The origin of similarity fields in steady elastoplastic crack propagation under K–T loading

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It has been inferred from computer simulations that the plastic-zone fields of a crack that propagates steadily under K–T loading are similarity fields. Here, we show theoretically that these similarity fields are but a manifestation of the existence of an invariant path integral. We also show that the attendant similarity variable involves an intrinsic length scale set by the specific fracture energy that flows into the crack tip. Finally, we show that where the crack is stationary there can be no similarity fields, even though there exists a (different) invariant path integral. Our results afford some new insights into the relation between similarity fields and invariant path integrals in mathematical physics.

Keywords: self-similar fields; elastoplastic crack; invariant path integral

1. Introduction

In mathematical physics, it is sometimes the case that a field that on first analysis depends on two independent variables is found on further analysis to depend on a single similarity variable—a suitable combination of the two original variables. Such a field is known as a similarity field. A simple example is the temperature in a long heat-conducting bar in which a dose of energy is quickly released at \( x = 0 \): the temperature \( \eta \) is found to depend on the elapsed time \( t \) and the position \( x \) in the form \( \eta = g(\tau) \), where \( \tau = t/\sqrt{x} \) is the similarity variable. In many problems, a similarity field has been associated with the lack of a characteristic length. More profoundly, in some problems, the origin of a similarity field has been traced to a conservation law, where the conservation law is often embodied by an invariant path integral (Barenblatt 1986). Thus, the origin of the similarity field \( \eta = g(t/\sqrt{x}) \) has been traced to the law of conservation of energy, i.e. to the fact that \( \int_{-\infty}^{\infty} c \eta \, dx = Q \) for all \( t > 0 \), where \( c \) is the specific heat per unit length of bar and \( Q \) is the dose of energy released at \( x = 0 \) and \( t = 0 \) (Barenblatt 1986). The establishment of a connection between a similarity field and an invariant path integral affords valuable physical insight, especially in problems where the field equations are
unknown or do not lend themselves to analytical treatment, and the existence of a similarity field can be inferred only empirically, either from experiments or simulations. One such problem will concern us here: the propagation of a crack in an elastoplastic material.

2. The Varias–Shih similarity fields

Consider a crack that propagates steadily and quasi-statically within a body made of an elastoplastic material (a metal, say). Attached to the crack tip is a zone in which the material undergoes plastic deformation—the plastic zone (figure 1). As we shall see, it is possible to show via a dimensional analysis that the plastic-zone fields (i.e. the fields that prevail within the plastic zone) can be expressed as functions of three dimensionless variables (Varias & Shih 1993): \(r/(K/\sigma_0)^2\), \(T/\sigma_0\), and \(\theta\), where \(r\) and \(\theta\) are polar coordinates with origin at the crack tip; \(\sigma_0\) is the yield stress of the material; and \(K\) (the stress intensity factor) and \(T\) (the \(t\)-stress) are scalar variables (Williams 1957; Betegon & Hancock 1991; Wang 1991) that depend on the geometry of the body and on the boundary conditions (including the loading). Thus, for example, the mean-stress plastic-zone field can be written in the form

\[
\bar{\sigma} = \sigma_0 F \left( \frac{r}{(K/\sigma_0)^2}, \frac{T}{\sigma_0}, \theta \right),
\]

where \(\bar{\sigma}\) is the mean stress—the first invariant of the stress tensor—and \(F\) is a dimensionless function of the three dimensionless variables defined above.

In a 1993 paper, Varias & Shih (1993) probed the outcome of extensive computer simulations to surmise a fact beyond the reach of dimensional analysis. They found that the plastic-zone fields appear to be similarity fields that can be expressed as functions of only two dimensionless variables: \(\theta\) and \(r/L\) (the Varias–Shih similarity variable), where \(L\) is an intrinsic length scale. The physical meaning of the intrinsic length scale \(L\) was not ascertained, but it was argued that \(L = (K/\sigma_0)^2 \Omega(T/\sigma_0)\), where \(\Omega(T/\sigma_0)\) is a dimensionless function of \(T/\sigma_0\) that Varias and Shih were able to determine computationally, \textit{albeit only up to an unknown constant factor} (Varias & Shih 1993). Thus, equation (2.1) may be written in the form

\[
\bar{\sigma} = \sigma_0 G \left( \frac{r}{(K/\sigma_0)^2 \Omega(T/\sigma_0)}, \theta \right),
\]

where \(G\) is a dimensionless function of two dimensionless variables.

The implications of Varias and Shih’s empirical finding are momentous because the plastic-zone fields govern the fracture processes that take place at the crack tip (which the plastic-zone fields embed; Wang 1991; Varias & Shih 1993). If the plastic-zone fields are similarity fields, the type of fracture (brittle or ductile) associated with any given combination of \(K\) and \(T\) will depend on the value of a single function of \(K\) and \(T\): the intrinsic length scale \(L\). Thus, where \(N\) experiments would be required to study the brittle–ductile transition in a specific material, \(\sqrt{N}\) experiments will suffice if the plastic-zone fields are similarity fields. (Note that this conclusion applies even where \(L\) can be calculated only up to an unknown constant factor, because the constant factor can be absorbed.
Similarity fields in crack propagation

Figure 1. Schematic of the vicinity of a crack tip that propagates steadily from left to right in the horizontal direction (the $x_1$ direction). $C$, crack; $P$, plastic zone (shaded); $W$, wake of the plastic zone; $S$, path that encircles the crack tip; $ds$ is an arc-length differential along the path, and $n$ is an outward-pointing unit vector normal to the path.

in the function $G$ of equation (2.2).) Nevertheless, many theoretical questions remain open: Why are the plastic-zone fields similarity fields? What is the physical meaning of $L$? In what follows we answer these questions by establishing a connection between the plastic-zone fields and a well-known invariant path integral (Freund & Hutchinson 1985).

3. Dimensional analysis

We start by performing a dimensional analysis of the plastic-zone fields. For concreteness, we focus on the mean-stress plastic-zone field, and seek to derive equation (2.1). (Other plastic-zone fields may be dealt with in an analogous manner.)

In many applications, the plastic zone is very small compared with the length of the crack and the size of the body. If this condition is met, it can be shown that the plastic zone is totally surrounded by an annular region in which the Williams stress field (Williams 1957) is dominant. (The inner radius of this annular region is $\gg$ than the largest linear dimension of the plastic zone; the outer radius is $\ll$ than the length of the crack (Varias & Shih 1993).) The Williams field reads

$$\sigma_W = K f(\theta) \frac{1}{\sqrt{2\pi r}} + T e_1 e_1, \quad (3.1)$$

where $\sigma_W$ is the stress tensor (of order 2); $f$ is a dimensionless universal tensor function of $\theta$; $e_1$ is the unit vector in the $x_1$ direction; and $e_1 e_1$ is a tensor dyad (figure 1). The leading term in the Williams field (the term that contains the stress intensity factor $K$) is singular at the crack tip, whereas the second term (the term that contains the t-stress $T$) is finite and corresponds to a uniaxial stress parallel to the crack plane. As we noted earlier, $K$ and $T$ are set by the boundary conditions (including the loading) and the geometry of the body.
The Williams field acts as a remote applied stress on the outer limits of the plastic zone; therefore, the plastic-zone fields depend on the boundary conditions and the geometry of the body only through the stress intensity factor and the t-stress, $K$ and $T$ (Betegon & Hancock 1991; Wang 1991). The plastic-zone fields depend also on the material properties, but we need concern ourselves only with the yield stress, $\sigma_0$, and treat all other material properties as constant parameters, not variables. In addition, the plastic-zone fields depend of course on $r$ and $q$.

Thus, the mean-stress plastic-zone field can be described using six variables: $\bar{\sigma}$ (i.e. the mean stress itself), $\sigma_0$, $r$, $\theta$, $K$ and $T$. The dimensional equations $[\bar{\sigma}] = [\sigma_0]$, $[r] = [K^2 \sigma_0^{-2}]$, $[\theta] = [K^0 \sigma_0^0]$ and $[T] = [\sigma_0]$ indicate that the dimensions of four of the variables (the dimensionally dependent variables) can be expressed as products of powers of the dimensions of the other two variables (the dimensionally independent variables, in this case $K$ and $\sigma_0$); it follows from Buckingham’s $\Pi$ theorem (Barenblatt 1986) that the functional relation among $\bar{\sigma}$, $\sigma_0$, $r$, $\theta$, $K$ and $T$ can be reduced to an equivalent functional relation among $6 - 2 = 4$ dimensionless variables. A sensible choice of dimensionless variables is

$$\Pi_0 \equiv \frac{\bar{\sigma}}{\sigma_0}, \quad \Pi_1 \equiv \frac{r}{(K/\sigma_0)^2}, \quad \Pi_2 \equiv \frac{T}{\sigma_0} \quad \text{and} \quad \Pi_3 \equiv \theta. \quad (3.2)$$

We can therefore write $\Pi_0 = F(\Pi_1, \Pi_2, \Pi_3)$, where $F$ is a dimensionless function of $\Pi_1$, $\Pi_2$ and $\Pi_3$. This expression coincides with equation (2.1), as expected.

4. Path integral

Consider now the path integral

$$I \equiv \int_S n \cdot \left( w e_1 - \sigma \cdot \frac{\partial u}{\partial x_1} \right) \, ds, \quad (4.1)$$

where $S$ is a path that encircles the crack tip; $n$ is an outward-pointing unit vector normal to $S$; $\cdot$ denotes inner product; $w$ is the work-of-stress density, $w \equiv \int_0^\epsilon \sigma \, d\epsilon$, where $\epsilon$ is the strain tensor and $\sigma$ the stress tensor; $e_1$ is the unit vector in the $x_1$ direction; and $u$ is the displacement vector (figure 1). The integral of equation (4.1) is an invariant path integral; in other words, $I$ is the same regardless of the chosen path around the crack tip (Freund & Hutchinson 1985). Further, if the crack propagates with a constant speed $U > 0$ in the $x_1$ direction (so that $U \partial / \partial x_1 = -\partial / \partial t$), then $I = G_{\text{tip}}$ (Freund & Hutchinson 1985), where $G_{\text{tip}}$ is the specific fracture energy, i.e. the energy per unit area of crack that flows into the crack tip and is consumed there in effecting fracture. Note that $I = G_{\text{tip}}$ regardless of the material properties, as long as these properties remain independent of $x_1$.

5. Specific fracture energy

It has been shown that for a perfectly elastoplastic material $G_{\text{tip}} = 0$, and no energy is available to effect fracture at the crack tip (Rice 1966). It is thought that the same paradox obtains for elastoplastic materials with hardening (Kfouri & Miller 1976; Drugan et al. 1982; Castaneda 1987). (For a dissenting stand, e.g.
Vadier (2004).) The paradox can be dispelled by incorporating viscoelastic or cohesive effects, allowing for crack propagation in finite steps, substituting a notch for the sharp crack tip—or, in general, by introducing a characteristic length scale other than \((K/\sigma_0)^2\) (Rice 1966; Kfouri & Miller 1976; Castaneda 1987; Tvergaard & Hutchinson 1992).

Now, Varias & Shih did not explicitly introduce such a characteristic length scale in their computational simulations. And yet, as we have seen, the denominator of the Varias–Shih similarity variable represents an intrinsic length other than \((K/\sigma_0)^2\). This intrinsic length scale might be construed as evidence that the paradox does not obtain; alternatively, it might be that the paradox is readily dispelled in computational simulations, where an intrinsic length scale could be the inevitable by-product of discretization. Be that as it may, in the analysis that follows we assume that \(G_{\text{tip}} \neq 0\), and show that under this assumption it is possible to trace the origin of the Varias–Shih similarity fields to the existence of the invariant path integral of equation (4.1). Note that the assumption is in keeping with Varias and Shih’s statement that ‘the energy released at the tip can be neglected in subsequent considerations since it is small compared to the plastic work and the elastic residual strain energy’ (Varias & Shih 1993; p. 841).

6. Analysis

In general, the integrand of equation (4.1) is not a field (because \(n\) depends on the path, not just on the position). Nevertheless, if we specialize (4.1) to circular paths of radius \(r > 0\), then we can write

\[
I = r \int_{-\pi}^{\pi} \left( e_1 \cos \theta + e_2 \sin \theta \right) \cdot \left( w e_1 - \sigma \cdot \frac{\partial u}{\partial x_1} \right) \, d\theta,
\]

where \(e_2\) is the unit vector in the \(x_2\) direction. The integrand of equation (6.1) is now a field with units of stress. Let us denote this field by \(y\). If we perform on \(y\) the same dimensional analysis that we performed on \(\bar{\sigma}\), we obtain

\[
y = \sigma_0 Y(\Pi_1, \Pi_2, \theta),
\]

where \(Y\) is a dimensionless function of the dimensionless variables \(\Pi_1, \Pi_2\) and \(\theta\) defined in equation (3.2). Thus we can rewrite equation (6.1) in the form

\[
I = r \sigma_0 \int_{-\pi}^{\pi} Y(\Pi_1, \Pi_2, \theta) \, d\theta.
\]

Next, consider the specific fracture energy, \(G_{\text{tip}}\). \(G_{\text{tip}}\) may depend only on \(\sigma_0\), \(K\) and \(T\). A simple dimensional analysis allows us to write

\[
G_{\text{tip}} = \frac{K^2}{\sigma_0} \Omega(\Pi_2),
\]

where \(\Omega\) is a dimensionless function of \(\Pi_2 \equiv T/\sigma_0\).

Suppose now that we effect mutually independent, small changes in $K^2$ and $T$ of magnitude $dK^2$ and $dT$, respectively. Since $I = G_{\text{tip}}$, the attendant changes in $I$ and $G_{\text{tip}}$ must satisfy $dI = dG_{\text{tip}}$. From equation (6.3) we have

$$dI = r\sigma_0 \int_{-\pi}^{\pi} \left( \frac{\partial Y}{\partial \Pi_1} \frac{\partial \Pi_1}{\partial K^2} dK^2 + \frac{\partial Y}{\partial \Pi_2} \frac{\partial \Pi_2}{\partial T} dT \right) d\theta. \quad (6.5)$$

Substituting $\partial \Pi_1/\partial K^2 = -\Pi_1/K^2$ and $\partial \Pi_2/\partial T = 1/\sigma_0$, we obtain

$$dI = r\sigma_0 \int_{-\pi}^{\pi} \left( -\frac{\partial Y}{\partial \Pi_1} \frac{\Pi_1}{K^2} dK^2 + \frac{\partial Y}{\partial \Pi_2} \frac{1}{\sigma_0} dT \right) d\theta. \quad (6.6)$$

On the other hand, from equation (6.4) we have

$$dG_{\text{tip}} = \frac{\Omega'}{\sigma_0} dK^2 + \frac{\Omega'K^2}{\sigma_0^2} dT, \quad (6.7)$$

where $\Omega' \equiv d\Omega/d\Pi_2$. Since $dK^2$ and $dT$ are mutually independent, $dI = dG_{\text{tip}}$ together with equations (6.6) and (6.7) lead to two independent equations, namely

$$-\frac{\Pi_1}{\Omega} \int_{-\pi}^{\pi} \frac{\partial Y}{\partial \Pi_1} d\theta = \frac{1}{\Pi_1}, \quad (6.8)$$

and

$$\frac{1}{\Omega'} \int_{-\pi}^{\pi} \frac{\partial Y}{\partial \Pi_2} d\theta = \frac{1}{\Pi_1}, \quad (6.9)$$

where we have taken into account that $\Pi_1 \equiv r/(K/\sigma_0)^2$. These equations must be satisfied on account of the invariant character of the path integral $I$.

We now try to represent the field $y$ as a similarity field of the form

$$y = \sigma_0 Y_s(\Pi_s, \theta) \equiv \sigma_0 Y(\Pi_1, \Pi_2, \theta), \quad (6.10)$$

where $\Pi_s$ is a similarity variable that depends on $\Pi_1$ and $\Pi_2$. We seek to ascertain whether equation (6.10) is compatible with equations (6.8) and (6.9), and to determine the form of $\Pi_s$. We start by taking a partial derivative with respect to $\Pi_1$ on both sides of equation (6.10), and then a partial derivative with respect to $\Pi_2$ on both sides of equation (6.10), with the results

$$\frac{\partial Y_s}{\partial \Pi_s} \frac{\partial \Pi_s}{\partial \Pi_1} = \frac{\partial Y}{\partial \Pi_1} \quad \text{and} \quad \frac{\partial Y_s}{\partial \Pi_s} \frac{\partial \Pi_s}{\partial \Pi_2} = \frac{\partial Y}{\partial \Pi_2}. \quad (6.11)$$

If we now substitute equation (6.11) into equations (6.8) and (6.9), we obtain

$$-\frac{\Pi_1}{\Omega} \int_{-\pi}^{\pi} \frac{\partial Y_s}{\partial \Pi_s} \frac{\partial \Pi_s}{\partial \Pi_1} d\theta = \frac{1}{\Pi_1}, \quad (6.12)$$

and

$$\frac{1}{\Omega'} \int_{-\pi}^{\pi} \frac{\partial Y_s}{\partial \Pi_s} \frac{\partial \Pi_s}{\partial \Pi_2} d\theta = \frac{1}{\Pi_1}, \quad (6.13)$$

respectively. From both equations (6.12) and (6.13), it is apparent that \( \int_{-\pi}^{\pi} \frac{\partial Y_s}{\partial \Pi_s} \, d\theta \neq 0 \). Keeping this fact in mind, we equate the left side of equation (6.12) to the left side of equation (6.13) to obtain

\[
\left( \frac{\Pi_1}{Q} \frac{\partial \Pi_s}{\partial \Pi_1} + \frac{1}{Q} \frac{\partial \Pi_s}{\partial \Pi_2} \right) \int_{-\pi}^{\pi} \frac{\partial Y_s}{\partial \Pi_s} \, d\theta = 0; \tag{6.14}
\]

since the integral on the left side of equation (6.14) is non-zero, we conclude that

\[
\frac{\Pi_1}{Q} \frac{\partial \Pi_s}{\partial \Pi_1} + \frac{1}{Q} \frac{\partial \Pi_s}{\partial \Pi_2} = 0. \tag{6.15}
\]

We attempt to satisfy this equation with a similarity variable of the form \( \Pi_s = f_1(\Pi_1)f_2(\Pi_2) \); by substituting this latter expression in equation (6.15), we obtain

\[
f_1 \frac{f_1'}{f_1} = - \frac{Q}{Q'} \frac{f_2'}{f_2} = \mu, \tag{6.16}
\]

where \( \mu \) is a constant. It follows that \( f_1(\Pi_1) = \Pi_1^\mu \) and \( f_2(\Pi_2) = (\Omega(\Pi_2))^{-\mu} \). If we now set (without loss of generality) \( \mu = 1 \), we have \( \Pi_s = \Pi_1/Q \), and can therefore write the similarity field in the form \( y = \sigma_0 Y_s(\Pi_1/\Omega(\Pi_2), \theta) \) or

\[
y = \sigma_0 Y_s \left( \frac{r}{(K/\sigma_0)^2 \Omega(T/\sigma_0)} \right) \tag{6.17}.
\]

From our results, we conclude that, under the assumption \( G_{\text{tip}} \neq 0 \), there exists a similarity field which (i) is compatible with equations (6.12) and (6.13), as required by the invariant character of the path integral \( I \); (ii) preserves the dependence on \( \Pi_1, \Pi_2 \) and \( \theta \); and (iii) has the same form as the Vairas–Shih similarity fields. Further, from equation (6.4) we recognize that the dimensionless function \( \Omega(T/\sigma_0) \) that appears in the expression of the intrinsic length scale \( L, L = (K/\sigma_0)^2 \Omega(T/\sigma_0) \), is but a dimensionless form of the specific fracture energy, \( \Omega(T/\sigma_0) = G_{\text{tip}} \sigma_0/K^2 \). Thus, the intrinsic length scale \( L \) is set by \( G_{\text{tip}} \).

We seek to estimate the order of magnitude of \( L \). The energy that flows into the plastic zone per unit area of crack is given by the expression, \( G_{\text{pz}} = K^2(1 - v^2)/E \), where \( v \) is the Poisson ratio of the material, and \( E \) the Young modulus. Form \( \Omega(T/\sigma_0) = G_{\text{tip}} \sigma_0/K^2 \), we have \( \Omega(T/\sigma_0) = (G_{\text{tip}}/G_{\text{pz}}) \sigma_0(1 - v^2)/E \). With \( E/\sigma_0 = 300 \) and \( v = 0.3 \) (the values used by Vairas and Shih), we estimate \( \Omega(T/\sigma_0) \approx 0.003 G_{\text{tip}}/G_{\text{pz}} \), and therefore \( L \approx 0.003 (G_{\text{tip}}/G_{\text{pz}}) (K/\sigma_0)^2 \). As \( G_{\text{tip}} \) is presumably only a fraction of \( G_{\text{pz}} \), \( L \) may be as small as a few times the radius \( (\approx 10^{-4}(K/\sigma_0)^2) \) of the crack-tip zone in which finite-strain effects are sizeable (Ritchie & Thompson 1985; Vairas & Shih 1993). It is remarkable, then, that although in some cases, especially for \( T > 0 \) and \( |\theta| > \pi/2 \), the similarity fields have been found to hold only up to a value of \( r \) on the order of \( 10^{-4}(K/\sigma_0)^2 \) (Vairas & Shih 1993), in most cases the similarity fields appear to hold up to distances comparable to the extent of the plastic zone immediately ahead of the crack tip, or \( 10^{-2}(K/\sigma_0)^2 \) (Vairas & Shih 1993).
7. Separable fields

In this section, we explore the possibility of substituting equation (6.2) by \( y = \sigma_0 Y_1(\Pi_1) Y_2(\Pi_2) Y_\theta(\theta) \). We can rewrite equations (6.8) and (6.9) in the form

\[
-\frac{\Pi_1}{Q} Y_1' Y_2 \int_{-\pi}^{\pi} Y_\theta \, d\theta = \frac{1}{\Pi_1}
\]

and

\[
\frac{1}{Q'} Y_1' Y_2' \int_{-\pi}^{\pi} Y_\theta \, d\theta = \frac{1}{\Pi_1}.
\]

By proceeding from these equations in much the same manner as we proceeded from equations (6.12) and (6.13), we conclude that \( y = C(\Pi_1 / Q(\Pi_2))^2 Y_\theta(\theta) \), where \( C \) and \( \lambda \) are constants. Interestingly, fields of this form have been used to obtain an analytical solution for the case of \( T = 0 \) (and, therefore, \( Q(\Pi_2) = Q(0) = \text{const.} \)); e.g. Amazigo & Hutchinson (1977). Nevertheless, the computational simulations of Varias and Shih appear to indicate that for the general case \( T \neq 0 \), the plastic-zone fields are not separable in the form \( y = \sigma_0 Y_1(\Pi_1) Y_2(\Pi_2) Y_\theta(\theta) \) (Varias & Shih 1993).

8. Stationary crack

To gain additional insight into the relation between similarity fields and invariant path integrals, we now turn to the problem of a stationary crack in an elastoplastic body (Wang 1991). Computational simulations (Varias & Shih 1993) indicate that the plastic-zone fields of stationary cracks are not similarity fields of the form (6.17). This fact does not contradict our conclusions so far, because the equality \( I = G_{\text{tip}} \) must be satisfied only when \( I \) is computed for a propagating crack, not for a stationary crack. Yet \( I \) remains an invariant path integral when computed for a stationary crack. In fact, if the loading is proportional and monotonic everywhere in the plastic zone (this is a strong condition, but it is widely thought to hold), then \( I = K^2(1 - \nu^2)/E \) regardless of the chosen path around the crack tip (Rice 1968), where \( \nu \) is the Poisson ratio and \( E \) the modulus of elasticity of the material. Therefore, in stationary cracks there exists an invariant path integral (at least under a condition that is widely thought to hold), but the plastic-zone fields are not similarity fields. We now endeavour to elucidate this fact.

We start by writing the integral \( I \) for a stationary crack in the form \( I = r \int_{-\pi}^{\pi} z \, d\theta \), where \( z \) is a field with units of stress. From dimensional analysis, we know that \( z \) can be written in the form \( z = \sigma_0 Z(\Pi_1, \Pi_2, \theta) \), where \( Z \) is a dimensionless function of the dimensionless variables defined in equation (3.2). Next, we try to represent the field \( z \) as a similarity field of the form

\[
z = \sigma_0 Z_s(\Pi_s, \theta) \equiv \sigma_0 Z(\Pi_1, \Pi_2, \theta),
\]
where \( \Pi_s \) is a similarity variable that depends on \( \Pi_1 \) and \( \Pi_2 \). Last, we apply arbitrary increments \( dK^2 \) and \( dT \) to \( I = (1 - \nu^2)K^2/E \), as we did before to \( I = G_{\text{tip}} \), and obtain the equations

\[
-\Pi_1^2 \frac{\partial \Pi_s}{\partial \Pi_1} \int_{-\pi}^{\pi} \frac{\partial Z_s}{\partial \Pi_s} \, d\theta = \kappa
\]  

(8.2)

and

\[
\frac{\partial \Pi_s}{\partial \Pi_2} \int_{-\pi}^{\pi} \frac{\partial Z_s}{\partial \Pi_s} \, d\theta = 0,
\]  

(8.3)

where we have defined \( \kappa \equiv (1 - \nu^2)\sigma_0/E \). Equations (8.2) and (8.3) are the counterparts of equations (6.12) and (6.13). From equation (8.2), it is apparent that \( \int_{-\pi}^{\pi} \frac{\partial Z_s}{\partial \Pi_s} \, d\theta \neq 0 \); then, it follows from equation (8.3) that \( \partial \Pi_s/\partial \Pi_2 = 0 \), which in turn implies that \( \Pi_s = \Pi_1 \). Thus, if the path integral \( I \) is invariant, the plastic-zone fields of a stationary crack can be similarity fields only at the cost of losing their dependence on \( \Pi_2 \), and therefore on \( T \). Yet, computational simulations indicate that the plastic-zone fields do depend on the t-stress (O’Dowd & Shih 1991; Faleskog 1995). We conclude that either (i) the path integral \( I \) is not invariant after all or (ii) the plastic-zone fields of a stationary crack cannot be similarity fields of the form (8.1). Note that, in the unlikely case that the path integral \( I \) be not invariant, there is no reason to expect similarity fields in the first place.

9. Conclusions

We have traced the origin of similarity fields in a steady elastoplastic crack propagation to the existence of an invariant path integral. We have also elucidated the form of the similarity variable as well as its physical meaning. To that end, we have developed a method that evinces the close connection between similarity fields and invariant path integrals. This method may be broadly applicable to establish the existence of similarity fields in numerous other problems of mathematical physics; it could prove especially useful in problems where similarity fields have been inferred only empirically, but an invariant path integral is known to exist. If no such path integral is known to exist, the connection between similarity fields and invariant path integrals suggests that an invariant path integral may exist and be worth finding. On the other hand, our results on stationary elastoplastic cracks demonstrate that the existence of an invariant path integral does not necessarily entail similarity fields. Thus, even where an invariant path integral is known to exist, the method developed here will not invariably lead to the discovery of similarity fields.

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