superconductivity (Abrikosov 1957; Ginzburg & Landau 1965; Jaffe & Taubes 1980), electroweak theory (Ambjorn & Olesen 1988, 1989a,b, 1990) and cosmology (Vilenkin & Shellard 1994). The mathematical existence and properties of such vortices have been well studied (Jaffe & Taubes 1980; Berger & Chen 1989; Neu 1990; Du et al. 1992; Spruck & Yang 1992a,b; Bethuel et al. 1994; Weinan 1994; Bethuel & Riviè re 1995; Lin 1995, 1998; Ovchinnikov & Sigal 1997; Bauman et al. 1998; Serfaty 1999; Pacard & Riviè re 2000; Yang 2001; Montero et al. 2004; Tarantello 2008; B. J. Plohr 1980, unpublished data). Naturally, it will be interesting and important to establish the existence of dyon-like vortices, simply called electrically charged vortices, carrying both electric and magnetic charges. Such dually charged vortices have applications in a wide range of areas including high-temperature superconductivity (Khomskii & Freimuth 1995; Matsuda et al. 2002), optics (Bezryadina et al. 2006), the Bose–Einstein condensates (Inouye et al. 2001; Kawaguchi & Ohmi 2004), the quantum Hall effect (Sokoloff 1985) and superfluids.

Surprisingly, unlike static gauge field theory in three-space dimensions, it is recognized that there can be no finite-energy electrically charged vortex solutions in two-space dimensions for the classical Yang–Mills–Higgs equations, Abelian or non-Abelian. The impossibility of finite-energy electrically charged solutions is known as the Julia–Zee theorem (Julia & Zee 1975; Spruck & Yang 2009). Owing to the pioneering studies of Jackiw & Templeton (1981), Schonfeld (1981), Deser et al. (1982a,b), Paul & Khare (1986), de Vega & Schaposnik (1986a,b) and Kumar & Khare (1986), it has become accepted that, in order to accommodate electrically charged vortices, one needs to introduce into the action Lagrangian a Chern–Simons topological term (Chern & Simons 1971, 1974), which has become a central structure in anyon physics (Wilczek 1982, 1990; Fröhlich & Marchetti 1989). Therefore, an imperative problem one encounters is to develop an existence theory for the solutions of the full Chern–Simons–Higgs equations (de Vega & Schaposnik 1986a,b; Paul & Khare 1986) governing such electrically charged vortices.

This basic existence problem, however, has not yet been tackled in the literature, despite some successful numerical solutions reported (Jacobs et al. 1991). In fact, the lack of understanding of the solutions of the full system of equations has led to some dramatic trade-wind changes in the research on the Chern–Simons vortices, starting from the seminal papers of Hong et al. (1990) and Jackiw & Weinberg (1990), in which the Maxwell term is removed from the Lagrangian density while the Chern–Simons term stands out alone to govern the dynamics of electromagnetism. Physically, this procedure recognizes the dominance of the Chern–Simons term over the Maxwell term over large distances; mathematically, it allows one to pursue a Bogomol'nyi reduction (Bogomol'nyi 1976) when the Higgs potential takes a critical form as that in the classical Abelian Higgs model (Bogomol'nyi 1976; Jaffe & Taubes 1980). Such an approach triggered a wide range of explorations on the reduction of numerous
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Chern–Simons models, Abelian and non-Abelian, relativistic and non-relativistic (see Dunne (1995) for a review), and a rich spectrum of mathematical existence results for the Bogomol’nyi-type Chern–Simons vortex equations has been obtained (Spruck & Yang 1992c, 1995; Caffarelli & Yang 1995; Tarantello 1996, 2008; Chae & Kim 1997; Yang 1997; Nolasco & Tarantello 1999; Chae & Yu 2000; Ricciardi & Tarantello 2000; Chan et al. 2002; Nolasco 2003; Lin et al. 2007). We note that the existence of planar Abelian Chern–Simons models with no Maxwell term for non-Bogomol’nyi regimes has been recently established in Chen & Spirn (2009) and Spirn & Yan (2009). Although these contributions lead to considerable understanding of the properties of charged vortices when interaction between vortices is absent, the original problem of the existence of charged vortices, which are necessarily subject to interaction owing to the lack of a Bogomol’nyi structure, in the Chern–Simons–Higgs theory containing a Maxwell term (de Vega & Schaposnik 1986a,b; Kumar & Khare 1986; Paul & Khare 1986) remains unsolved.

In the present paper, we will establish the existence of charged vortices in the full Chern–Simons–Higgs theory with the Maxwell term (de Vega & Schaposnik 1986a,b; Kumar & Khare 1986; Paul & Khare 1986) in both Abelian and non-Abelian cases.

The rest of the paper is organized as follows. In §2, we review the Abelian Chern–Simons–Higgs theory, discuss some basic properties of charged vortices and their governing equations and state our main existence theorem. Then, we discuss the methods used in our proofs. In §3, we describe the basic set-up of our problem and introduce our constraint space. In §4 to §6, we prove the existence of weak solutions. In §7, we show that our weak solutions are, in fact, classical solutions. In §8, we establish the quantization formulas (2.16) and (2.17) expected for the magnetic and electric charges. Finally, in §9, we apply our methods to solve the non-Abelian Chern–Simons–Higgs equations.

2. Abelian Chern–Simons–Higgs equations and main existence theorem

After adding a Chern–Simons term to the classical Abelian Higgs Lagrangian density (Nielsen & Olesen 1973; Jaffe & Taubes 1980) and taking normalized units, the minimally extended action density, or the Chern–Simons–Higgs Lagrangian density introduced in Paul & Khare (1986) and de Vega & Schaposnik (1986a), defined over the Minkowski space–time $\mathbb{R}^{2,1}$ with metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$, may be written in the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\alpha} A_{\mu} F_{\nu\alpha} + \frac{1}{2} D_{\mu} \phi \overline{D_{\mu} \phi} - \frac{\lambda}{8} (|\phi|^2 - 1)^2, \quad (2.1)$$

where $\phi$ is a complex scalar function, the Higgs field, $A_{\mu}$ ($\mu = 0, 1, 2$), is a real-valued vector field, the Abelian gauge field, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, is the induced electromagnetic field, $D_{\mu} = \partial_{\mu} + i A_{\mu}$ is the gauge-covariant derivative, $\kappa > 0$ is a constant referred to as the Chern–Simons coupling parameter, $\epsilon^{\mu\nu\gamma}$ is the Kronecker skew-symmetric tensor with $\epsilon^{012} = 1$ and summation convention over repeated indices is observed. The extremals of the Lagrangian density (2.1) formally satisfy its Euler–Lagrange equations or the Abelian Chern–Simons–Higgs equations (Paul & Khare 1986),

$$D_{\mu} D^{\mu} \phi = \frac{\lambda}{2} \phi (1 - |\phi|^2) \quad (2.2)$$

and
\[ \partial_{\nu} F_{\mu\nu} - \frac{\mathcal{K}}{2} \varepsilon^{\mu\nu\alpha} F_{\nu\alpha} = -J^\mu, \tag{2.3} \]
in which equation (2.3) expresses the modified Maxwell equations so that the current density \( J^\mu \) is given by
\[ J^\mu = \frac{i}{2}(\overline{\phi} D^\mu \phi - \phi \overline{D}^\mu \phi). \tag{2.4} \]
Recall that we may rewrite \( J^\mu \) into a decomposed form
\[ J^\mu = (\rho, \mathbf{J}), \tag{2.4} \]
such that \( \rho \) represents electric charge density and \( \mathbf{J} = J^k \) represents electric current density.

Here, and in the sequel, we use the Latin letters \( j, k = 1, 2 \) to denote the indices of spatial components.

Therefore, since we will consider static configurations only so that all the fields are independent of the temporal coordinate, \( t = x^0 \), we have
\[ \rho = J^0 = \frac{i}{2}(\overline{\phi} D^0 \phi - \phi \overline{D}^0 \phi) = -A_0|\phi|^2, \tag{2.5} \]
which indicates that a non-trivial temporal component, \( A_0 \), of the gauge field \( A_\mu \) is essential for the presence of electric charge. Besides, also recall that the electric field \( \mathbf{E} = E^j \) (in the spatial plane) and magnetic fields \( \mathbf{H} \) (perpendicular to the spatial plane) induced from the gauge field \( A_\mu \) are
\[ E^j = \partial_j A_0, \quad j = 1, 2; \quad \mathbf{H} = F_{12}, \tag{2.6} \]
respectively. The static version of the Chern–Simons–Higgs equations (2.2) and (2.3) takes the explicit form
\[ D^2_j \phi = \frac{\lambda}{2}(|\phi|^2 - 1)\phi - A_0^2 \phi, \tag{2.7} \]
\[ \partial_k F_{jk} - \kappa \varepsilon_{jk} \partial_k A_0 = \frac{i}{2}(\overline{\phi} D_j \phi - \phi \overline{D}_j \phi) \tag{2.8} \]
and
\[ \Delta A_0 = \kappa F_{12} + |\phi|^2 A_0. \tag{2.9} \]

On the other hand, since the Chern–Simons term gives rise to a topological invariant, it makes no contribution to the energy–momentum tensor \( T_{\mu\nu} \) of the action density (2.1), which may be calculated as
\[ T_{\mu\nu} = -\eta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + \frac{1}{2}([D_\mu \phi][D_\nu \overline{\phi}] + [D_\nu \phi][D_\mu \overline{\phi}]) - \eta_{\mu\nu} \mathcal{L}_0, \tag{2.10} \]
where \( \mathcal{L}_0 \) is obtained from the Lagrangian (2.1) by setting \( \kappa = 0 \). Hence, it follows that the Hamiltonian \( \mathcal{H} = T_{00} \) or the energy density of the theory is given by
\[ \mathcal{H} = \frac{1}{2}F_{01}^2 + \frac{1}{2}F_{02}^2 + \frac{1}{2}|\phi|^2 A_0^2 + \frac{1}{2}F_{12}^2 + \frac{1}{2}(|D_1 \phi|^2 + |D_2 \phi|^2) + \frac{\lambda}{8}(|\phi|^2 - 1)^2 \]
\[ = \frac{1}{2}|
abla A_0|^2 + \frac{1}{2}|\phi|^2 A_0^2 + \frac{1}{2}F_{12}^2 + \frac{1}{2}(|D_1 \phi|^2 + |D_2 \phi|^2) + \frac{\lambda}{8}(|\phi|^2 - 1)^2, \tag{2.11} \]
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which is positive definite and the terms in equation (2.11) not containing \( A_0 \) are exactly those appearing in the classical Abelian Higgs model (Nielsen & Olesen 1973; Jaffe & Taubes 1980). Thus, the finite-energy condition

\[
E(A_0, A_j, \phi) = \int_{\mathbb{R}^2} \mathcal{H}(A_0, A_j, \phi)(x) \, dx < \infty
\]  

(2.12)

leads us to arrive at the following natural asymptotic behaviour of the fields \( A_0, A_j \) and \( \phi \):

\[
A_0, \partial_j A_0 \rightarrow 0, \quad F_{12} \rightarrow 0 \quad (2.13)
\]

\[
|\phi| \rightarrow 1 \quad \text{and} \quad |D_A \phi| \rightarrow 0, \quad (2.15)
\]

and

as \(|x| \rightarrow \infty\). In analogue to the Abelian Higgs model (Nielsen & Olesen 1973; Jaffe & Taubes 1980), we see that a finite-energy solution of the Chern–Simons–Higgs equations (2.7)–(2.9) should be classified by the winding number, say \( N \in \mathbb{Z} \), of the complex scalar field \( \phi \) near infinity, which is expected to give rise to the total quantized magnetic charge (or magnetic flux).

The resolution of the aforementioned open problem for the existence of charged vortices in the full Chern–Simons–Higgs theory amounts to prove that, for any integer \( N \), the coupled nonlinear elliptic equations (2.7)–(2.9) over \( \mathbb{R}^2 \) possess a smooth solution \((A_0, A_j, \phi)\) satisfying the finite-energy condition (2.12) and natural boundary conditions (2.13)–(2.15) so that the winding number of \( \phi \) near infinity is \( N \).

Here is our main existence theorem, which solves the above problem.

**Theorem 2.1.** For any given integer \( N \), the Chern–Simons–Higgs equations (2.7)–(2.9) over \( \mathbb{R}^2 \) have a smooth finite-energy solution \((A_0, A_j, \phi)\) satisfying the asymptotic properties (2.13)–(2.15) as \(|x| \rightarrow \infty\), such that the winding number of \( \phi \) near infinity is \( N \), which is also the algebraic multiplicity of zeros of \( \phi \) in \( \mathbb{R}^2 \), and the total magnetic charge or flux \( \Phi \) and electric charge \( Q \) are given by the quantization formulas

\[
\Phi = \int_{\mathbb{R}^2} F_{12} \, dx = 2\pi N
\]  

(2.16)

and

\[
Q = \int_{\mathbb{R}^2} \rho \, dx = 2\pi N \kappa.
\]  

(2.17)

Such a solution represents an \( N \)-vortex soliton, which is indeed both magnetically and electrically charged.

The proof of theorem 2.1 is contained in the proofs of propositions 5.2, 6.1 and 8.1. In the subsequent sections, we shall establish this theorem.

**Methodology.** We use the following standard ansatz to represent a radially symmetric \( N \)-vortex solution of the Abelian Chern–Simons–Higgs equations so that the \( N \) vortices are clustered at the origin:

\[
\phi(x) = u(r)e^{iN\theta},
\]  

(2.18)

\[
A_j(x) = Nv(r)\epsilon_{kj} \frac{x_k}{r^2}, \quad j, k = 1, 2
\]  

(2.19)

and

\[
A_0(x) = w(r).
\]  

(2.20)
As derived by Paul & Khare (1986) (and also de Vega & Schaposnik 1986a), the equations of motion (2.7)–(2.9) become

\[ u'' + \frac{1}{r} u' = \frac{N^2}{r^2} (v - 1)^2 u - w^2 u + \frac{\lambda}{2} u (u^2 - 1), \]  

(2.21)

\[ v'' - \frac{1}{r} v' = (v - 1) u^2 + \frac{\kappa r}{N} w', \]  

(2.22)

and

\[ w'' + \frac{1}{r} w' = u^2 w + \frac{\kappa N}{r} v'. \]  

(2.23)

Regularity and finite-energy condition prompt us to impose the boundary conditions

\[ \lim_{r \to 0} u(r) = \lim_{r \to 0} v(r) = \lim_{r \to \infty} w(r) = 0, \]  

(2.24)

\[ \lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = 1 \]  

(2.25)

and

\[ \lim_{r \to 0} w(r) = w_0. \]  

(2.26)

Here \( w_0 \) is some finite constant, depending on \( N, \lambda \) and \( \kappa \), that should arise from our constrained minimization procedure.

In order to establish existence, we note that equations (2.21)–(2.23) are the Euler–Lagrange equations of the indefinite action functional

\[ I(u, v, w) = \int_0^\infty \left( r (u')^2 + \frac{N^2}{r} u^2 (v - 1)^2 + \frac{\lambda}{4} (1 - u^2)^2 r + \frac{N^2}{r} (v')^2 \right) \, dr \]

\[ - \int_0^\infty (r w')^2 + ru^2 w^2 + 2\kappa N v' w) \, dr \]

\[ = G(u, v) - J_{u,v}(w). \]

Here \( G(u, v) \) is the standard Ginzburg–Landau functional for radially symmetric vortices, studied by B. J. Plohr (1980, unpublished data) and Berger & Chen (1989). The functional \( J_{u,v}(w) \) is indefinite and a source of difficulty in our existence problem.

In §3, we discuss some general notation and definitions used throughout the paper and set up our constrained minimization space. In particular, we will minimize \( I(u, v, w) \) over the space \( \mathcal{C} \), consisting of triples \( (u, v, w) \) such that \( w \) is a weak solution to equation (2.23) with \( u \) and \( v \) given. This approach is similar to those of Schechter & Weder (1981) and Yang (2001) for the dyon problem in three spaces.

Note that, in view of the radially symmetric ansatz (2.18)–(2.20), the total energy calculated from the Hamiltonian density (2.11) is

\[ E(u, v, w) = \pi \int_0^\infty \left( r (u')^2 + \frac{N^2}{r} (v')^2 + r (w')^2 + \frac{N^2}{r} u^2 (v - 1)^2 \right) \, dr \]

\[ + ru^2 w^2 + \frac{\lambda}{4} (u^2 - 1)^2 r \]  

(2.27)

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In §4, we assume bounded \( G(u, v) \) energy and we show that \( J_{u,v}(w) \) has a minimizer, say \( w_{u,v} \), among \( H^1_r \) functions, and this minimizer is the unique critical point of \( J_{u,v}(w) \). Here we first show that we have a uniform control of the radius \( R \),
such that $|u(r)| > \frac{1}{2}$ (say) outside the ball $B_R$, which implies both the boundedness of $J_{u,v}(w)$, $C \geq J_{u,v}(w) \geq -C$ and the control of the $H^1$ norm of $w$. Such boundedness and $H^1$ control give us the existence of a minimizer for $J_{u,v}$.

We prove the existence of weak solutions of equations (2.21)–(2.23) in §5 and §6. To do so we show that $I(u,v,w) \geq G(u,v)$ for $(u,v,w) \in C$, which implies the coercivity of $I(u,v,w)$. Once we have this coercivity behaviour, we can take a minimizing sequence in $C$ and obtain a constrained minimizer. Such a minimizer can be shown to solve the equations (2.21)–(2.23) at least in a weak sense. Here some extra attention will be given to proving the existence of a Fréchet derivative.

In §7, we establish the boundary conditions and expected full regularity of our solutions. In §8, we obtain the quantization formulas for the magnetic and electric charges. In §9, we construct non-Abelian Chern–Simons–Higgs vortex solutions using our methods presented in the previous sections.

3. Radial equations action principle and the constrained admissible space

Recall that a radially symmetric solution of the Chern–Simons–Higgs theory with $N$ vortices clustered at the origin satisfies equations (2.21)–(2.23), which can be derived from the indefinite action functional

$$I(u,v,w) = \int_0^\infty \left( r(u')^2 + \frac{N^2}{r} u^2(v - 1)^2 + \frac{\lambda}{4} (1 - u^2)^2 r + \frac{N^2}{r} (v')^2 \right) dr. \quad (3.1)$$

Let

$$G(u,v) = \int_0^\infty \left( r(u')^2 + \frac{N^2}{r} u^2(v - 1)^2 + \frac{\lambda}{4} (1 - u^2)^2 r + \frac{N^2}{r} (v')^2 \right) dr \quad (3.2)$$

and

$$J_{u,v}(w) = \int_0^\infty (r(u')^2 + ru^2w^2 + 2\kappa N v' w) dr. \quad (3.3)$$

Then $I(u,v,w) = G(u,v) - J_{u,v}(w)$. Note that $G(u,v)$ does not depend on $w$ and has the form of the Ginzburg–Landau energy, whereas $J_{u,v}(w)$ contains an indefinite part $\int_0^\infty 2\kappa N v' w dr$.

The natural admissible space $A$ is defined by

$$A = \{(u,v,w) | E(u,v,w) < \infty \text{ and } u,v,w \text{ satisfy (2.24), (2.25)}\}. \quad (3.4)$$

Note that here we leave out the boundary condition (2.26) in the admissible set because it cannot be simply recovered from a finite energy requirement. However, condition (2.26) will be obtained when we construct a constrained admissible space.

Our goal is to find a critical point of the functional (3.1) in the admissible space $A$. However, the difficulty comes from both the negative definite energy part and the indefinite energy part, which is an obstacle to getting the coerciveness of $I(u,v,w)$. Motivated by the idea of the constrained minimization methods by Schechter & Weder (1981) and Yang (2001), we look for a suitable set of constraints to restrict the consideration of (3.1) over a smaller admissible space,
say \( C \). With this choice of \( C \), \( I(u, v, w) \) becomes coercive on \( C \) and the minimizer of \( I(u, v, w) \) over \( C \) can be shown to be a critical point over the original admissible space \( A \), and thus is a solution of equations (2.21)–(2.23).

In order to make \( I(u, v, w) \) coercive over a properly constrained admissible space \( C \), we need to control \( J_{u,v}(w) \). To do so, we need to ‘freeze’ the unknown \( w \), which certainly cannot be done arbitrarily since we are looking for a solution of equations (2.21)–(2.23) eventually. Hence, we naturally require \( w \) to satisfy equation (2.23) in a suitable weak sense for given \( u \) and \( v \). In this way, we are led to considering seeking, for each fixed pair \((u, v)\), a critical point of the functional \( J_{u,v}(w) \). In order to get a good convergence result, we restrict further to considering \( w \in H^1_r(\mathbb{R}^2) \), where

\[
H^1_r(\mathbb{R}^2) = \{ f \in H^1(\mathbb{R}^2) \mid f \text{ is radially symmetric about the origin} \}. \tag{3.5}
\]

We often use \( f(r) \) to unambiguously denote the radial dependence of the function \( f \) over \( \mathbb{R}^2 \), which is symmetric about the origin of \( \mathbb{R}^2 \).

Note that \( w \in H^1_r(\mathbb{R}^2) \) implies \( w(\infty) = 0 \) (Strauss 1977). In fact, it is easily seen that the set of all \( w \in H^1_r(\mathbb{R}^2) \) so that \( J_{u,v}(w) < \infty \) is an affine linear space. Besides, since \( J_{u,v} \) is strictly convex with respect to \( w \) for each given pair \((u, v)\),

\[
J_{u,v} \text{ can at most have one critical point.} \tag{3.6}
\]

If \( w \) is a critical point, then

\[
\int_0^\infty (rw'\tilde{w}' + ru^2w\tilde{w} + \kappa Nv'\tilde{w}) \, dr = 0, \tag{3.7}
\]

for all \( \tilde{w} \in H^1_r(\mathbb{R}^2) \) such that

\[
J_{u,v}(w + \tilde{w}) < \infty.
\]

In this way, we may define the constrained admissible space

\[
C = \{(u, v, w) \in A \mid w \in H^1_r(\mathbb{R}^2), (u, v, w) \text{satisfies (3.7)} \}. \tag{3.8}
\]

We need to make sure that \( C \) is not empty. A natural way is to use the variational approach, that is, to consider minimizing \( J_{u,v}(w) \) over \( w \in H^1_r(\mathbb{R}^2) \) for certain fixed \((u, v)\). The major difficulty is that, when it comes to minimizing \( I(u, v, w) \), one is looking at a class of \((u, v)\). Moreover, \( J_{u,v}(w) \) contains an indefinite part, which, after applying Cauchy–Schwarz, introduces a term \( \|w\|^2_{L^2(\mathbb{R}^2)} \) that cannot be controlled by \( \|w'\|^2_{L^2(\mathbb{R}^2)} \) only. Therefore, we have to enlist the second term \( \|uw\|^2_{L^2(\mathbb{R}^2)} \) in \( J_{u,v}(w) \) to help control the \( H^1 \) norm of \( w \).

4. Minimization of \( J_{u,v}(w) \)

Since \( u \) may vanish in a finite-energy setting, we need to control the size of the set in which \(|u| \leq \frac{1}{2}\).

**Proposition 4.1.** Suppose that \((u, v)\) satisfies that \( G(u, v) \leq M < \infty \). Then there exists an \( R \) independent of \( u \) such that \( \{x : |u(x)| \leq \frac{1}{2}\} \subset B_R \), where \( B_R \) is a ball in \( \mathbb{R}^2 \) of radius \( R \) centred at the origin.
Proof. Consider a pair \((u, v)\) such that \(G(u, v) \leq M < \infty\). Then using the result in Ginzburg–Landau theory (Berger & Chen 1989), we know that \(1 - |u| \in H^1_r(\mathbb{R}^2)\). We also know that \(||u'|| \leq |u'|\) a.e. (Gilbarg & Trudinger 1983). Hence, we have

\[
(1 - |u(r)|)^2 \leq 2 \int_r^\infty |1 - |u(r)||u'(r)|| \, d\rho
\]

\[
\leq \frac{4}{r} G(u, v) \leq \frac{4M}{r \sqrt{\lambda}}.
\]

In this way, we may choose

\[
R = \frac{16M}{\sqrt{\lambda}}
\]

so that \(|u(x)| > \frac{1}{2}\) for \(|x| \geq R\). 

We are now ready to study the minimization problem for \(J_{u,v}(w)\) over \(H^1_r(\mathbb{R}^2)\), for a fixed pair \((u, v)\) such that \(G(u, v) < \infty\), \(u(0) = v(0) = 0\) and \(u(\infty) = v(\infty) = 1\).

**Lemma 4.2.** For each \((u, v)\) with \(G(u, v) < \infty\), the following minimization problem

\[
\min\{J_{u,v}(w) \mid w \in H^1_r(\mathbb{R}^2)\}
\]

has a unique solution. Hence \(C \neq \emptyset\).

**Proof.** The uniqueness of the minimizer can be seen from the fact that the functional \(J_{u,v}(w)\) is strictly convex.

In order to prove the existence of the minimizer, we need to first show that \(J_{u,v}(w)\) is bounded from below, provided that \(G(u, v) \leq M < \infty\). Using Cauchy–Schwartz, we have

\[
J_{u,v}(w) \geq \int_0^\infty \left( r(w')^2 + ru^2 w^2 - \varepsilon rw^2 - \frac{1}{\varepsilon} \frac{\kappa^2 N^2}{r} (v')^2 \right) \, dr
\]

\[
= \int_0^\infty r(w')^2 \, dr + \int_{|u| > \frac{1}{2}} r(u^2 - \varepsilon) w^2 \, dr
\]

\[
+ \int_{|u| \leq \frac{1}{2}} r(u^2 - \varepsilon) w^2 \, dr - \frac{1}{\varepsilon} \int_0^\infty \frac{\kappa^2 N^2}{r} (v')^2 \, dr
\]

\[
\geq \int_0^\infty r(w')^2 \, dr + \int_{|u| > \frac{1}{2}} \left( \frac{1}{4} - \varepsilon \right) rw^2 \, dr
\]

\[
- \int_{|u| \leq \frac{1}{2}} \varepsilon rw^2 \, dr - \frac{1}{\varepsilon} \int_0^\infty \frac{\kappa^2 N^2}{r} (v')^2 \, dr
\]

\[
\geq \int_0^\infty r(w')^2 \, dr + \int_R^\infty \left( \frac{1}{4} - \varepsilon \right) rw^2 \, dr - \int_0^R \varepsilon rw^2 \, dr - \frac{1}{\varepsilon} \int_0^\infty \frac{\kappa^2 N^2}{r} (v')^2 \, dr,
\]

where \(R\) in the last inequality is defined by equation (4.1).

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Take a smooth function \( \eta(x) \) on \( \mathbb{R}^2 \) such that \( \text{supp} \ \eta \subseteq B_{2R}, \ 0 \leq \eta(x) \leq 1 \) and \( \eta \equiv 1 \) on \( B_R \). Let \( \tilde{w} = \eta w \). Then \( \tilde{w} \in H_0^1(\mathbb{R}^2) \). Hence using Poincaré’s inequality, we have
\[
\int_{B_R} \tilde{w}^2 \, dx \leq \int_{B_R} \eta \tilde{w}^2 \, dx \leq CR \int_{B_{2R}} |\nabla \tilde{w}|^2 \, dx \leq CR \| \nabla \tilde{w} \|_{L^2(\mathbb{R}^2)}^2.
\]
However,
\[
\int_{\mathbb{R}^2} |\nabla \tilde{w}|^2 \, dx = \int_{\mathbb{R}^2} (|\nabla \eta| \, w + \eta |\nabla w|^2) \, dx \leq 2 \int_{\mathbb{R}^2} (|\nabla \eta|^2 + |\eta \nabla w|^2) \, dx \leq 2 \left( C \int_{B_R} w^2 \, dx + \int_{\mathbb{R}^2} |\nabla w|^2 \, dx \right).
\]
Therefore,
\[
\int_{B_R} w^2 \, dx = \int_0^R r^2 \, d(r \, d) \leq CR \left( \int_0^\infty (w')^2 \, d(r \, d) + \int_R^\infty r w^2 \, d(r \, d) \right). \tag{4.3}
\]
Hence, we obtain
\[
J_{u,v}(w) \geq (1 - \varepsilon CR) \int_0^\infty (w')^2 \, d(r \, d) + \left( \frac{1}{4} - (1 + CR)\varepsilon \right) \int_R^\infty r w^2 \, d(r \, d)
\]
\[
- \frac{1}{\varepsilon} \int_0^\infty \frac{\kappa^2 N^2}{r} (v')^2 \, d(r \, d) \quad \text{(now choosing } \varepsilon = 1/(1 + CR))
\]
\[
\geq \frac{7}{8} \int_0^\infty (w')^2 \, d(r \, d) + \frac{1}{8} \int_R^\infty r w^2 \, d(r \, d) - 8(1 + CR) \int_0^\infty \frac{\kappa^2 N^2}{r} (v')^2 \, d(r \, d)
\]
\[
\geq \frac{7}{8} \int_0^\infty (w')^2 \, d(r \, d) + \frac{1}{8} \int_R^\infty r w^2 \, d(r \, d) - 8(1 + CR)\kappa^2 M.
\]
From equation (4.3) and the above inequality, we can also derive the following control of \( H^1 \) norm of \( w \) in terms of \( J_{u,v}(w) \):
\[
\| w \|^2_{H^1(\mathbb{R}^2)} = \int_0^\infty (w')^2 \, d(r \, d) + \int_0^\infty r w^2 \, d(r \, d)
\]
\[
\leq (1 + CR) \left( \int_0^\infty (w')^2 \, d(r \, d) + \int_R^\infty r w^2 \, d(r \, d) \right)
\]
\[
\leq 8(1 + CR)[J_{u,v}(w) + 8(1 + CR)\kappa^2 M]. \tag{4.4}
\]
Now we can take a minimizing sequence \( \{ w_n \} \) in \( H^1_r(\mathbb{R}^2) \). Then, by equation (4.4), \( \| w_n \|_{H^1(\mathbb{R}^2)} \) is uniformly bounded. Hence (up to a subsequence)
\[ w_n \rightharpoonup w \ \text{in} \ H^1_r(\mathbb{R}^2). \]
Then by the compactness lemma in Strauss (1977), we know that
\[ w_n \longrightarrow w \ \text{a.e. on} \ (0, \infty). \]
From Berger & Chen (1989), we know that \( G(u, v) < \infty \) implies that \( 1 - |u| \in H^1_r(\mathbb{R}^2) \) and \( v' / r \in L^2(\mathbb{R}^2) \). Thus, the weak lower semicontinuity of the \( L^2 \) norm,
the Fatou’s lemma and the weak convergence of \( w_n \) imply that

\[
J_{u,v}(w) = \int_0^\infty (r(w')^2 + ru^2 w^2 - 2\kappa N v' w) \, dr \leq \liminf_{n \to \infty} J_{u,v}(w_n).
\]

Therefore, \( w \) solves equation (4.2).

Critical points of \( J_{u,v}(w) \), of course, satisfy equation (3.7), \( C \neq \emptyset \), and the lemma is proved. \( \blacksquare \)

Remark. From the above proposition, we understand the structure of \( C \) explicitly: for any pair \((u, v)\) satisfying \( G(u, v) < \infty \), equations (2.4) and (2.5), then \((u, v, w) \in C\) is the unique triplet such that \( w \) is the unique solution to equation (2.5) and in fact minimizing \( J_{u,v}(w) \). Thus each pair \( u, v \) unambiguously defines \( w = w_{(u,v)} \) and \( C \) looks like the image of the map \((u, v) \mapsto w_{(u,v)} \) in \( \mathcal{A} \).

5. Minimization of \( I(u, v, w) \)

In this section, we try to solve the minimization problem of the full energy \( I(u, v, w) \) over the constrained admissible space \( C \). We first show that \( I(u, v, w) \) is positive definite and coercive with respect to \( u \) and \( v \) on \( C \).

Proposition 5.1. For \((u, v, w) \in C\),

\[
I(u, v, w) \geq G(u, v). \tag{5.1}
\]

Proof. Considering equation (3.7) for \((u, v, w)\) and taking \( \bar{w} = w \), we get

\[
\int_0^\infty (r(w')^2 + ru^2 w^2 + \kappa N v' w) \, dr = 0.
\]

Therefore,

\[
J_{u,v}(w) = -\int_0^\infty (r(w')^2 + ru^2 w^2) \, dr \leq 0. \tag{5.2}
\]

Hence, we have equation (5.1). \( \blacksquare \)

Proposition 5.2. The minimization problem

\[
\min\{I(u, v, w) \mid (u, v, w) \in C\} \tag{5.3}
\]

has a solution.

Proof. By lemma 4.2, we can take a minimizing sequence \( \{(u_n, v_n, w_n)\} \) of equation (5.3). Since all terms involving function \( u \) appear in a quadratic form, we may take all \( u_n \geq 0 \). From equation (5.1), we know that \( \{G(u_n, v_n)\} \) is uniformly bounded. Therefore, from Berger & Chen (1989), we know that \( \|1-u_n\|_{H^1_0(\mathbb{R}^2)} \) and \( \|v_n\|_{C_S} = \|(1/r)v'\|_{L^2(\mathbb{R}^2)} \) are uniformly bounded, where

\[
C_S = \left\{ \text{the set of real-valued radially symmetric functions } v(|x|) \text{ on } \mathbb{R}^2 \right\}
\]

such that \((1/r)v \in L^2_{\text{loc}}(\mathbb{R}^2)\) and \((1/r)v' \in L^2(\mathbb{R}^2)\), where the derivative \( v' \) is in the distributional sense.

Hence,

\[
1 - u_n \rightharpoonup 1 - u \text{ in } H^1_0(\mathbb{R}^2), \quad v_n \rightharpoonup v \text{ in } C_S.
\]
Moreover, we have

\[ H \rightarrow w \text{ in } H^1_\ell (\mathbb{R}^2). \]

Moreover, we have

\[ u_n \rightarrow u, \quad v_n \rightarrow v, \quad w_n \rightarrow w, \text{ a.e. on } (0, \infty). \]

Next we check that \((u, v, w) \in \mathcal{C}\), that is, equation (3.7) is satisfied for all \(\bar{w} \in H^1_\ell (\mathbb{R}^2)\) with \(J_{u, v}(w + \bar{w}) < \infty\).

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Weak convergence of \(\{w'_n\}\) in \(L^2\) and \(\{v_n\}\) in \(C_S\) implies that

\[
\lim_{n \to \infty} \int_0^\infty rw'_n \bar{w}' \, dr = \int_0^\infty rw' \bar{w}' \, dr \quad \text{and} \quad \lim_{n \to \infty} \int_0^\infty v'_n \bar{w} \, dr = \int_0^\infty v' \bar{w} \, dr.
\]

As for the second term in equation (3.7),

\[
\int_0^\infty ru^2_n w_n \bar{w} \, dr - \int_0^\infty ru^2 w \bar{w} \, dr \\
= \int_0^\infty r(u_n - u)u_n w_n \bar{w} \, dr + \int_0^\infty ruu_n(w_n - w) \bar{w} \, dr + \int_0^\infty ru(u_n - u)w \bar{w} \, dr \\
= T_1 + T_2 + T_3.
\]

Using the compact embedding of \(H^1_\ell (\mathbb{R}^2) \subset \subset L^p_\ell (\mathbb{R}^2)\) for any \(p > 2\) (Chabrowski 1992),

\[
|T_1| = \left| \int_0^\infty r(u_n - u)(u_n - 1)w_n \bar{w} \, dr + \int_0^\infty r(u_n - u)w_n \bar{w} \, dr \right| \\
\leq \|1 - u_n\|_L^1 \|1 - u\|_L^1 \|w_n\|_L^1 \|\bar{w}\|_L^1 \\
+ \|1 - u_n\|_L^1 \|w_n\|_L^1 \|\bar{w}\|_L^1 \\
\rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Similarly,

\[
|T_2| = \left| \int_0^\infty r(1 - u)(1 - u_n)(w_n - w) \bar{w} \, dr + \int_0^\infty r(w_n - w) \bar{w} \, dr \\
+ \int_0^\infty r(u - 1)(w_n - w) \bar{w} \, dr + \int_0^\infty r(u_n - 1)(w_n - w) \bar{w} \, dr \right| \\
\rightarrow 0, \text{ as } n \rightarrow \infty.
\]

and

\[
|T_3| = \left| \int_0^\infty r(u - 1)((1 - u) - (1 - u_n))w \bar{w} \, dr + \int_0^\infty r((1 - u) - (1 - u_n))w \bar{w} \, dr \right| \\
\rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Therefore, we have proved that \((u, v, w) \in \mathcal{C}\). To show that the limiting configuration \((u, v, w)\) is a minimizer of equation (5.3), we consider equation (3.7)
for \((u, v, w) = (u_n, v_n, w_n)\) and take \(\tilde{w} = w_n\). Then
\[
\int_0^\infty (r(w'_n)^2 + ru_n^2 w_n^2 + \kappa N v'_n w_n) \, dr = 0.
\]
In the same way, considering equation (3.7) for \((u, v, w)\) and taking \(\tilde{w} = w\), we get
\[
\int_0^\infty (r(w')^2 + ru^2 w^2 + \kappa N v'w) \, dr = 0.
\]
Therefore,
\[
J_{\tilde{u}, \tilde{v}}(w_n) = -\int_0^\infty (r(w'_n)^2 + ru_n^2 w_n^2) \, dr
\]
and
\[
J_{u, v}(w) = -\int_0^\infty (r(w')^2 + ru^2 w^2) \, dr.
\]
Thus, using weak lower semicontinuity and Fatou’s lemma, we have
\[
I(u, v, w) = G(u, v) + \int_0^\infty (r(w')^2 + ru^2 w^2) \, dr
\]
\[
\leq \liminf_{n \to \infty} \left( G(u_n, v_n) + \int_0^\infty (r(w'_n)^2 + ru_n^2 w_n^2) \, dr \right)
\]
\[
= \liminf_{n \to \infty} I(u_n, v_n, w_n).
\]
Hence, we conclude that such a limit \((u, v, w)\) satisfies equation (5.3). 

6. Weak solutions of the governing equations

We use the idea developed in Yang (2001) to establish the existence of weak solutions.

**Proposition 6.1.** The action minimizing solution \((u, v, w)\) of problem (5.3) is a weak solution of equations (2.21)–(2.23), subject to the partial boundary conditions (2.24) and (2.25).

**Proof.** As discussed earlier, with
\[
\mathcal{S} = \{(u, v) \mid G(u, v) < \infty, \quad u(0) = v(0) = 0, \quad u(\infty) = v(\infty) = 1\},
\]
the constrained set \(\mathcal{C}\) may be viewed as the image of the map \(\chi : \mathcal{S} \to \mathcal{A}, (u, v) \mapsto (u, v, w(u, v))\) with \(w = w(u, v)\) being determined by equation (3.7), which is the weak form of equation (2.23). Consequently, \(\chi\) is a differentiable map in an obvious sense. Besides, the minimizer \((u, v, w)\) of the constrained problem (5.3) obtained in proposition 5.2 may simply be viewed as the image under \(\chi\) of an absolute minimizer \((u, v)\) of the functional \(I(u, v, w(u, v))\) over the unconstrained class \(\mathcal{S}\).

Let \(h\) be a real parameter confined in a small interval, say, \(|h| < 1\) and \(\tilde{u} \in C^1_0(0, \infty)\) (functions with compact supports). Set \(w_h = w(u+h\tilde{u}, v)\). We use the following notations:
\[
\Delta w = w_h - w \quad \text{and} \quad Dw = \lim_{h \to 0} \frac{\Delta w}{h} = \frac{d\Delta w}{dh} \bigg|_{h=0}.
\]
Then we use $\Delta w$ as a test function in equation (3.7) to get
\[
\int_0^\infty (rw'(\Delta w)' + ru^2 w\Delta w + \kappa N u'\Delta w) \, dr = 0.
\]
We also have
\[
\int_0^\infty (rw'_h(\Delta w)' + r(u + h\tilde{u})^2 w_h\Delta w + \kappa N u'\Delta w) \, dr = 0.
\]
Subtracting the first equality from the second, we get
\[
\int_0^\infty (r[(\Delta w)']^2 + ru^2(\Delta w)^2) \, dr = -h \int_0^\infty (2ru\tilde{u}w_h\Delta w + hr\tilde{u}^2 w_h\Delta w) \, dr.
\]
Using Cauchy–Schwartz, we obtain
\[
\int_0^\infty \left(r \left[\frac{(\Delta w)'}{h}\right]^2 + ru^2 \left(\frac{\Delta w}{h}\right)^2\right) \, dr \leq \int_0^\infty \left(\frac{1}{2}ru^2 \left(\frac{\Delta w}{h}\right)^2 + 2r\tilde{u}^2 w_h^2 + r\tilde{u}^2 |w_h\Delta w|\right) \, dr.
\]
Hence,
\[
\int_0^\infty \left(r \left[\frac{(\Delta w)'}{h}\right]^2 + ru^2 \left(\frac{\Delta w}{h}\right)^2\right) \, dr \leq 15 \int_0^\infty (r\tilde{u}^2 w_h^2 + r\tilde{u}^2 w^2) \, dr. \tag{6.2}
\]
From equations (4.4) and (5.2), we know that
\[
\|w_h\|_{L^2}^2 \leq \|w_h\|_{H^1}^2 \leq C,
\]
where $C$ depends on $G(u + h\tilde{u}, v)$. By assumption, we know $|h| < 1$. Then
\[
\int_0^\infty r((u + h\tilde{u})'')^2 \, dr \leq 2\int_0^\infty (r(u')^2 + rh^2(\tilde{u}'')^2) \, dr
\]
\[
\leq 2\int_0^\infty (r(u')^2 + r(\tilde{u}'')^2) \, dr,
\]
\[
\int_0^\infty \frac{N^2}{r} (u + h\tilde{u})^2(v - 1)^2 \, dr \leq 2\int_0^\infty \frac{N^2}{r} (u^2 + \tilde{u}^2)(v - 1)^2 \, dr
\]
and
\[
\int_0^\infty (1 - (u + h\tilde{u})^2) \, dr = \int_0^\infty ((1 - u^2) + 2h\tilde{u}(1 - u) - 2h\tilde{u} - h^2 \tilde{u}^2)^2 \, dr
\]
\[
\leq 2(1 + \|\tilde{u}\|_{L^\infty}^2)^2 \int_0^\infty (1 - u^2)^2 \, dr + 2\|\tilde{u}\|_{L^\infty}^2 \int_{\text{supp } \tilde{u}} (2 + |\tilde{u}|)^2 \, dr.
\]
Hence,
\[
G(u + h\tilde{u}, v) \leq CG(u, v) + C,
\]
where $C$ depends on $\tilde{u}$, not on $h$.

Similarly, we obtain that
\[
\|w\|_{L^2}^2 \leq C,
\]
where $C$ is independent of $h$. Thus,
\[
\int_0^\infty \left(r \left[\frac{(\Delta w)'}{h}\right]^2 + ru^2 \left(\frac{\Delta w}{h}\right)^2\right) \, dr \leq C, \tag{6.3}
\]
where $C$ is independent of $h$. Taking $h \to 0$ in equation (6.3), we have

$$\int_0^\infty (r[(Dw)'])^2 + ru^2(Dw)^2 \, dr \leq C.$$  

The third term in $J_{u,v}(Dw)$ can be bounded as follows by using proposition 9.1 and equation (4.4):

$$\int_0^\infty 2\kappa N v'Dw \, dr \leq \int_0^\infty \left( \kappa^2 \frac{N^2}{r} (v')^2 + r(Dw)^2 \right) \, dr$$

$$\leq \kappa^2 G(u, v) + \| Dw \|^2_{H^1(\mathbb{R}^2)}$$

$$\leq \kappa^2 G(u, v) + (1 + CR) \left( \int_0^\infty r[(Dw)']^2 \, dr + \int_R^\infty r(Dw)^2 \, dr \right)$$

$$\leq \kappa^2 G(u, v) + (1 + CR) \left( \int_0^\infty r[(Dw)']^2 \, dr + 4 \int_R^\infty ru^2(Dw)^2 \, dr \right)$$

$$\leq \kappa^2 G(u, v) + 4(1 + CR) \int_0^\infty (r[(Dw)']^2 + ru^2(Dw)^2) \, dr \leq C,$$

where we have used equation (4.4) from the second inequality to the third and proposition 9.1 from the third to the fourth. Therefore, we get $J_{u,v}(Dw) < \infty$. Hence,

$$J_{u,v}(w + Dw) < \infty.$$  

Furthermore from the above estimates, we also obtain that $Dw \in H^1_r(\mathbb{R}^2)$. Therefore, equation (3.7) is satisfied with $\tilde{w} = Dw$.

Since $(u, v)$ minimizes $I(u, v, w_{(u,v)})$, we have

$$\frac{d}{dh} I(u + h\tilde{u}, v, w_h) \bigg|_{h=0} = 0,$$

which gives

$$\int_0^\infty \left( ru' \tilde{u} - \frac{\lambda}{2} ru(1 - u^2) \tilde{u} + \frac{N^2}{r} u(v - 1)^2 \tilde{u} - ruw^2 \tilde{u} \right) \, dr$$

$$= \int_0^\infty (r\tilde{u}'(Dw)' + ru^2 w Dw + \kappa N v'Dw) \, dr = 0. \quad (6.4)$$

The left-hand side of the above leads to the validity of a weak form of equation (2.21).

Similarly, we fix a compactly supported test function $\tilde{v}$ and consider $w_h = w_{(u,v+h\tilde{v})}$ as before. We can show in a similar way as we did for equation (6.4) that

$$\int_0^\infty \left( \frac{N^2}{r} v' \tilde{v}' + \frac{N^2}{r} u^2(v - 1) \tilde{v} + \kappa N w' \tilde{v} \right) \, dr = 0, \quad (6.5)$$

which is the weak form of equation (2.22). Therefore, the proof of the proposition is complete. 

7. Full set of boundary conditions and regularity

In this section, we show that the remaining boundary condition (2.26) also holds for the solution \((u, v, w)\) obtained in the last section and then prove that the solution \((u, v, w)\) is indeed a classical solution to equations (2.21)–(2.23).

**Lemma 7.1.** Let \((u, v, w)\) be the solution of equations (2.21)–(2.23) obtained in the last section. Then equation (2.26) holds for a certain suitable \(w_0\).

**Proof.** From the finite-energy configuration, we know
\[
\int_0^\infty \frac{(v')^2}{r} \, dr < \infty,
\]
and we have
\[
\liminf_{r \to 0} |v'(r)| = 0. \tag{7.1}
\]
We rewrite equation (2.22) as
\[
(rv')' = 2v' + r(v - 1)u^2 + \frac{\kappa}{N} r^2 w'. \tag{7.2}
\]
Integrating equation (7.2) and using equation (7.1), we have
\[
r v'(r) = 2v(r) + \int_0^r \rho (v(\rho) - 1)u^2(\rho) \, d\rho + \frac{\kappa}{N} \int_0^r \rho^2 w'(\rho) \, d\rho. \tag{7.3}
\]
On the other hand, the condition
\[
\int_0^\infty r(w')^2 \, dr < \infty
\]
implies that
\[
\liminf_{r \to 0} |r| |w'(r)| = 0. \tag{7.4}
\]
Using equation (7.4) to integrate equation (2.23), we obtain, in view of equation (7.2), that
\[
w'(r) = \frac{1}{r} \int_0^r \rho u^2(\rho) w(\rho) \, d\rho + \frac{\kappa}{N} \frac{v(r)}{r}
\]
\[
= \frac{1}{r} \int_0^r \rho u^2(\rho) w(\rho) \, d\rho + \frac{\kappa N}{2} v'(r) - \frac{\kappa N}{2r} \int_0^r (v(\rho) - 1) u^2(\rho) \, d\rho
\]
\[
+ \frac{\kappa^2}{2r} \int_0^r \rho^2 w'(\rho) \, d\rho
\]
\[
\equiv \frac{1}{r} I_1(r) + \frac{\kappa N}{2} v'(r) - \frac{\kappa N}{2r} I_2(r) + \frac{\kappa^2}{2r} I_3(r), \quad r > 0. \tag{7.5}
\]
For \(I_1(r)\), the Schwartz inequality gives us
\[
|I_1(r)| \leq Cr \left( \int_0^r \rho u^2(\rho) w^2(\rho) \, d\rho \right)^{1/2}, \tag{7.6}
\]
where $C$ may depend on the upper bound of $|u|$. Similarly, for $I_2(r)$ and $I_3(r)$, we have

$$|I_2(r)| \leq Cr^2 \left( \int_0^r \frac{1}{\rho} (v(\rho) - 1)^2 u^2(\rho) \, d\rho \right)^{1/2} \quad (7.7)$$

and

$$|I_3(r)| \leq Cr^2 \left( \int_0^r \rho (w'(\rho))^2 \, d\rho \right)^{1/2}. \quad (7.8)$$

Note that each of the right-hand sides of equations (7.6)–(7.8) appears in the energy functionals.

Integrating equation (7.5) and using equations (7.6)–(7.8), we see that the limit $w_0 = \lim_{r \to 0} w(r)$ exists as hoped.

Lemma 7.2. Through the ansatz (2.18)–(2.20), the solution $(u, v, w)$ of the radial equations (2.21)–(2.23) obtained in the last section gives rise to a classical (smooth) solution $(\phi, A_1, A_0)$ of the static Chern–Simons–Higgs equations (2.7)–(2.9) over $\mathbb{R}^2$.

Proof. We first prove the interior regularity of solutions. From the minimization procedure, we obtain that the weak solution lives in the space: $1 - u \in H^1_r$, $v \in C_S$, where $C_S$ is defined in equation (5.4), and $w \in H^1_r$. For any $0 < \delta < R$, let $\Omega = B_R \setminus B_{\delta}$, then $(u, v, w)$ is a generalized solution of the system

$$-\Delta u = \frac{\lambda}{2} u(1 - u^2) - \frac{N^2}{r^2} (v - 1)^2 u + w^2 u,$$

$$-\Delta v = (1 - v) u^2 - \frac{2}{r} v' - \frac{\kappa r}{N} w'$$

and

$$- \Delta w = -u^2 w - \frac{\kappa N}{r} v',$$

on $\Omega$. The right-hand side of the third equation is in $L^2(\Omega)$. Hence, $w \in H^2(\Omega)$. In the second equation,

$$\|(1 - v)u^2\|^2_{L^2(\Omega)} = \int_\Omega (v - 1)^2 u^4 \, dx \leq \|ru\|^2_{L^\infty(\Omega)} \int_{\mathbb{R}^2} \frac{(v - 1)^2}{r^2} u^2 \, dx < \infty.$$ 

Hence, we also have $v \in H^2(\Omega)$. In the first equation,

$$\left\| \frac{(v - 1)^2}{r^2} u \right\|^2_{L^2(\Omega)} = \int_\Omega \frac{(v - 1)^2}{r^2} u^2 \cdot \frac{(v - 1)^2}{r^2} \, dx \leq \sup_{\delta < r < R} \left| \frac{v - 1}{r} \right|^2 \int_{\mathbb{R}^2} \frac{(v - 1)^2}{r^2} u^2 \, dx$$

$$\leq (C + \|v'/r\|^2_{L^2(\mathbb{R}^2)}) \int_{\mathbb{R}^2} \frac{(v - 1)^2}{r^2} u^2 \, dx < \infty,$$
where we have used the fact (Berger & Chen 1989)

\[ \sup_{0 < r < \infty} \left| \frac{1}{r} - v \right| \leq \|v' / r\|^2_{L^2(\mathbb{R}^2)}. \]

In this way, \( u \in H^2(\Omega) \). Therefore, by standard regularity theory of elliptic equations and using the iterative bootstrap argument, we conclude that \((u, v, w)\) is a classical solution of equations (2.21)–(2.23) on \( \Omega \).

Since both \( u \) and \( w \) satisfy the property that

\[ \lim_{r \to 0} \frac{u}{\ln r} = \lim_{r \to 0} \frac{w}{\ln r} = 0, \]

by the removable singularity theorem, the regularity of \( u \) and \( w \) extends to the origin, so does the regularity of \( \phi(x) \) and \( A_0(x) \) as in equations (2.18) and (2.20).

As for \( v \), we first look at \( A_j(x) \). Since \( \partial_j A_j(x) = 0 \) (divergence free) away from the origin, we know that, in \( \Omega \), \( A_j(x) \) satisfies

\[ \Delta A_j = h_j, \]

for some \( h_j \in L^2(\Omega) \). Since \( A_j(x) \) is an \( H^1(\Omega) \) solution, from the previous interior regularity argument, we know that it is also an \( H^2(\Omega) \) solution. Hence, we may apply the same removable singularity theorem to extend the regularity of \( A_j(x) \) to the origin.

Bootstrap then shows that \( \phi, A_j \) and \( A_0 \) are all smooth across the origin. For example, for \( \phi \), we notice that equation (2.7) may be rewritten as

\[ \Delta \phi - i2A_j \partial_j \phi - i(\partial_j A_j)\phi = (A_1^2 + A_2^2 - A_0^2)\phi + \frac{\lambda}{2}(|\phi|^2 - 1)\phi. \]

Therefore, we know that \((u, v, w)\) gives rise to a classical solution.

\[ \Box \]

8. Quantization of magnetic flux and electrostatic charge

We finish with the proof of (2.16) and (2.17).

**Proposition 8.1.** The solution satisfies the quantization relationship

\[ Q = \kappa \Phi = 2\pi N, \]

where \( Q \) is the electrostatic charge and \( \Phi \) is the magnetic flux.

**Proof.** In the static case, the \( \mu = 0 \) component of equation (2.3) is the Gauss law,

\[ \Delta A_0 = \kappa F_{12} + |\phi|^2 A_0, \]

where \( \rho = J^0 = -|\phi|^2 A_0 \) = electric charge density. (8.1)

On the other hand, within the radial ansatz (2.19), we know that the magnetic field is represented by

\[ F_{12} = N \frac{v'(r)}{r}, \quad r > 0. \]

Therefore, equation (2.23) is exactly the radial form of the Gauss law (8.1), which correctly relates the magnetic field \( F_{12} \) to the electric charge density \( J^0 \) and implies that electricity and magnetism must coexist when the Chern–Simons
coupling parameter is non-trivial, $\kappa \neq 0$. Thus, the total magnetic charge (flux) is given by

$$\Phi = \int_{\mathbb{R}^2} F_{12} \, dx = 2\pi N \int_0^\infty v'(r) \, dr = 2\pi N. \quad (8.2)$$

Since $\int_0^\infty r(w'(r))^2 \, dr < \infty$, then

$$\liminf_{r \to 0} \{ r \mid w'(r) \} = \liminf_{r \to \infty} \{ r \mid w'(r) \} = 0. \quad (8.3)$$

Multiplying equation (2.23) by $r$, integrating and using equation (8.3), we get

$$\int_0^\infty ru^2(r) w(r) \, dr = \kappa N \int_0^\infty v'(r) \, dr = \kappa N.$$  

In particular,

$$Q = \int_{\mathbb{R}^2} J^0 \, dx = \kappa \int_{\mathbb{R}^2} F_{12} \, dx = \kappa \Phi = 2\pi \kappa N,$$

which explicitly shows how electric charge is proportional to magnetic flux.

9. Application to non-Abelian Chern–Simons–Higgs equations

We start from the simplest non-Abelian case (de Vega & Schaposnik 1986a), where the gauge group is $SU(2)$ and the scalar fields are two scalar fields represented adjointly. For convenience, use isovectors. The Chern–Simons–Higgs field-theoretical Lagrangian density reads (de Vega & Schaposnik 1986a)

$$L = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} D_\mu \phi \cdot D^\mu \phi + \frac{1}{2} D_\mu \psi \cdot D^\mu \psi + \frac{1}{4} \kappa \varepsilon_{\mu\nu\rho\sigma} \left( F^{\alpha\mu} \cdot A^\rho - \frac{2}{3} A^\alpha \cdot [A^\mu \times A^\nu] \right) - V(\phi, \psi), \quad (9.1)$$

where $A_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$ ($\mu = 0, 1, 2$), $\phi = (\phi^1, \phi^2, \phi^3)$ and $\psi = (\psi^1, \psi^2, \psi^3)$ are isovectors,

$$D_\mu \phi = \partial_\mu \phi + A_\mu \times \phi,$$

$$D_\mu \psi = \partial_\mu \psi + A_\mu \times \psi$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu,$$

and the Higgs potential density is chosen to be

$$V(\phi, \psi) = \frac{1}{8} \lambda (|\phi|^2 - 1)^2 + \frac{1}{8} \lambda_1 (|\psi|^2 - 1)^2 + \frac{1}{2} \lambda_2 (\phi \cdot \psi)^2. \quad (9.2)$$

The equations of motion of (9.1) are

$$D_\mu D^\mu \phi = -\frac{\delta V}{\delta \phi} \quad \text{and} \quad D_\mu D^\mu \psi = -\frac{\delta V}{\delta \psi} \quad (9.3)$$

and

$$D_\mu F^{\mu\nu} = D^\nu \phi \times \phi + D^\nu \psi \times \psi + \frac{1}{2} \kappa \varepsilon^{\nu\alpha\beta} F_{\alpha\beta}. \quad (9.4)$$

Following de Vega & Schaposnik (1986a), we take the following radially symmetric ansatz for an electrically charged static vortex solution so that $\phi$ and $\psi$
are orthogonal in isospace,
\[
\phi = r(u)(\cos \theta, \sin \theta, 0) \quad \text{and} \quad \psi = (0, 0, 1)
\]
and
\[
A_r = 0, \quad A_0 = -v(r)(0, 0, 1) \quad \text{and} \quad A_0 = w(r)(0, 0, 1),
\]
where \(u, v\) and \(w\) are real-valued functions. Then the governing equations (9.3) and (9.4) become (2.21)–(2.23) when \(N = 1\),
\[
u'' - \frac{1}{r} u' = \frac{1}{r^2} (v - 1)^2 u - w^2 u + \frac{\lambda}{2} (u^2 - 1) u, \tag{9.7}
\]
\[
\nu' - \frac{1}{r} v = (v - 1) u^2 + \kappa rw', \tag{9.8}
\]
and
\[
\omega'' - \frac{1}{r} \omega' = u^2 \omega + \frac{\kappa}{r} \omega', \tag{9.9}
\]
subject to the boundary conditions (2.24)–(2.26). (Note that, in de Vega & Schaposnik (1986a), equation (2.26) is stated in a stronger form that the constant \(w_0\) assumes zero value. However, we have seen in our present study that \(w_0\) cannot be determined by the structure of the governing equations. This undeterminedness does affect the regularity, finiteness of energy and quantization of electric and magnetic charges of solutions.) Thus, the existence of electrically and magnetically charged static vortex solutions as described in de Vega & Schaposnik (1986a) follows.

We next describe how to apply our work to the study of the dually charged vortex solutions in the general non-Abelian Chern–Simons–Higgs gauge field theory. To be specific, we consider the \(SU(n) \quad (n \geq 3)\) theory formulated in de Vega & Schaposnik (1986b). We use \(su(n)\) to denote the Lie algebra of \(SU(n)\) consisting of \(n\) by \(n\) Hermitian matrices with vanishing trace. The inner product over \(su(n)\) is then defined by \((A, B) = \text{Tr}(AB^\dagger) = \text{Tr}(AB)\quad (A, B \in su(n))\). Recall that the dimension of the Cartan subalgebra, or the rank, of \(SU(n)\) is \(n - 1\). Following de Vega & Schaposnik (1986b), we consider the Chern–Simons–Higgs field theory housing \(2(n - 1)\) Higgs scalar particles \(\phi^a\) and \(\psi^a\) \((a = 1, 2, \ldots, n - 1)\) in the adjoint representation of \(SU(n)\) given by the Lagrangian density

\[
\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu \nu} F^{\mu \nu} + \text{Tr} \sum_{a=1}^{n-1} D_\mu \phi^a D^\mu \phi^a + \text{Tr} \sum_{a=1}^{n-1} D_\mu \psi^a D^\mu \psi^a + \frac{\kappa}{2} e^{\mu \nu \alpha} \text{Tr} \left( F_{\mu \nu} A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha \right) - V(\phi, \psi), \tag{9.10}
\]
where \(A_\mu \in su(n), F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\) and \(D_\mu = \partial_\mu + [A_\mu, \cdot]\), and the potential density may be chosen to take the typical form

\[
V(\phi, \psi) = \sum_{a=1}^{n-1} \frac{\lambda_a}{8} (|\phi^a|^2 - \eta_a^2)^2 + \sum_{a=1}^{n-1} \frac{\mu_a}{8} (|\psi^a|^2 - \gamma_a^2)^2 + \sum_{a, b=1}^{n-1} V_{ab}(\text{Tr}(\phi^a \psi^b)), \tag{9.11}
\]
in which \(V_{ab}\)'s are some functions satisfying \(V_{ab} \geq 0\) and \(V_{ab}(0) = 0 \quad (1 \leq a, b \leq n - 1)\) and \(\lambda_a, \eta_a, \mu_a, \gamma_a \quad (1 \leq a \leq n - 1)\) are positive coupling constants.

Recall that we can use the Cartan–Chevalley–Weyl basis \(\{H_\alpha, E_R\}\) to decompose \(su(n)\), where \(\{H_\alpha | a = 1, 2, \ldots, n - 1\}\) is a basis of the (Abelian) Cartan subalgebra and \(R = (R_1, \ldots, R_{n-1})\) are root vectors, so that the spaces \(H\) and \(E\), spanned by \(\{H_\alpha\}\) and \(\{E_R\}\), respectively, satisfy \(H \perp E, [H, H] = \{0\}, [H, E] \subset\)

$E, [E, E] \subset H$. With these facts, it is consistent to impose the condition that the
gauge field $A_\mu$ lies in $H$ and $\phi^a$ and $\psi^a$ stay in $E$ and $H$, respectively, for which
$\psi^a$ takes a constant value in $H$ $(a = 1, 2, \ldots, n - 1)$. Therefore, the equations of motion of (9.10) contain $A_\mu$ and $\Phi^a$ only, which are rewritten as (de Vega &
Schaposnik 1986b)

\[ D_\mu D^\mu \phi^a = \frac{\delta V}{\delta \phi^a} \quad (9.12) \]

and

\[ D_\nu F^{\mu \nu} - \frac{\kappa}{2} \varepsilon^{\mu \nu \alpha} F_{\nu \alpha} = J^\mu, \quad (9.13) \]

where $J^\mu = i \sum_{a=1}^{n-1} [D_\mu \phi^a, \phi^a]$ is the matter current generated from the Higgs
particles.

To proceed, we follow de Vega & Schaposnik (1986b) to write down the group
element

\[ \Omega_m(\theta) = \text{diag}\{ e^{im\theta/n}, e^{im\theta/n}, \ldots, e^{im\theta/n}, e^{-i[(n-1)/n]m\theta} \}, \quad (9.14) \]

$m = 0, 1, \ldots, n - 1$, which lies in the Cartan subgroup and is responsible for the
degeneracy of vacuum space. Then set

\[ M = -\frac{i}{m} \Omega_m^{-1} \partial_\theta \Omega_m = \text{diag}\{ \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, \frac{1-n}{n} \}, \quad 1 \leq m \leq n - 1. \quad (9.15) \]

The radially symmetric static vortex solutions of the $SU(n)$ Chern–Simons–
Higgs theory formulated in de Vega & Schaposnik (1986b) are given by the ansatz

\[ \phi^a = \frac{u_a(r)}{\sqrt{n}} \Omega_m^{-1}(\theta) E_{R_a} \Omega_m(\theta), \quad a = 1, 2, \ldots, n - 1 \quad (9.16) \]

and

\[ A_\theta = v(r) m M, \quad A_r = 0 \quad \text{and} \quad A_0 = w(r) m M, \quad (9.17) \]

realizing a solution asymptotically associated with the $m$th non-trivial vacuum
state represented by an integral class in the fundamental group of the coset
space of centre $Z_n$ of $SU(n)$, that is, by $m \in \pi_1(SU(n)/Z_n) = \mathbb{Z}_n$, where the
ladder generators $\{E_{R_a}\}$ are chosen to assume the normalized forms (de Vega &
Schaposnik 1986b)

\[ (E_{R_a})_{jk} = \frac{1}{\sqrt{2}} \delta_{ja} \delta_{kn}, \quad a = 1, 2, \ldots, n - 1, \quad j, k = 1, 2, \ldots, n. \quad (9.18) \]

Substituting equations (9.16) and (9.17) into equations (9.12) and (9.13)
and using equation (9.18), we arrive at the radial version of the equations of motion (de Vega & Schaposnik 1986b)

\[ u'' + \frac{1}{r} u' = \frac{m^2}{r^2} (v - 1)^2 u_a - m^2 w^2 u_a + \frac{\lambda_a}{2} (u_a^2 - \eta_a^2) u_a, \quad (9.19) \]

\[ v'' - \frac{1}{r} v' = \frac{1}{n - 1} \left( \sum_{a=1}^{n-1} u_a^2 \right) (v - 1) + \kappa r w' \quad (9.20) \]

and

\[ w'' + \frac{1}{r} w' = \frac{1}{n - 1} \left( \sum_{a=1}^{n-1} u_a^2 \right) w + \frac{\kappa}{r} v', \quad (9.21) \]
subject to the boundary condition consisting of
\[
\lim_{r \to 0} u_a(r) = \lim_{r \to 0} v(r) = \lim_{r \to \infty} w(r) = 0, \quad \text{for } a = 1, 2, \ldots, n - 1, \quad (9.22)
\]
\[
\lim_{r \to \infty} u_a(r) = \eta_a, \quad \text{for } a = 1, 2, \ldots, n - 1, \quad \lim_{r \to \infty} v(r) = 1 \quad (9.23)
\]
and
\[
\lim_{r \to 0} w(r) = w_0. \quad (9.24)
\]

The associated action functional to the above equations is
\[
\tilde{I}(u, v, w) = \int_0^\infty \left( r \left( \sum_{a=1}^{n-1} \left( u'_a \right)^2 \right) + \frac{m^2}{r} \left( \sum_{a=1}^{n-1} \frac{u_a^2(v - 1)^2}{n - 1} \right) 
\right.
\]
\[
+ \sum_{a=1}^{n-1} \frac{\lambda_a}{4} \left( \eta_a^2 - u_a^2 \right)^2 \frac{r}{n - 1} + \frac{m^2}{r} (v')^2 
\]
\[
- rm^2 (w')^2 - r \left( \sum_{a=1}^{n-1} \frac{u_a^2}{n - 1} \right) m^2 w^2 - 2\kappa m^2 v' w \right) \, dr, \quad (9.25)
\]

which is again indefinite of course. Here and in the sequel, we use the vector notation \( u = (u_a) = (u_1, \ldots, u_{n-1}) \). Let
\[
\tilde{G}(u, v) = \int_0^\infty \left( r \left( \sum_{a=1}^{n-1} \left( u'_a \right)^2 \right) + \frac{m^2}{r} \left( \sum_{a=1}^{n-1} \frac{u_a^2(v - 1)^2}{n - 1} \right) 
\right.
\]
\[
+ \sum_{a=1}^{n-1} \frac{\lambda_a}{4} \left( \eta_a^2 - u_a^2 \right)^2 \frac{r}{n - 1} + \frac{m^2}{r} (v')^2 \right) \, dr
\]
and
\[
\tilde{J}_{u,v}(w) = m^2 \int_0^\infty \left( r (w')^2 + r \left( \sum_{a=1}^{n-1} \frac{u_a^2}{n - 1} \right) w^2 + 2\kappa v' w \right) \, dr.
\]

Then it is clear that \( \tilde{I}(u, v, w) = \tilde{G}(u, v) - \tilde{J}_{u,v}(w) \).

The total energy is
\[
\tilde{E}(u, v, w) = \int_0^\infty \left( r \left( \sum_{a=1}^{n-1} \left( u'_a \right)^2 \right) + \frac{m^2}{r} (v')^2 + rm^2 (w')^2 
\right.
\]
\[
+ \frac{m^2}{r} \left( \sum_{a=1}^{n-1} \frac{u_a^2}{n - 1} \right) (v - 1)^2 + r \left( \sum_{a=1}^{n-1} \frac{u_a^2}{n - 1} \right) m^2 w^2 
\]
\[
+ r \sum_{a=1}^{n-1} \frac{\lambda_a}{4} \left( \eta_a^2 - u_a^2 \right)^2 \right) \, dr. \quad (9.26)
\]

Thus the natural admissible space \( \tilde{A} \) is
\[
\tilde{A} = \{ (u, v, w) \mid \tilde{E}(u, v, w) < \infty \text{ and } (9.22), (9.23) \text{ hold} \}. \quad (9.27)
\]

We will first minimize \( \tilde{J}_{u,v}(w) \) for \( (u, v) \) such that \( \tilde{G}(u, v) \leq M < \infty \) in order to construct our constraint set. From the argument before, we need to control the size of the set where \( |u_a(x)| \leq \eta_a/2 \) for each \( a = 1, 2, \ldots, n - 1 \).

Proposition 9.1. Suppose that \((u, v)\) satisfies that \(\tilde{G}(u, v) \leq M < \infty\). Then there exists an \(R\) independent of \(u_a\) such that \(\{x : |u_a(x)| \leq \eta_a/2\} \subset B_R\) for all \(a = 1, 2, \ldots, n - 1\), where \(B_R\) is a ball in \(\mathbb{R}^2\) of radius \(R\) centred at the origin.

Proof. Consider \((u, v)\) such that \(\tilde{G}(u, v) \leq M < \infty\). From the result on Ginzburg–Landau theory, we know that \(\eta_a - |u_a| \in H^1(\mathbb{R}^2)\). Hence, we have

\[
(1 - \frac{|u_a(r)|}{\eta_a})^2 \leq 2 \int_r^\infty \left(1 - \frac{|u_a(\rho)|}{\eta_a}\right) \rho \ d\rho \\
\leq \frac{2}{r} \left(\int_r^\infty \left(1 - \frac{|u_a(\rho)|}{\eta_a}\right)^2 \rho \ d\rho\right) \left(\int_r^\infty \frac{|u_a'(\rho)|^2}{\eta_a^2} \rho \ d\rho\right)^{1/2} \\
\leq \frac{2}{r} \left(\int_r^\infty \left(1 - \frac{|u_a(\rho)|}{\eta_a^2}\right)^2 \rho \ d\rho\right) \left(\int_r^\infty \frac{|u_a'(\rho)|^2}{\eta_a^2} \rho \ d\rho\right)^{1/2} \\
\leq \frac{4(n-1)}{\eta_a^3 \sqrt{\kappa} R} \tilde{G}(u, v) \leq \frac{4(n-1)M}{r \eta_a^3 \sqrt{\kappa}}.
\]

In this way, we may choose

\[
R = \max_{a=1,2,\ldots,n-1} \frac{16(n-1)M}{\eta_a^3 \sqrt{\kappa}}. \tag{9.28}
\]

Then \(|u_a(x)| > \eta_a/2\) for \(|x| \geq R\), \(a = 1, \ldots, n - 1\).

In particular, denote

\[
\eta = \min_{a=1,2,\ldots,n-1} \eta_a.
\]

Hence from the above proposition \(|u_a| > \eta/2\) for \(|x| \geq R\). Then using the same argument as in lemma 4.2, we have the following.

Lemma 9.2. For each \((u, v)\) with \(\tilde{G}(u, v) \leq M < \infty\), the following minimization problem

\[
\min \{\tilde{J}_{u,v}(w) \mid w \in H^1(\mathbb{R}^2)\}
\]

has a unique solution.

Proof. The uniqueness of the minimizer can be seen from the fact that the functional \(\tilde{J}_{u,v}(w)\) is strictly convex in \(w\).

First, we derive the lower bound for \(\tilde{J}_{u,v}(w)\). Using Cauchy–Schwartz, we have

\[
\frac{1}{m^2} \tilde{J}_{u,v}(w) \geq \int_0^\infty \left(r (w')^2 + r \sum_{a=1}^{n-1} \frac{u_a^2}{n-1} w^2 - \varepsilon rw^2 - \frac{\kappa^2 (v')^2}{\varepsilon r}\right) dr \\
\geq \int_0^\infty r (w')^2 dr + \sum_{a=1}^{n-1} \int_{|u_a| > \frac{\eta}{2}} \left(\frac{\eta^2}{4(n-1)}\right) rw^2 dr \\
- \varepsilon \int_0^\infty rw^2 dr - \frac{\kappa^2}{\varepsilon} \int_0^\infty \frac{(v')^2}{r} dr \\
\geq \int_0^\infty r (w')^2 dr + \int_0^R \left(\frac{\eta^2}{4} - \varepsilon\right) rw^2 dr - \int_0^R \varepsilon rw^2 dr - \frac{\kappa^2}{m^2 \varepsilon} \tilde{G}(u, v) \\
\geq (1 - \varepsilon CR) \int_0^\infty r (w')^2 dr + \left(\frac{\eta^2}{4} - (1 + CR)\varepsilon\right) \int_R^\infty rw^2 dr - \frac{\kappa^2 M}{m^2 \varepsilon},
\]
where \( R \) in the last inequality is defined by equation (9.28) and we have used equation (4.3) to obtain the last inequality. Choosing \( \varepsilon \) to satisfy

\[
\varepsilon = \min \left\{ \frac{\eta^2}{8(1+CR)}, \frac{1}{2CR} \right\},
\]

we obtain

\[
\frac{1}{m^2} \tilde{J}_{u,v}(w) \geq \frac{1}{2} \int_0^\infty r(w')^2 \, dr + \frac{\eta^2}{8} \int_R^\infty r w^2 \, dr - \frac{\kappa^2 M}{m^2 \varepsilon}.
\]

From this and equation (4.3), we also obtain the control of \( \|w\|_{H^1(\mathbb{R}^2)} \)

\[
\|w\|_{H^1(\mathbb{R}^2)}^2 \leq \frac{1 + CR}{\min \left\{ \frac{1}{2}, \frac{\eta^2}{8} \right\}} \left( \frac{1}{m^2} \tilde{J}_{u,v}(w) + \frac{\kappa^2 M}{m^2 \varepsilon} \right),
\]

where \( \varepsilon \) satisfies equation (9.30). The rest of the proof will be the same as in lemma 4.2.

In this way, we can construct the constraint set to the minimization problem of \( \tilde{I}(u, v, w) \) to be

\[
\tilde{C} = \left\{ (u, v, w) \in \tilde{A} \mid w \in H^1_r(\mathbb{R}^2), \int_0^\infty \left( r w' \tilde{w}' + r \sum_{a=1}^{n-1} \frac{u_a^2}{n-a} w \tilde{w} + \kappa v' \tilde{w} \right) \, dr = 0 \right\}
\]

for all \( \tilde{w} \in H^1_r(\mathbb{R}^2) \) such that \( J_{a,v}(w + \tilde{w}) < \infty \).

Therefore, from lemma 9.2, we know that \( \tilde{C} \neq \emptyset \), and for each minimizer \( w \), we have

\[
\tilde{J}_{u,v}(w) = -m^2 \int_0^\infty \left( r(w')^2 + \sum_{a=1}^{n-1} \frac{u_a^2}{n-a} w^2 \right) \, dr \leq 0.
\]

Therefore, the minimization of \( \tilde{I}(u, v, w) \) can be done via the similar method as before.

**Proposition 9.3.** The minimization problem

\[
\min \{ \tilde{I}(u, v, w) \mid (u, v, w) \in \tilde{C} \}
\]

has a solution.

Furthermore, the existence and regularity of solutions and the verification of the boundary conditions can all be achieved by the same argument as in §§6 and 7. As for the quantization relations, following de Vega & Schaposnik (1986b), we introduce an electromagnetic tensor

\[
\mathcal{F}_{\mu\nu} = \frac{\text{Tr}[MF_{\mu\nu}]}{\text{Tr}[M^2]}.
\]

Then the quantization of the magnetic flux and the electric charge can also be obtained by the same method as in §8.

Summing up all of the above, we have the following.
Theorem 9.4. For any given integer \( m \in \{1, \ldots, n - 1\} \), the non-Abelian Chern–Simons–Higgs equations expressed in equations (9.12) and (9.13) over \( \mathbb{R}^2 \) have a smooth finite-energy solution \((A_0, A, \phi)\), where \( \phi = (\phi^a) \) represents a multiplet of \( n - 1 \) Higgs fields each lying in the Cartan subalgebra of \( \text{su}(n) \), satisfying the asymptotic properties
\[
F_{\mu\nu} \to 0, \quad D_\mu \phi^a \to 0, \quad |\phi^a| \to \eta_a, \quad a = 1, \ldots, n - 1, \quad A_0 \to 0, \quad \partial_j A_0 \to 0,
\]
as \(|x| \to \infty\). Moreover, the total magnetic flux \( \Phi \) and electric charge \( Q \) are given, respectively, by the quantization formulas
\[
\Phi = \int_{\mathbb{R}^2} \frac{\text{Tr}[MF_{12}]}{\text{Tr}[M^2]} \, dx = 2\pi m \tag{9.36}
\]
and
\[
Q = \int_{\mathbb{R}^2} \frac{\text{Tr}[MJ^0]}{\text{Tr}[M^2]} \, dx = 2\pi mk. \tag{9.37}
\]
Such a magnetically and electrically charged solution realizes an \( \text{SU}(n) \) vortex configuration asymptotically and topologically represented by the \( m \)th integral class in the classification space of the vortex vacuum manifold \( \text{SU}(n)/\mathbb{Z}_n \), that is, by \( m \in \pi_1(\text{SU}(n)/\mathbb{Z}_n) = \mathbb{Z}_n \) for \( m = 1, \ldots, n - 1 \).

To conclude, in this paper, we have developed an existence theory for the electrically and magnetically charged vortex solutions arising in the classical Abelian and non-Abelian Chern–Simons–Higgs models using a constrained variational approach. Such a construction is of a general nature and does not rely on exploring the self-dual or BPS formulation of the problem.

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References


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