Hyper-Kähler geometry and semi-geostrophic theory

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We use the formalism of Monge–Ampère operators to study the geometric properties of the Monge–Ampère equations arising in semi-geostrophic (SG) theory and related models of geophysical fluid dynamics. We show how Kähler and hyper-Kähler structures arise, and the Legendre duality arising in SG theory is generalized to other models of nearly geostrophic flows.

Keywords: semi-geostrophic theory; balanced models; Monge–Ampère equations; hyper-Kähler geometry

1. Introduction

Semi-geostrophic (SG) theory is an approximation to the Navier–Stokes-based equations for atmosphere–ocean dynamics that has proven especially useful in the study of weather fronts, land/sea breezes, flow over orography, monsoons and large-scale ocean currents. The utility of SG theory is a consequence of several elegant mathematical properties, which include Hamiltonian structure, contact (Legendre) duality and the ubiquity of a Monge–Ampère equation. The latter is the second-order nonlinear partial differential equation that, in the context of SG theory, relates the potential vorticity to the wind and temperature fields. Contact transformations of the Monge–Ampère equation allow the SG equations to be solved analytically for certain idealized flows, for example, Blumen (1981), Hoskins (1975) and Shutts (1991). The Legendre duality can be used to construct novel numerical techniques, including finite-element methods, which have proven especially useful in exploiting the Lagrangian formulation of the equations (Cullen & Purser 1984, 1989). SG theory is one of the many so-called ‘balanced models’ (balance in this context refers to the geostrophic balance between fluid velocity and pressure gradient in flows on a rotating domain), although SG theory retains a special significance because of its elegant geometrical properties. Roubtsov & Roulstone (1997, 2001) showed that a hierarchy of balanced models possess symplectic, contact and Monge–Ampère structures akin to SG theory, and that these hitherto apparently disconnected features can be viewed as the component parts of a hyper-Kähler structure. However, it was believed that SG theory itself could not be formulated in terms of hyper-Kähler geometry,

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and several other questions concerning the relevance of hyper-Kähler structures to balanced models remained open. For an extensive discussion of these issues, see McIntyre & Roulstone (2002).

The purpose of this paper is to address these issues. The approach adopted by Roubtsov & Roulstone (1997, 2001) was based on the special significance attached to the Jacobian of the map between local symplectic coordinates and the Lagrangian configuration coordinates of the fluid. In this paper, we adopt a different approach. Instead of focusing on the map between local symplectic coordinates and the Lagrangian fluid coordinates, we apply the methods of Kushner et al. (2007) and re-derive the geometric properties using the theory of Monge–Ampère operators. The salient difference between the two approaches may be summarized as follows: Roubtsov & Roulstone (1997, 2001) studied the geometry associated with \( D(\phi) \), where \( D \) denotes the determinant of the Hessian matrix of a dependent variable \( \phi \) (this determinant is the Jacobian discussed above), whereas we study the Monge–Ampère equation \( D(\phi) = q \), where \( q \) is a given function of the independent variables. As a consequence, we are able to show that SG theory does indeed possess a hyper-Kähler structure, and the issues raised by McIntyre & Roulstone (2002) can be resolved.

Although a thermal structure is crucial for the formation of phenomena such as fronts and sea breezes, the salient mathematical features of SG theory can be studied within the context of shallow-water equations (Roulstone & Sewell 1997; McIntyre & Roulstone 2002), which is the approach we adopt in this paper. This paper is organized as follows. In §2, after reviewing SG theory and its essential mathematical features, we discuss the Hamiltonian balanced models (HBMs) derived in McIntyre & Roulstone (2002). In §3, we introduce hyper-Kähler structures and we show how the phase space of HBMs is endowed with such structures. In §4, we follow the approach of Kushner et al. (2007) to study the Monge–Ampère equation relating the geopotential to the absolute vorticity from a geometric perspective; we derive a new example of a hyper-Kähler structure, extending the hyper-Kähler structure described in §3 for HBMs, which incorporates SG theory. In §5, we use this new example to extend the Legendre duality arising in SG theory to HBMs.

2. Balanced models

(a) Semi-geostrophic theory

In the SG regime, the motion of a shallow layer of inviscid fluid of depth \( \eta(x, y, t) \), rotating with constant angular frequency \( f/2 \), can be approximated by replacing the acceleration \( \dot{u} \) in the momentum equations with the Lagrangian time derivative of the geostrophic wind

\[
\dot{u}_g + f k \times \dot{x} + g \nabla \eta = 0.
\]  

(2.1)

Here, \( x = (x, y)^T \in \mathbb{R}^2 \) denotes the positions of the fluid particles, \( g \) is a constant representing the acceleration due to gravity, \( k \) is the unit vertical vector, and we assume no bottom topography. Throughout this paper \( f \), which denotes

\[1\] The singularities of the map between these coordinate systems are precisely the feature that, via Legendre duality, can be interpreted in the context of fronts (Chynoweth & Sewell 1989).
the Coriolis parameter, is assumed to be a constant. The geostrophic wind
\( u_g = (u_g, v_g)^T \) is defined by
\[
\begin{align*}
u_x &= -f \frac{\partial \phi}{\partial y} \quad \text{and} \quad v_y = f \frac{\partial \phi}{\partial x},
\end{align*}
\]
with \( \phi \) denoting the geopotential \( \phi = g \eta/f^2 \). The positions \( x \) are functions of the Lagrangian mass coordinates \( a \) and \( b \), and the time \( t \),
\[
x = x(a, b, t) \quad \text{and} \quad y = y(a, b, t),
\]
where Lagrangian mass coordinates are chosen so that \( a = x(a, b, 0), \quad b = y(a, b, 0) \). The superposed dot denotes the Lagrangian time derivative following a particle, that is, \( \partial / \partial t \) with \( a \) and \( b \) held fixed. The incompressibility hypothesis requires \( \eta \) to satisfy the relation \( \eta_0 \, da \, db = \eta(x, y, t) \, dx \, dy \), where \( \eta_0 \) is a constant initial state. If we further assume that \( \eta_0 = 1 \), then the height field \( \eta \) is given by
\[
\eta(x, y, t) = \frac{\partial (a, b)}{\partial (x, y)}. \quad (2.3)
\]
This equation provides us with an implicit form of the continuity equation. The momentum equations (2.1), together with the continuity equation (2.3), constitute the shallow-water version of the SG equations. Using equation (2.1) and taking the Lagrangian time derivative of equation (2.3), we can show that the potential vorticity \( Q_{sg} \) defined by
\[
Q_{sg} = \frac{g}{f \phi} \left[ 1 + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] \quad (2.4)
\]
is conserved, i.e. \( \dot{Q}_{sg} = 0 \). Equation (2.4) is a Monge–Ampère type equation for \( \phi \), given \( Q_{sg} \) and appropriate boundary conditions. This equation is elliptic when \( Q_{sg} > 0 \).

Hoskins (1975) showed that the integration of the SG equations is facilitated by the use of the coordinate transformation \( x \mapsto X = (X, Y) \) given by
\[
X = x + \frac{\partial \phi}{\partial x} \quad \text{and} \quad Y = y + \frac{\partial \phi}{\partial y}, \quad (2.5)
\]
in terms of which the potential vorticity (2.4) can then be expressed in Jacobian form
\[
Q_{sg} = \frac{g}{f \phi} \frac{\partial (X, Y)}{\partial (x, y)}. \quad (2.6)
\]
The coordinates \( (X, Y) \) are often referred to as geostrophic coordinates because, using the definition (2.5), we can express equation (2.1) as \( \dot{X} = u_g \); that is, the motion in these coordinates is exactly geostrophic. The so-called geostrophic coordinate transformation \( x \mapsto X \) can be interpreted in terms of Legendre duality.
That is, given the two dual functions $P(x, y, t)$ and $R(X, Y, t)$ defined by

$$P(x, y, t) = \phi(x, y, t) + \frac{1}{2}(x^2 + y^2) \quad (2.7)$$

and

$$R(X, Y, t) = xX + yY - P, \quad (2.8)$$

we see that the transformation $(x, y) \mapsto (X, Y)$ is a Legendre transformation because, using equations (2.7), (2.8) and (2.5), we have

$$\frac{\partial P}{\partial x} = X, \quad \frac{\partial P}{\partial y} = Y \quad \text{and the dual relations} \quad \frac{\partial R}{\partial X} = x, \quad \frac{\partial R}{\partial Y} = y. \quad (2.9)$$

We define a dual potential $\Phi$, referred to as the Bernoulli potential, as the function

$$\Phi(X, Y, t) = -R + \frac{1}{2}(X^2 + Y^2) = \phi(x, y, t) + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right]. \quad (2.10)$$

Using equation (2.9), we can show that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial X} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{\partial \Phi}{\partial Y}. \quad (2.11)$$

This property, referred to by Roulstone & Sewell (1997) as the gradient transformation property of the geostrophic coordinate transformation, allows us to write the momentum equations (2.1) in the Hamiltonian form,

$$\dot{X} = -f \frac{\partial \Phi}{\partial Y} \quad \text{and} \quad \dot{Y} = f \frac{\partial \Phi}{\partial X}. \quad (2.12)$$

Finally, as first pointed out by Blumen (1981), the transformation

$$\left( x, y, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) \mapsto \left( X, Y, \Phi, \frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial Y} \right)$$

defines a strict contact transformation: using equations (2.5), (2.10) and (2.11), we see that

$$d\Phi = \frac{\partial \Phi}{\partial X} dX + \frac{\partial \Phi}{\partial Y} dY = d\phi - \frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial y} dy.$$

See McIntyre & Roulstone (2002, §5), for further details.

(b) More accurate balanced models

Despite its remarkable conceptual simplicity and its useful mathematical properties, the SG approximation is formally correct only to leading order in Rossby number (e.g. McIntyre & Roulstone 2002 for discussion). Therefore, the question of how to derive more accurate models that would retain the essential features of SG theory has been much studied since its introduction in the mid-Seventies. Using the framework of constrained Hamiltonian dynamics, pioneered by Salmon (1983, 1985) and Allen & Holm (1996 and references therein), McIntyre & Roulstone (1996) attempted to provide an answer to this question.
by systematically deriving a class of HBMs. These models conserve the potential vorticity, $Q_c$, defined by

$$Q_c = \frac{g}{f\phi} \left[ 1 + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + (1 - c^2) \left( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \right)^2 \right) \right], \quad (2.13)$$

where $c \in \mathbb{R}$. Different values of $c$ define different balanced models. For example, when $c = 0$, $Q_c$ corresponds to equation (2.4)—the functional form of the potential vorticity conserved by the SG equations. When $c = 1$, the nonlinear term is eliminated in equation (2.13), and this is the conserved potential vorticity of Salmon’s $L_1$ dynamics (Salmon 1985). A third value of $c$ is of particular interest: when $c = \sqrt{3}$, equation (2.13) agrees to order two with an asymptotic expansion in terms of the Rossby number of the absolute vorticity for shallow-water dynamics. By comparison, $L_1$-dynamics ($c = 1$) is only accurate to order one and SG theory ($c = 0$) introduces the correct second-order term, but with the wrong coefficient (+1 instead of −2: see Snyder et al. (1991) for further discussion; also Delahaies (2009, §2.2.4)). The so-called $\sqrt{3}$-model is the most accurate model within the class of models derived by McIntyre and Roulstone.

McIntyre & Roulstone (1996) showed that

$$X = (X, Y)^T = x + \nabla \phi - i c k \times \nabla \phi, \quad (2.14)$$

where $i = \sqrt{-1}$ are canonical coordinates for the Hamiltonian models. Note that $(X, Y)$ was used earlier to denote the geostrophic coordinates; we adopt the same notation, since setting $c = 0$ in equation (2.14) we obtain equation (2.5). As is the case of SG theory, the potential vorticity $Q_c$ can be expressed in terms of the Jacobian of the coordinate transformation

$$Q_c = \frac{g}{f\phi} \frac{\partial(X, Y)}{\partial(x, y)}. \quad (2.15)$$

Roubtsov & Roulstone (2001) introduced a dual complex potential $\Phi$ given by

$$\Phi(X, \bar{Y}, t) = \phi(x, y, t) + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] + i c \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y}, \quad (2.16)$$

which satisfies

$$\frac{\partial \Phi}{\partial X} = 0, \quad \frac{\partial \Phi}{\partial Y} = 0, \quad \frac{\partial \Phi}{\partial X} = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad \frac{\partial \Phi}{\partial Y} = \frac{\partial \Phi}{\partial y}, \quad (2.17)$$

where the overbar is used to denote the complex conjugate. The first two equations ((2.17)$_1$ and (2.17)$_2$) show that $\Phi$ is holomorphic in $X$ and $\bar{Y}$, and the last two equations ((2.17)$_3$ and (2.17)$_4$) are reminiscent of the gradient transformation property of SG theory (cf. (2.11)). Introducing the space spanned by the coordinate system $(X, \bar{Y}, \Phi, \partial \Phi/\partial X, \partial \Phi/\partial \bar{Y})$, endowed with the contact form $d\Phi - (\partial \Phi/\partial X) dX - (\partial \Phi/\partial \bar{Y}) d\bar{Y}$—referred to in Roubtsov & Roulstone (2001) as the semi-holomorphic contact bundle corresponding to the
coordinates \((X, \bar{Y})\)—the transformation
\[
\begin{pmatrix}
x, y, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}
\end{pmatrix}
\mapsto
\begin{pmatrix}
X, \bar{Y}, \Phi, \frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial \bar{Y}}
\end{pmatrix}
\]
defines a strict contact transformation for, using equations (2.14), (2.16) and (2.17), we have
\[
d\Phi - \frac{\partial \Phi}{\partial X} dX - \frac{\partial \Phi}{\partial \bar{Y}} d\bar{Y} = d\phi - \frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial y} dy.
\]
Note that \(\Phi\) was used earlier to denote the Bernoulli potential (2.10); again, we adopt the same notation since, by setting \(c = 0\), the coordinates \(X\) and \(\bar{Y}\) are nothing but the geostrophic coordinates and equation (2.16) reduces to equation (2.10). Just as for SG theory, the dynamics of HBMs can be formulated in terms of \(X, \bar{Y}, \Phi, \frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial \bar{Y}}\); however, as stated in McIntyre & Roulstone (2002) ‘there is no reason to expect \(\Phi\) to enter into the evolution equations with anything like the simplicity of (2.12)’. In the following sections, we introduce the geometry that explains why complex coordinates arise naturally in the Jacobian of equation (2.15), and, in turn, this explains why SG theory does not possess complex structure. Then, by introducing the theory of Monge–Ampère operators, which associates geometrical structures to equation (2.15), we derive new complex coordinates that incorporate SG theory.

3. Kähler and hyper-Kähler geometry

The discovery of complex canonical coordinates for HBMs (McIntyre & Roulstone 1996) brings a new geometric framework in which to study these models (Roubtsov & Roulstone 1997, 2001). We now introduce some notation and concepts of differential geometry, not all of which have appeared in our previous papers on this subject, that we shall require in subsequent sections. Further details can be found in McDuff & Salamon (1998) and Kushner et al. (2007).

(a) Definitions and properties

Let \(M\) be a \(2n\)-dimensional manifold. An almost-complex structure \(I\) on \(M\) is a field of endomorphisms of the tangent bundle \(TM\), such that \(I^2 = -1_{2n}\), where \(1_{2n}\) denotes the \(2n \times 2n\) identity matrix. The pair \((M, I)\) is called an almost-complex manifold. A map \(F\) between two almost-complex manifolds \((M, I)\) and \((M', I')\) is said to be \((I, I')\)-holomorphic if, and only if, for all \(m \in M\), the tangent map \(dF_m\) is complex linear; that is, \(dF_m \circ I(m) = I'(F(m)) \circ dF_m\). The almost-complex structure \(I\) is integrable if, at any point \(m \in M\), there exists a neighbourhood \(U_m\) of \(m\) such that we can define \((I, i)\)-holomorphic coordinates \(z_l^l : U_m \rightarrow \mathbb{C}, l = 1 \ldots n\), in which \(I\) takes the form
\[
I = \begin{pmatrix}
i_{1n} & 0 \\
0 & -i_{1n}
\end{pmatrix},
\]
where 0 denotes the $n \times n$ zero matrix. This allows the identification of the manifold $M$ as a complex manifold: in the coordinate system \(\{z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n\}\), where the overbar denotes the complex conjugate, operating with $I$ is equivalent to transforming $dz^I$ into $i dz^I$ and $d\bar{z}^I$ into $-i d\bar{z}^I$. When $I$ is integrable, it is called a complex structure, and we will say that a coordinate system $(z^I, \bar{z}^I)$, which realizes the identification of the manifold $M$ as a complex manifold, is an adapted coordinate system for $I$, or that it is induced by the complex structure $I$.

A Riemannian metric $h$, together with an almost-complex structure $I$, and a non-degenerate 2-form $\sigma$, satisfying the compatibility condition $\sigma(\cdot, \cdot) = h(I\cdot, \cdot)$, defines an almost-Hermitian structure on $M$. When $\sigma$ is closed, this structure is called almost-Kähler; in addition, if $I$ is integrable, this structure defines a Kähler structure on $M$. The metric $h$ is then called a Kähler metric and $\sigma$ a Kähler form. Note that any two objects of the triple $(h, I, \sigma)$ determine the third one from the compatibility condition.

Assume now that $M$ is a $4n$-dimensional manifold and let $h$ be a Riemannian metric on $M$. A hyper-Kähler structure on $(M, h)$ is prescribed by a triple of linearly independent complex structures $(I_1, I_2, I_3)$ satisfying the quaternion relations

\[ I_1^2 = I_2^2 = I_3^2 = -1_{4n} \quad \text{and} \quad I_1 I_2 I_3 = -1_{4n}, \]

and such that $h$ is Kähler with respect to all three complex structures. Alternatively, the hyper-Kähler structure can be prescribed by the triple of symplectic 2-forms $(\omega_1, \omega_2, \omega_3)$ defined by $\omega_i(\cdot, \cdot) = h(I_i \cdot, \cdot)$, $i = 1, 2, 3$. When endowed with a hyper-Kähler structure, the Riemannian manifold $(M, h)$ is called a hyper-Kähler manifold and $h$ is said to define a hyper-Kähler metric.

Let us consider a simple example that will prove useful in the subsequent sections to describe the geometric structures arising in HBMs and SG theory: take $\mathbb{C}^2$ endowed with the coordinate system $\{X, Y, \bar{X}, \bar{Y}\}$. Then, $\mathbb{C}^2$ is endowed with a natural Kähler structure $(I, h)$, where $I$ is the canonical complex structure on $\mathbb{C}^2$ given by equation (3.1) and $h$ is the standard complex metric given by

\[ h = dX \otimes d\bar{X} + dY \otimes d\bar{Y}. \]

(3.2)

Then, the Kähler 2-form is given by

\[ \sigma = -\frac{i}{2} (dX \wedge d\bar{X} + dY \wedge d\bar{Y}). \]

(3.3)

This example will be referred to as the Kähler structure induced by the coordinate system $(X, Y)$. We further obtain the hyper-Kähler structure induced by the coordinate system $(X, Y)$ by considering the standard complex metric $h$ together with the triple of 2-forms $(\omega_1, \omega_2, \omega_3)$ given by

\[ \omega_1 = \text{Re} \ dX \wedge dY, \quad \omega_2 = -\frac{i}{2} (dX \wedge d\bar{X} + dY \wedge d\bar{Y}) \quad \text{and} \quad \omega_3 = \text{Im} \ dX \wedge dY. \]

(3.4)

In this example, the hyper-Kähler structure is expressed relative to the Kähler structure defined by $(h, \omega_2)$. By definition, a hyper-Kähler manifold $M$ is Kähler with respect to all three complex structures; then, at any point $m \in M$, there exists a neighbourhood in which three complex-coordinate systems coexist, each of them providing an identification of $M$ as a complex manifold.
using the relation 

\[ \omega = \cdots \] 

as discussed by McIntyre \& Roulstone (2002). We observe that this ties together the geometrical and the dynamical aspects of balanced models, and the hyper-Kähler metric

\[ \mathcal{H} = \cdots \] 

is given, in terms of the coordinate system \( (x, y, p, q) \), by

\[ h = dx \otimes dx + dy \otimes dy + dp \otimes dp + dq \otimes dq + (1 + c^2)(dp \otimes dp + dq \otimes dq) \]

and

\[ \omega = c(dx \wedge dq + dp \wedge dy + 2dp \wedge dq). \]

Using the relation \( \omega(\cdot, \cdot) = h(I_\omega \cdot, \cdot) \), the complex structure \( I_\omega \) is given by

\[ I_\omega = \frac{1}{c} \begin{pmatrix} 0 & -1 & 0 & -1 + c^2 \\ 1 & 0 & 1 - c^2 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \]

Furthermore, the hyper-Kähler structure \( (h, \omega_1, \omega_2, \omega_3) \) induced on \( T^*\mathbb{R}^2 \) by the coordinate system \( X = (X, Y)^T \) given by equation (2.14)

\[ X = x + p + icq \quad \text{and} \quad Y = y + q - icp, \quad (3.5) \]

where \( (p, q) = (\partial \phi / \partial x, \partial \phi / \partial y) \) defines an adapted coordinate system for a complex structure on the cotangent bundle \( T^*\mathbb{R}^2 \). The Kähler structure \( (h, \omega) \) induced on \( T^*\mathbb{R}^2 \) by \( X \) is given, in terms of the coordinate system \( \{x, y, p, q\} \), by

\[ \omega = \cdots \] 

where the absolute constrained vorticity \( \zeta_c \) is defined by

\[ \zeta_c(x, y) = f^{-1} g^{-1} \phi(x, y) Q_c(x, y), \]

the section \( d\phi : \mathbb{R}^2 \rightarrow T^*\mathbb{R}^2 \) maps \( (x, y) \) to \( (x, y, \phi_x, \phi_y) \) and the asterisk denotes pull-back of differential forms, which, in this case, consists of replacing \( p \) and \( q \) by \( \phi_x \) and \( \phi_y \). However, for SG theory, which corresponds to the case \( c = 0 \), the coordinate system (3.5) becomes real and \( I_\omega \) is not defined. Furthermore, we see from equations (3.10) and (3.11) that \( \omega_2 \) and \( \omega_3 \) vanish identically. In the following sections, we show how to recover a complex structure for SG.
4. Monge–Ampère theory

In the previous section, following the approach of McIntyre & Roulstone (2002), we expressed the Monge–Ampère equation relating the potential vorticity to the geopotential in terms of the hyper-Kähler structure induced by the complex-coordinate system \((X, Y)\) (equation (3.12)). The purpose of this section is to invert this strategy: that is, we make the Monge–Ampère equation the object of primary interest, and we apply the theory of Monge–Ampère operators developed in Lychagin (1979), Lychagin et al. (1993) and Kushner et al. (2007) to define a new hyper-Kähler structure. The first part of this section contains the necessary concepts to develop this approach in the second part, further details can be found in Kushner et al. (2007).

(a) Symplectic Monge–Ampère equations in \(\mathbb{R}^2\)

A symplectic Monge–Ampère equation in \(\mathbb{R}^2\) is a second-order nonlinear differential equation of the form

\[
A \frac{\partial^2 \phi}{\partial x^2} + 2B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \left( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right) + E = 0,
\]

where \(A, B, C, D\) and \(E\) are smooth functions on the cotangent bundle \(T^*\mathbb{R}^2\). We endow \(T^*\mathbb{R}^2\) with the coordinate system \(\{x, y, p, q\}\) and we denote by \(\Omega \in \Omega^2(T^*\mathbb{R}^2)\) the canonical symplectic form on \(T^*\mathbb{R}^2\) given by

\[
\Omega = dx \wedge dp + dy \wedge dq.
\]

A 2-form \(\omega \in \Omega^2(T^*\mathbb{R}^2)\) is said to be effective if \(\omega \wedge \Omega = 0\). The Hodge–Lepage–Lychagin theorem (Lychagin 1979) shows that there is a one-to-one correspondence between effective 2-forms on \(T^*\mathbb{R}^2\) and symplectic Monge–Ampère equations in \(\mathbb{R}^2\) given by the map

\[
\omega \mapsto \Delta_\omega \phi = 0,
\]

where, for any smooth function \(\phi\) on \(\mathbb{R}^2\), the Monge–Ampère operator \(\Delta_\omega\) is defined by

\[
\Delta_\omega \phi = (d\phi)^* \omega.
\]

As previously, \(d\phi : \mathbb{R}^2 \to T^*\mathbb{R}^2\) is the section defined by \((x, y) \mapsto (x, y, \phi_x, \phi_y)\) and the asterisk denotes the pull-back of differential forms, which, again, consists of replacing \(p\) and \(q\) by \(\phi_x\) and \(\phi_y\) in this case. We denote the Monge–Ampère equation \(\Delta_\omega \phi = 0\) by \(\mathcal{E}_\omega\). A smooth function \(\phi : \mathbb{R}^2 \to \mathbb{R}\) such that \(\Delta_\omega \phi = 0\) is called a classical solution of \(\mathcal{E}_\omega\). A generalized (or multivalued) solution of \(\mathcal{E}_\omega\) is a submanifold \(L \subset T^*\mathbb{R}^2\), such that \(L\) is bilagrangian with respect to \((\Omega, \omega)\), that is, \(\Omega|_L = 0\) and \(\omega|_L = 0\). The latter definition is compatible with the notion of classical solution; indeed, if \(\phi\) is a classical solution of the equation \(\mathcal{E}_\omega\), then the graph of \(d\phi\) defines a generalized solution of \(\mathcal{E}_\omega\). Conversely, a generalized solution \(L\) is locally the graph of a classical solution \(\phi\) if the projection \(L \to \mathbb{R}^2\) is a local diffeomorphism. Two symplectic Monge–Ampère equations \(\mathcal{E}_{\omega_1}\) and \(\mathcal{E}_{\omega_2}\) are said to be symplectically equivalent if there exists a symplectomorphism.
\( F : T^* \mathbb{R}^2 \to T^* \mathbb{R}^2 \), such that

\[ F^* \omega_1 = \omega_2, \]

and, in this case, if \( L \) is a generalized solution of \( \mathcal{E}_{F^* \omega_1} = \mathcal{E}_{\omega_2} \), then \( F(L) \) is a generalized solution of \( \mathcal{E}_{\omega_1} \).

To any effective 2-form \( \omega \) on \( T^* \mathbb{R}^2 \), and to any symplectic Monge–Ampère equation \( \mathcal{E}_\omega \), we associate the field of endomorphisms \( A_\omega : T^* \mathbb{R}^2 \to \text{End}(T^* \mathbb{R}^2) \), defined by \( \omega(\cdot, \cdot) = \Omega(A_\omega \cdot, \cdot) \). As shown in Kushner et al. (2007, §6.1), when \( \omega \) is effective, we have

\[ A_\omega^2 + \text{pf}(\omega) = 0, \quad (4.3) \]

where \( \text{pf}(\omega) \) is the scalar, called the pfaffian of \( \omega \), defined by

\[ \omega \wedge \omega = \text{pf}(\omega) \Omega \wedge \Omega. \quad (4.4) \]

For any effective 2-form \( \omega \), the equation \( \mathcal{E}_\omega \) is said to be: non-degenerate if and only if \( \text{pf}(\omega) \neq 0 \), elliptic if \( \text{pf}(\omega) > 0 \) and hyperbolic if \( \text{pf}(\omega) < 0 \). For reasons that we make explicit in the following subsection, we restrict ourselves to the study of elliptic equations. From equation (4.3), we then see that the field of endomorphisms \( I_\omega \) given by

\[ I_\omega = |\text{pf}(\omega)|^{-1/2} A_\omega \quad (4.5) \]

defines an almost-complex structure on \( T^* \mathbb{R}^2 \), and any generalized solution of \( \mathcal{E}_\omega \) defines an \( I_\omega \)-holomorphic curve. Finally, we have the following result due to Lychagin et al. (1993), and summarized by Banos (2006) as:

**Theorem 4.1.** Let \( \mathcal{E}_\omega \) be a symplectic Monge–Ampère equation on \( \mathbb{R}^2 \), then the following assertions are equivalent:

(i) \( \mathcal{E}_\omega \) is symplectically equivalent to the equation

\[ \phi_{xx} + \phi_{yy} = 0, \text{ when } \text{pf}(\omega) > 0, \]

\[ \phi_{xx} - \phi_{yy} = 0, \text{ when } \text{pf}(\omega) < 0, \]

(ii) the structure \( I_\omega \) is integrable, and

(iii) the normalized 2-form \( |\text{pf}(\omega)|^{-1/2} \omega \) is closed.

We now investigate SG theory and HBM, regarding the Monge–Ampère equation as the primary object of interest.

(b) Hyper-Kähler geometry and balanced models: Monge–Ampère operators

Using the absolute constrained vorticity \( \zeta_c \) defined by equation (3.13), the equation relating the potential vorticity \( \mathcal{Q}_c \) to the geopotential \( \phi \) becomes a symplectic Monge–Ampère equation for \( \phi \) given \( \zeta_c \), which can be written as

\[ 0 = (1 - f^{-1} \zeta_c) + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + (1 - c^2) \left( \frac{\partial^2 \phi}{\partial x \partial y} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right). \quad (4.6) \]

The corresponding effective 2-form \( \omega \in \Omega^2_\epsilon(T^* \mathbb{R}^2) \) is the closed 2-form given by

\[ \omega = (1 - f^{-1} \zeta_c) dx \wedge dy + dp \wedge dy + dx \wedge dq + (1 - c^2) dp \wedge dq, \quad (4.7) \]
whose pfaffian is given by

\[ \text{pf}(\omega) = 1 - (1 - c^2)(1 - f^{-1}\zeta_c). \] (4.8)

Recall that \( c \) is a real parameter taking certain specific values (in the range \( 0 \leq c \leq \sqrt{3} \)); moreover, the assumptions under which balanced models are valid are \( \epsilon \ll 1 \) and \( f^{-1}\zeta_c = 1 + \mathcal{O}(\epsilon) \). Therefore, we have \( \text{pf}(\omega) > 0 \) for the physical regimes of interest, then equation (4.6) is elliptic. The almost-complex structure \( I_\omega \) is given by

\[
I_\omega = \frac{1}{\sqrt{\text{pf}(\omega)}} \begin{pmatrix}
0 & -1 & 0 & -1 + c^2 \\
1 & 0 & 1 - c^2 & 0 \\
0 & 1 - f^{-1}\zeta_c & 0 & 1 \\
-1 + f^{-1}\zeta_c & 0 & -1 & 0
\end{pmatrix}. \] (4.9)

From theorem 4.1, we see that \( I_\omega \) is integrable if and only if \( \zeta_c \) is constant in space.

Although this condition is very restrictive for practical use, many important idealized features of SG have been studied in the context of constant-coefficient problems (e.g. Hoskins & West 1979; Chynoweth & Sewell 1989; Shutts 1991; see also McIntyre & Roulstone 2002, §14; Roulstone et al. 2009a).

Restricting ourselves to the case where \( \zeta_c \) is a constant denoted by \( \zeta_0 \), we seek an adapted coordinate system for \( I_\omega \) to associate a Kähler structure with the Monge–Ampère equation (4.6). Recall that, in the preceding section, the coordinate system was specified and the structures were derived by considering the standard complex metric (3.2). In the present case, there is a priori no reason to favour any metric among the set of all Hermitian metrics with respect to \( I_\omega \).

Noting equation (3.5), we choose to work with the complex-coordinate system \((X_2, Y_2)\) defined on \( T^*\mathbb{R}^2 \) by

\[
\begin{align*}
X_2 &= \alpha_1 x + i\alpha_2 y + \beta_1 p + i\beta_2 q \\
Y_2 &= \alpha_1 y - i\alpha_2 x + \beta_1 q - i\beta_2 p,
\end{align*}
\] (4.10)

with \( (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2 - \{(0,0)\} \). We use the subscript ‘2’ because, in our construction of the hyper-Kähler structure \((h, \omega_1, \omega_2, \omega_3)\) induced by a complex-coordinate system \((X, Y)\) and defined by equations (3.2) and (3.4), the coordinate system is an adapted coordinate system for the complex structure \( I_2 \). The subscript ‘2’ in the coordinate system \((X_2, Y_2)\) defined above is to remind us of this fact. This notation will prove useful in the following section, where we will need to distinguish between the three different identifications of \( T^*\mathbb{R}^2 \) as \( \mathbb{C}^2 \) induced by the three Kähler forms \( \omega_1, \omega_2 \) and \( \omega_3 \).

Inserting equation (4.10) in equation (3.2), we see that the hyper-Kähler metric induced by \((X_2, Y_2)\) is given by

\[
h = (\alpha_1^2 + \alpha_2^2)(dx \otimes dx + dy \otimes dy) + (\alpha_1 \beta_1 + \alpha_2 \beta_2)(dx \otimes dp + dy \otimes dq) + (\beta_1^2 + \beta_2^2)(dp \otimes dp + dq \otimes dq),
\]
and inserting equation (4.10) in equation (3.4), we obtain the triple of Kähler forms
\[
\omega_1 = (\alpha_1^2 - \alpha_2^2) \, dx \wedge dy + (\alpha_1 \beta_1 - \alpha_2 \beta_2)(dx \wedge dq + dp \wedge dy) + (\beta_1^2 - \beta_2^2) \, dp \wedge dq,
\]
\[
\omega_2 = 2\alpha_1 \alpha_2 \, dx \wedge dy + (\alpha_1 \beta_1 + \alpha_2 \beta_2)(dx \wedge dq + dp \wedge dy) + 2\beta_1 \beta_2 \, dp \wedge dq
\]
and
\[
\omega_3 = - (\alpha_1 \beta_2 - \alpha_2 \beta_1)(dx \wedge dp + dy \wedge dq).
\]
We identify the 2-form \( \omega_1 \) with the effective 2-form (4.7); therefore, we are led to consider the underdetermined system of equations for \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \),
\[
\begin{aligned}
1 - f^{-1} \xi_0 &= \alpha_1^2 - \alpha_2^2, \\
1 &= \alpha_1 \beta_1 - \alpha_2 \beta_2, \\
1 - c^2 &= \beta_1^2 - \beta_2^2.
\end{aligned}
\tag{4.11}
\]
By considering \( \gamma = \alpha_2 \) as a parameter and inverting equation (4.11) with the assumption that \( 1 - f^{-1} \xi_0 + \gamma^2 > 0 \), and taking the positive roots, we can write
\[
\begin{aligned}
\alpha_1 &= \sqrt{1 - f^{-1} \xi_0 + \gamma^2}, \\
\beta_1 &= \frac{\sqrt{1 - f^{-1} \xi_0 + \gamma^2 + \gamma \sqrt{pf(\omega)}}}{1 - f^{-1} \xi_0}, \\
\alpha_2 &= \gamma \quad \text{and} \quad \beta_2 = \frac{\gamma + \sqrt{1 - f^{-1} \xi_0 + \gamma^2 \sqrt{pf(\omega)}}}{1 - f^{-1} \xi_0}
\end{aligned}
\tag{4.12}
\]
(providing \( 1 - f^{-1} \xi_0 \neq 0 \)). Setting \( \gamma = 0 \), from equation (4.10) we obtain the coordinate system
\[
X_2 = \sqrt{1 - f^{-1} \xi_0} \, x + \frac{1}{\sqrt{1 - f^{-1} \xi_0}} \, p + i \sqrt{\frac{pf(\omega)}{1 - f^{-1} \xi_0}} \, q
\]
and
\[
Y_2 = \sqrt{1 - f^{-1} \xi_0} \, y + \frac{1}{\sqrt{1 - f^{-1} \xi_0}} \, q - i \sqrt{\frac{pf(\omega)}{1 - f^{-1} \xi_0}} \, p,
\tag{4.13}
\]
where \( pf(\omega) \) is given by equation (4.8). The hyper-Kähler structure \((h, \omega_1, \omega_2, \omega_3)\) induced by \((X_2, Y_2)\), defined by equation (4.13), is given by
\[
h = (1 - f^{-1} \xi_0)(dx \otimes dx + dy \otimes dy) + dx \otimes dp + dy \otimes dq + \frac{1 + \sqrt{pf(\omega)}}{1 - f^{-1} \xi_0}(dp \otimes dp + dq \otimes dq),
\]
\[
\omega_1 = (1 - f^{-1} \xi_0) \, dx \wedge dy + dp \wedge dq + (1 - c^2) \, dp \wedge dq,
\]
\[
\omega_2 = \sqrt{pf(\omega)} \left( dx \wedge dq + dp \wedge dy + \frac{2}{1 - f^{-1} \xi_0} \, dp \wedge dq \right)
\]
and
\[
\omega_3 = - \sqrt{pf(\omega)}(dx \wedge dp + dy \wedge dq).
\]
When \( c = 0 \), noting from equation (4.8) that \( pf(\omega) = f^{-1} \xi_0 \), we obtain a new hyper-Kähler structure for SG theory. Note (see also McIntyre & Roulstone 2002, §14) that the term \( 1 - f^{-1} \xi_0 \) is not sign-definite, it is a dimensionless
Hyper-Kähler geometry and SG theory

measure of minus the relative vorticity: when $1 - f^{-1}\zeta_0 > 0$, the dynamics is cyclonic, when $1 - f^{-1}\zeta_0 < 0$, the dynamics is anticyclonic and $\sqrt{1 - f^{-1}\zeta_0}$ is complex.

5. Legendre structure

Having found a hyper-Kähler structure for SG, we now ask if the Legendre duality of the geostrophic momentum coordinates (2.5) can be generalized to the new complex coordinates.

In §2b, we presented the holomorphic potential $\Phi$, given by equation (2.16) as a function of the coordinates $X$ and $\bar{Y}$ defined using equation (2.14). Using this complex potential, Roubtsov & Roulstone (2001) exhibited contact properties for HBMs. The hyper-Kähler structure $(\tilde{h}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ induced by the coordinate system $(X, \bar{Y})$ is given by

\[
\tilde{h} = dx \otimes dx + dy \otimes dy + dx \otimes dp + dy \otimes dq
\]
\[
+ (1 + c^2)(dp \otimes dp + dq \otimes dq),
\]
\[
\tilde{\omega}_1 = dx \wedge dy + dp \wedge dy + dx \wedge dq + (1 + c^2) dp \wedge d q,
\]
\[
\tilde{\omega}_2 = -c(dx \wedge dq - dp \wedge d y)
\]
and
\[
\tilde{\omega}_3 = c(dx \wedge dp - dy \wedge dq).
\]

This structure differs from the hyper-Kähler structure $(h, \omega_1, \omega_2, \omega_3)$ induced by the coordinate system $(X, Y)$ given by equations (3.6) and (3.9)–(3.11), and no longer corresponds to the dynamical properties, nor the geometrical properties, of HBMs (essentially, this aforementioned structure corresponds to a hyperbolic Monge–Ampère equation). In this section, by considering the hyper-Kähler structure induced by the coordinate system $(X_2, Y_2)$ given by equation (4.10), we shall extend the Legendre duality of SG theory and define a holomorphic potential, from which the contact properties of HBMs will be derived.

Let $A$ and $B$ be $2 \times 2$ non-degenerate commuting symmetric complex matrices. We define the complex coordinates $X = (X, Y)^T$ by

\[
X = Ax + Bp,
\]
with $p = (p, q)^T = (\partial_x \phi, \partial_y \phi)^T$. Let us define $\mathcal{P}(x)$, $\mathcal{R}(X)$ and $\Phi(X)$ by

\[
\mathcal{P}(x) = \phi(x) + \frac{1}{2} x^T B^{-1} A x,
\]
\[
\mathcal{R}(X) = X^T B^{-1} x - \mathcal{P}
\]
and
\[
\Phi(X) = -\mathcal{R} + \frac{1}{2} X^T B^{-1} A^{-1} X.
\]

We have

\[
\nabla_x \mathcal{P} = B^{-1} X \quad \text{and} \quad \nabla_X \mathcal{R} = B^{-1} x,
\]
where $\nabla_x = (\partial/\partial x, \partial/\partial y)^T$ and $\nabla_X = (\partial/\partial X, \partial/\partial Y)^T$. We obtain the gradient property

$$\nabla_X \Phi = A^{-1} \nabla_x \Phi. \quad (5.6)$$

Restricting ourselves to the case where $A$ or $B$ is of the form $aI_2$, $a \in \mathbb{C}$, we can show that

$$\frac{\partial \Phi}{\partial X} = 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial Y} = 0; \quad (5.7)$$

that is, $\Phi$ is a holomorphic function of $X$ and $Y$. Consider the semi-holomorphic contact bundle corresponding to the coordinates $(X, Y)$, that is, the space spanned by the coordinate system $(X, Y, \Phi, P, Q)$ endowed with the contact form $d\Phi - P \, dX - Q \, dY$, with $(P, Q)^T = (\partial \Phi/\partial X, \partial \Phi/\partial Y)^T$; we find that if $\Phi$ satisfies equation (5.7), then the transformation $(x, y, \phi, p, q) \mapsto (X, Y, \Phi, P, Q)$ is a strict contact transformation.

The coordinate system $(X_2, \bar{Y}_2)$ obtained from equation (4.10) is of the form of equation (5.1), and then we can derive a Legendre transformation using equations (5.2)–(5.4). However, as with the coordinate system $(X, \bar{Y})$ given by equation (2.14) discussed at the beginning of this section, the hyper-Kähler structure induced by $(X_2, \bar{Y}_2)$ does not correspond to the elliptic Monge–Ampère structures of HBM's. Nevertheless, we can obtain the appropriate structure by starting with the hyper-Kähler structure $(h, \omega_1, \omega_2, \omega_3)$ induced by the general coordinate system $(X_2, Y_2)$. This hyper-Kähler structure is, by construction, expressed relative to the complex structure $I_2$. The following proposition provides us with formulae for adapted coordinate systems $(X_i, Y_i)$, $i = 1, 2, 3$, relative to the complex structures $I_i$, respectively, and inducing the same hyper-Kähler structure, as we require.

**Proposition 5.1.** Let $(h, I_1, I_2, I_3)$ be the canonical hyper-Kähler structure induced by the coordinate system $(X_2, Y_2)$ defined by equation (4.10). Then, by construction $(X_2, Y_2)$ is an adapted coordinate system relative to $I_2$, and adapted coordinate systems $(X_1, Y_1)$, $(X_3, Y_3)$ relative to $I_1$ and $I_3$ are given by

$$X_1 = \alpha_1 (x - iy) + \beta_1 (p - iq),$$

$$Y_1 = \alpha_2 (x + iy) + \beta_2 (p + iq) \quad (5.8)$$

and

$$X_3 = (\alpha_1 + i\alpha_2) x + (\beta_1 + i\beta_2) p,$$

$$Y_3 = (\alpha_1 + i\alpha_2) y + (\beta_1 + i\beta_2) q. \quad (5.9)$$

This result is proved in Delahaies (2009) by showing that $I_i^* \, dX_i = i \, dX_i$, $I_i^* \, d\bar{X}_i = -i \, d\bar{X}_i$, for $i = 1, 3$, where $X_i = (X_i, Y_i)^T$. This proposition provides us with two new points of view to look at the same geometric structure.

We see that the coordinate system $X_3$ is of the form (5.1) with

$$A = (\alpha_1 + i\alpha_2) 1_2 \quad \text{and} \quad B = (\beta_1 + i\beta_2) 1_2, \quad (5.10)$$

we can therefore build a Legendre structure and show contact properties using formulae (5.2)–(5.4), and, by construction, the hyper-Kähler structure induced by $X_3$ is the same as the hyper-Kähler structure induced by $X_2$. In particular,
with the choice of parameter (4.11) with \( \gamma = 0 \), we have

\[
X_3 = \sqrt{1 - f^{-1}\zeta_0} \ x + \frac{1 + i\sqrt{pf(\omega)}}{\sqrt{1 - f^{-1}\zeta_0}} \ p
\]

(5.11)

and

\[
Y_3 = \sqrt{1 - f^{-1}\zeta_0} \ y + \frac{1 + i\sqrt{pf(\omega)}}{\sqrt{1 - f^{-1}\zeta_0}} \ q,
\]

(5.12)

and the construction above gives

\[
P(x, y) = \phi + \frac{1}{2} \ \frac{1 - f^{-1}\zeta_0}{1 + i\sqrt{pf(\omega)}} \ (x^2 + y^2),
\]

(5.13)

\[
R(X_3, Y_3) = \frac{\sqrt{1 - f^{-1}\zeta_0}}{1 + i\sqrt{pf(\omega)}} \ (xX_3 + yY_3) - P
\]

(5.14)

and

\[
\Phi(X_3, Y_3) = -R + \frac{1}{2} \ \frac{1}{1 + i\sqrt{pf(\omega)}}(X_3^2 + Y_3^2).
\]

(5.15)

These formulae generalize the Legendre duality of SG theory to the class of balanced models derived by McIntyre & Roulstone, in particular for \( L_1 \)-dynamics and the \( \sqrt{3} \)-model. For SG theory, for which \( c = 0 \) and then \( pf(\omega) = f^{-1}\zeta_0 \), the Legendre dual functions given by equations (2.7) and (2.8), the Bernoulli potential given by equation (2.10) and the geostrophic coordinates (2.5) are recovered by setting \( \zeta_0 = 0 \) in equations (5.11)–(5.15).

### 6. Concluding remarks

We have revisited the issues raised in McIntyre & Roulstone (2002) about hyper-Kähler structures and HBM. In particular, in §14 of McIntyre & Roulstone (2002), three issues were raised: (i) the role of one of the members of the hyper-Kähler triple of forms was unclear—ostensibly not corresponding to either a symplectic or a Monge–Ampère structure; this has now been clarified by equation (3.10), (ii) The sign definiteness of \( 1 - f^{-1}\zeta_c \) implied that the Kähler structure could only be exhibited for cyclonic flows; this conundrum has now been resolved by the approach adopted in §4, and (iii) Integrability conditions restrict our results to consideration of constant-coefficient Monge–Ampère equations. While this remains the case, it is perhaps not surprising that anything but the simplest idealized structures in geophysical flows should possess these highly abstract geometrical properties. Similar issues have been addressed in the context of incompressible Navier–Stokes flows in two and three dimensions (see Roulstone et al. 2009a,b).

In summary, using the formalism of Kushner et al. (2007), we have exhibited a new hyper-Kähler structure for HBM, which incorporates SG theory and, using the properties of the hyper-Kähler geometry, we show how Legendre duality is exhibited in other HBM.
An over-arching question remains: what is the significance of these complex structures in fluid mechanics, and how might the geometry enhance our understanding of balanced flows and/or turbulence? Although it would be premature to attempt to give an unequivocal answer, Roulstone et al. (2009b) have shown how the almost-complex structures can be used to describe coherent structures in Navier–Stokes flows in three dimensions. The salient point of this work is that the complex geometry provides a framework for studying coherent structures that is not readily accessible via the more traditional analysis of the underlying partial differential equations that can be performed for incompressible flows in two dimensions (e.g. Larchevêque 1993). The results of Roulstone et al. (2009a, b) are based on the work of Banos (2002) on Monge–Ampère structures for three independent variables. It is tempting to speculate that Banos's results could be applied to SG theory in three dimensions, but progress in this direction requires an understanding of how the complex geometry associated with Monge–Ampère structures can be applied to SG theory in two dimensions. In this paper, we have laid the foundations for such developments.

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