Elastohydrodynamics induced by a rapidly moving microscopic body

BY R. J. CLARKE1* AND S. POTNIS2

1Department of Engineering Science, University of Auckland, Auckland, New Zealand
2Department of Physics, University of Toronto, Toronto, Canada

The simple model involving a moving rigid particle, separated from a compliant wall by a thin film of viscous fluid, has previously been applied successfully to a number of important problems. For example, transport of blood cells, particle clearance in the lungs and the late stages of particle sedimentation. Considerable fluid forces are generated in the film, causing the compliant surface to deform. Hence, the usual goal is derivation and solution of an appropriate deformation evolution equation. In the applications considered to-date, however, flow inertia is neglected as flow speeds are not especially high. In this study, we are interested in regimes where unsteady flow inertia is significant, such as found in certain microdevices or thermal excitation of light particles. We present a novel model, which for the first time accounts for inertial effects in both the flow, and the deformable surface. The significant role that inertia plays is fully illustrated through surface deformation profiles, computed under a variety of parameter regimes, as well as calculations of associated hydrodynamic loading. Frequency response curves are seen to exhibit distinct shifts in resonant frequency and quality factor under different levels of inertia, a finding which we believe has important practical implications.

Keywords: elastohydrodynamics; unsteady Stokes flow; microscopic particles

1. Introduction

There are a number of important phenomena that can be understood by considering the motion of a rigid body that is separated from a nearby elastic wall by a viscous fluid. Examples in the existing literature include the resistance experienced by a body as it moves through a fluid-filled elastic tube (as an early model of a red blood cell negotiating a capillary (Lighthill 1968)), as well as particle clearance from the lung via the mucociliary transport mechanism (Weekley et al. 2006). Recently, this approach has also been shown to offer a means by which a sedimenting particle can make contact with a horizontal surface in finite time (Balmforth et al. 2010).

These examples have the common feature that the dynamics are controlled by a thin film of fluid separating the body from the wall. The flow within this layer can be well approximated using lubrication theory (sometimes including

*Author for correspondence (rj.clarke@auckland.ac.nz).
flow slippage effects (Vinogradova & Feuillebois 2000)). This approximation together with a suitable pressure–displacement law—for example, Hookean and beam models (Weekley et al. 2006; Balmforth et al. 2010), an elastic half-space (Balmforth et al. 2010), and a Hertz contact model (Davis et al. 1986; Vinogradova & Feuillebois 2000)—yields a non-linear partial differential equation for the evolution of film thickness. In these examples it is natural to neglect inertia, since the characteristic length scales in the gap are small, and flow speeds are not especially large.

There are a number of interesting situations, however, where this quasi-steady assumption does not necessarily always hold. For example, within many microdevices actuation amplitudes are typically too small for any significant convective inertia in the flow, but the motion of components can vary with sufficient rapidity that unsteady inertia is appreciable. We might also expect such considerations to be relevant in the case of small thermally excited particles in the presence of a deformable boundary. Describing such unsteady linearized flows is an important and increasingly relevant fluid dynamics problem, and one which has attracted much recent attention (Sader 1998; Paul & Cross 2004; Clarke et al. 2005, 2006). Crucially, the presence of a nearby surface has been shown to significantly alter the hydrodynamics (Clarke et al. 2005, 2006, 2008).

However, this recent work has assumed that the nearby surfaces are rigid. However, there are important occasions where softer material may be present. For example, atomic force microscope (AFM) measurements of a biological specimen where Young’s modulus may be in the kilopascal range (when compared with the high gigapascal modulus of a typical AFM cantilever). Under these circumstances there may be significant fluid–structure interactions. Indeed recent work by Gavara & Chadwick (2010) nicely confirms the presence of such elastohydrodynamic effects in AFM experiments using gels. Their accompanying theoretical analysis aims to exploit the observed shift in the AFM cantilever’s phase lag to determine the microrheology of their compliant samples, and assumes that purely oscillatory flow has been established. However, there may be many important situations in the AFM, and other microdevices, where a purely sinusoidal assumption is limiting. For example, when the oscillation amplitude is modulated over time (either deliberately, or as a consequence of measurement taking), or when the prescribed motion is not purely sinusoidal, for instance, during transient or intermittent dynamics, or perhaps when alternative motions are prescribed (e.g. square-wave motion).

Therefore in this study, we set out to gauge the effects of wall compliance in an unsteady, small-scale flow setting, accounting for inertia in both the flow and the compliant surface under general time-dependent conditions (§2). We consider two different constitutive laws for the surface’s elastic response: an unsteady Hookean model, and an unsteady Euler–Bernoulli model. Both choices of elasticity law result in an integro-differential equation for the surface deformation, rather than the more usual partial differential equation. This form for the evolution equation reflects the fact that memory effects often accompany the presence of flow inertia (Boussinesq 1885; Basset 1888). It is shown that the integro-differential equation does reduce down to the familiar nonlinear partial differential equation in the quasi-steady limit (appendix B), as would be expected. In general, the integro-differential equation is solved by taking Laplace transforms, with the resulting boundary-value problem and

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inverse transforms computed numerically. The findings presented in §3 reveal the importance of inertia in both the flow and the elastic surface, as well as the impact of elasticity model choice. Implications of these findings are discussed in §4.

2. Formulation

We consider a two-dimensional inelastic body of characteristic width $R^*$, immersed in a Newtonian fluid that has dynamic viscosity $\mu$ and density $\rho$. The body moves with prescribed vertical motion $h_0^*(t^*)$, and lies a small distance $H^*$ from a compliant wall. We assume that $h_0^*(t^*)$ has typical amplitude $A^* \ll H^* \ll R^*$ and that the characteristic time for the body to achieve an $O(A^*)$ displacement, $T_{0}^*$, is short. The wall deforms elastically in response to fluid pressures $p^*$, and this elastic deformation $d^*(x^*, t^*)$ is expected to scale with the driven amplitude $A^*$ (i.e. $|d^*| \sim A^* \ll H^* \ll R^*$). (Note that asterisks denote dimensional quantities.) The flow domain can be decomposed into an outer region (I), where unsteady flow inertia dominates owing to the rapid motion of the body, and a lubrication region (II), where both viscosity and unsteady inertia are important (figure 1). The flow $u^* = u^* \hat{x} + w^* \hat{z}$ (where $\hat{x}$ and $\hat{z}$ are the unit vectors in the $x$- and $z$-directions, respectively) satisfies

$$\rho(\partial_{z}^* u^* + u^* \cdot \nabla^* u^*) = -\nabla^* p^* + \mu \nabla^{*2} u^* \quad \text{and} \quad \partial_{x}^* u^* + \partial_{z}^* w^* = 0, \quad (2.1)$$

(2.2)

It also proves useful to define $D \equiv H^*/R^* \ll 1$ and $A \equiv A^*/H^* \ll 1$. The non-dimensional governing equations are therefore

$$\gamma^2 \partial_{x'} u' + Re u' \cdot \nabla u' = -\nabla p' + \nabla^2 u' \quad \text{and} \quad \partial_{x'} u' + \partial_{x'} w' = 0, \quad (2.3)$$

where $\gamma^2 \equiv \rho(R^*)^2/(\mu T_{0}^*)$ measures the importance of unsteady flow inertia against viscous effects in this outer region, whereas $Re \equiv A^* R^*/\nu T_{0}^*$ measures the importance of convective inertia. Table 1 provides some parameter values typical of the regime in which we are interested, and suggests that convective inertia is negligible. By contrast, $\gamma^2 \gg 1$, hence in the outer region (I), the flow is largely irrotational (to leading order in $\gamma$) and pressure assumes an inertial scale ($p' \sim \gamma^2$). If the rapid motion of the body ceases, the outer flow may become rotational, yet its contribution to the hydrodynamic loading on the wall will decrease and certainly become subdominant to that experienced in the thin lubrication region (II).
Figure 1. A stationary, rigid body lies close to a rapidly moving compliant wall. The flow domain can be decomposed into an outer inertia-dominated region (I) and an inner unsteady lubrication region (II).

Table 1. Example parameter values for a microdevice, such as an Atomic Force Microscope imaging a soft material of thickness $D^*$, with a Young’s modulus $E$ in the kilopascal range and density $\rho_e \sim 1000 \text{kgm}^{-3}$ (representative of biological tissue Radmacher 1997). The immersing fluid is assumed to be water ($\rho \sim \rho_e$, $\mu = 10^{-3} \text{kgm}^{-1} \text{s}^{-1}$).

$$
\begin{array}{cccccccccc}
D^* \text{ (m)} & E \text{ (kgms}^{-2} \text{)} & R^* \text{ (m)} & H^* \text{ (m)} & A^* \text{ (m)} & T_0^* \text{ (s)} & \Delta & \gamma^2 & Re & \alpha & \beta^{(1.2)} & \delta \\
10^{-4} & 10^4 & 10^{-4} & 10^{-5} & 10^{-7} & 10^{-4} & 10^{-1} & 10^2 & 10^{-1} & 1 & 10 & 10 \\
\end{array}
$$

In the lubrication region (II), we rescale according to

$$
x' = \Delta^a x, \quad z' = \Delta z, \quad u' = \Delta^{a-1} u, \quad p' = \Delta^{2a-3} p, \quad d' = d,
$$

(2.4)

where $a$ depends upon the local geometry of the body. Under these rescalings, the following flow lubrication approximation is appropriate:

$$
\alpha^{-1} \partial_t u = -\partial_x p + \partial_{zz} u, \quad \partial_x u + \partial_z w = 0 \quad \text{and} \quad \partial_z p = 0,
$$

(2.5)

where $\alpha^{-1} \equiv (\gamma \Delta)^2$ measures the importance of unsteady flow inertia in this inner lubrication region. (Of course, convective inertia is also negligible in this inner region, since the nonlinear terms are now multiplied by $\Delta^2 Re \ll 1$.) Whereas earlier studies (Lighthill 1968; Weekley et al. 2006) considered the regime where unsteady inertia is unimportant ($\alpha \to \infty$), table 1 suggests that we find ourselves in the novel regime where $\alpha \sim 1$ (i.e. $\gamma^2 \sim \Delta^{-2}$).
If the body’s surface in the lubrication region (II) is designated as lying at \( z = -s_1(x) \) when stationary, we note that the body shape can be represented locally as a power series expansion about \( x = 0 \), i.e.

\[
s_1(x) = x \partial_x s_1(0) + \left( \frac{x^2}{2} \right) \partial_{xx} s_1(0) + \left( \frac{x^3}{6} \right) \partial_{xxx} s_1(0) + \ldots
\]

(noting that we are free to set \( s_1(0) = 0 \)). We restrict ourselves to the study of bodies that are symmetric, hence \( \partial_x s_1(0) = 0 \), and therefore locally we have

\[
s_1(x) \propto |x|^q \quad (q \geq 2).
\]

(2.6)

The compliant wall is located at \( z = s_2(x, t) = 1 + Ad(x, t) \) (figure 2). The non-dimensional lubrication equations (2.5) are then subject to boundary conditions on two moving surfaces:

\[
\begin{aligned}
&u(z = -s_1(x) + Ah_0(t)) = 0, \\
&w(z = -s_1(x) + Ah_0(t)) = \partial_t h_0, \\
&\quad w(z = s_2(x, t)) = \partial_t d.
\end{aligned}
\]

(2.7)

By transforming vertical distances \( z = -s_1(x) + \alpha^{1/2} Z \), the governing equations (2.5) assume the form:

\[
\begin{aligned}
&\partial_t u = -\alpha \partial_x p + \partial_{ZZ}^2 u \quad (2.8a) \\
&\partial_x u + \alpha^{-1/2} \partial_Z w = 0, \quad (2.8b)
\end{aligned}
\]

and the boundary conditions become

\[
\begin{aligned}
&u(Z = \alpha^{-1/2} Ah_0(t)) = 0, \\
&w(Z = \alpha^{-1/2} Ah_0(t)) = \alpha^{-1/2} \partial_t h_0, \\
&\quad w(Z = h'(x, t)) = 0, \\
&w(Z = h'(x, t)) = \alpha^{-1/2} \partial_t d.
\end{aligned}
\]

(2.8d)

where \( h'(x, t) = \alpha^{-1/2} (S(x) + Ad(x, t)) \) and \( S(x) = 1 + s_1(x) \). Since pressure acts as a source term here, owing to its ultimate dependence on wall deformation \( d \),
$u$ can be solved independently of $w$. Moreover, owing to the fact that $A \ll 1$, the boundary conditions can be linearized to

$$u(Z = 0) = 0, \quad u(Z = \alpha^{-1/2} S(x)) = 0,$$  \hspace{1cm} (2.9a)

and

$$w(Z = 0) = \alpha^{-1/2} \partial_t h_0, \quad w(Z = \alpha^{-1/2} S(x)) = \alpha^{-1/2} \partial_t d.$$  \hspace{1cm} (2.9b)

Now equation (2.8a) subject to equation (2.9a) is solved by (Kartashov 1987)

$$u(x, Z, t) = \infty \sum_{n=1} b_n(x, t) \sin \left( \frac{\alpha^{1/2} n \pi Z}{S(x)} \right),$$  \hspace{1cm} (2.10)

(up to $O(A^2)$ accuracy) where

$$b_n = \begin{cases} -4 \alpha/(n \pi) \int_0^t \exp \left( \frac{-\alpha n^2 \pi^2 (t - \tau)}{S(x)^2} \right) \partial_x p(x, \tau) \, d\tau & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}$$  \hspace{1cm} (2.11)

The evolution equation for surface deformation, $d(x, t)$, then arises by integrating the continuity equation (2.8b) over the fluid layer and applying no-penetration boundary conditions (2.8d)

$$\alpha^{-1/2} (\partial_t d - \partial_t h_0) = - \int_0^{\alpha^{-1/2} S(x)} \partial_x u \, dZ = - \sum_{n=1}^{\text{odd}} \frac{2 \alpha^{-1/2}}{n \pi} \partial_x (S(x) b_n).$$  \hspace{1cm} (2.12)

We obtain the following linearized integro-differential equation for surface deformation:

$$\partial_t d = \sum_{n=1}^{\text{odd}} \frac{8 \alpha}{n^2 \pi^2} \partial_x \left( S(x) \int_0^t \partial_x p(x, \tau) \exp \left[ \frac{-\alpha n^2 \pi^2 (t - \tau)}{S(x)^2} \right] \, d\tau \right) + \partial_t h_0,$$  \hspace{1cm} (2.13)

up to $O(A^2)$ corrections. Note that when imposing boundary conditions for (2.13), we must be aware of a subtlety. Pressures within the lubrication region are $O(\Delta^{(2/q)-3})$ (since $a = 1/q$), while pressures in the outer irrotational region are $O(\gamma^2) = O(\Delta^{-2})$. Hence when the body’s shape is locally non-parabolic (i.e. $q > 2$) then the pressures and localized deformations that arise directly beneath the body (i.e. in the lubrication region) are an order of magnitude larger than any deformations occurring outside of this region, hence

$$d, \ p \to 0 \quad \text{as } |x| \to \infty.$$  \hspace{1cm} (2.14)

If the body’s shape is locally parabolic ($a = 1/2$), however, then the pressures inside and outside the gap are commensurate. Pressure and displacement then take non-zero limiting values, but can be determined by adapting the simple solution for the irrotational outer flow given by Clarke et al. (2005). Our next task is to specify a constitutive law for the surface compliance.

(a) Elasticity law

Currently, the evolution of the surface’s deformation, as governed by equation (2.13), is a function of both flow pressures $p$ and deformations $d$. In order to fully
specify the dynamics, we need to prescribe a relationship between flow pressure $p$ and wall deformation $d$. Two possible elasticity laws are considered:

\[ p^* = \left( \frac{c_0 E}{D^*} \right) d^* + \rho_e D^* \partial_{tt}^* d^* \quad \text{(Hookean),} \quad (2.15a) \]

which is appropriate under circumstances where the material compresses under hydrodynamic loading, for instances, a thin layer of elastic material supported by an inelastic substrate; alternatively

\[ p^* = c_1 E D^* \partial_{xxx}^* d^* + \rho_e D^* \partial_{tt}^* d^* \quad \text{(Euler–Bernoulli beam),} \quad (2.15b) \]

which is applicable when we are interested in a compliant material that bends in response to hydrodynamic loading. In both cases, $\rho_e$ is the density of the elastic surface, $E$ is its Young’s modulus, $D^*$ is its thickness, while constants $c_0 = (1 - \nu)/(1 + \nu)(1 - 2\nu)$ and $c_1 = 1/12 (1 - \nu^2)$ depend upon the material’s Poisson ratio $\nu$ (Balmforth et al. 2010). Under the non-dimensionalizations and rescalings given by equations (2.2) and (2.4), respectively, these elasticity laws take the form:

\[ \text{I. } p = \beta^{(1)} d + \delta \partial_{tt} d \quad \text{(Hookean)} \]

and

\[ \text{II. } p = \beta^{(2)} \partial_{xxx} d + \delta \partial_{tt} d \quad \text{(Euler–Bernoulli beam)}, \]

where $\beta^{(1)} = T_0^*/\tau_1$, $\beta^{(2)} = T_0^*/\tau_2$ and $\delta = \tau_3/T_0^*$. Here $\tau_1 \equiv \mu D^* \Delta^{2a-2}/c_0 E H^*$, $\tau_2 \equiv \mu H^{*3}/c_1 E D^* \Delta^{6-6a}$ and $\tau_3 \equiv \rho_e D^* H^*/\Delta^{2a-2} \mu$ are the characteristic times over which the rigid body must undergo its $O(A^*)$ prescribed displacement in order to generate the pressures necessary to deform the compliant wall by a comparable amount, through elastic compression, elastic bending and wall inertia, respectively. Table 1 provides typical parameter values for a relevant microdevice application.

Substituting these pressure–displacement laws (2.16) into (2.13), and integrating $\partial_{tt} d$ terms by parts twice, gives

\[
\begin{align*}
\partial_t d &= \partial_t h_0 + \sum_n \frac{8\alpha \delta}{\pi^2 n^2} \partial_x \left( S(x) \left[ \partial_{xx} d - \frac{\alpha n^2 \pi^2}{S(x)^2} \partial_x d \right] \right) \\
&\quad + \sum_n \frac{8n^2 \pi^2 \alpha^3 \delta}{\pi} \partial_x \left( \frac{1}{S(x)^3} \int_0^t \partial_x d(x, \tau) \exp \left[ -\frac{\alpha n^2 \pi^2 (t - \tau)}{S(x)^2} \right] d\tau \right) \\
&\quad + \sum_n \frac{8\alpha \delta}{\pi^3} \partial_x \left( S(x) \int_0^t \partial_x^k d(x, \tau) \exp \left[ -\frac{\alpha n^2 \pi^2 (t - \tau)}{S(x)^2} \right] d\tau \right),
\end{align*}
\]

where $j = 1$ corresponds to the Hookean law, whereas $j = 2$ relates to the Euler–Bernoulli beam law, and $k = 4(j - 1) + 1$. The number of required boundary conditions also depends upon the choice of elasticity model, for instance, under the Hookean assumption

\[ \text{I. } d \to 0 \quad \text{as } |x| \to \infty \quad \text{(Hookean),} \quad (2.18a) \]
which states that pressure asymptotically tends to zero outside of the lubrication region. Alternatively, under the beam model

\begin{equation}
\begin{align*}
 II. \quad d \to 0, \quad \partial_x d \to 0, \quad \partial_{xxxx} d \to 0 \quad \text{as} \quad |x| \to \infty \quad (\text{Euler–Bernoulli}). 
\end{align*}
\end{equation}

The first two conditions state that the compliant wall is clamped (horizontally) in the far-field, whereas the condition on the fourth derivative of \( d \) derives from the fact that the hydrodynamic (pressure) loading is asymptotically small well outside the gap (\( p \to 0 \)), hence the wall experiences no elastic shear in the far-field. In addition, we specify zero initial deformation, i.e.

\begin{equation}
\begin{align*}
 d(x,0) = 0. \tag{2.19}
\end{align*}
\end{equation}

The integrals in equation (2.17) are convolutions, hence under a Laplace transform

\begin{equation}
\begin{align*}
 \hat{H}(x,s) &\equiv \int_0^\infty H(x,t) \exp(-st) \, dt, \tag{2.20}
\end{align*}
\end{equation}

the integro-differential equation (2.17) reduces to the ordinary differential equation

\begin{equation}
\begin{align*}
 s\hat{d} &= \sum_{n=0}^{\text{odd}} \frac{8\alpha}{n^2 \pi^2} \partial_x \left( (\beta^{(j)} \partial_x^k \hat{d}(x,s) + \delta s^2 \partial_x \hat{d}(x,s)) \left( \frac{S(x)^3}{sS(x)^2 + \alpha n^2 \pi^2} \right) \right) + \partial_t \hat{h}_0. \tag{2.21}
\end{align*}
\end{equation}

Solutions to equation (2.21), subject to equation (2.18) must, in general, be solved numerically, so too their inverse Laplace transforms. The boundary-value problem (2.21) was solved using MATLAB’s \textit{bvp4c} routine. Once deformations have been determined, and given that in the lubrication region pressures are an order of magnitude larger than shear stresses, it is relatively straightforward to determine the hydrodynamic forces on the body in transformed space

\begin{equation}
\begin{align*}
 \hat{F} = \int_{-\infty}^\infty \hat{p} \, dx = \int_{-\infty}^\infty \left( \beta^{(j)} \partial_x^k \hat{d} + \delta(s^2 \hat{d} - s \hat{d}(0) - \partial_t \hat{h}_0(0)) \right) \, dx, \tag{2.22}
\end{align*}
\end{equation}

\((j = 1, 2 \text{ and } k = 4(j - 1) + 1)\). Inverse Laplace transforms of both deformations \( \hat{d} \) and forces \( \hat{F} \) were computed using an implementation of the de Hoog et al. algorithm (de Hoog \textit{et al.} 1982; Hollenbeck 1998).

Note that there are quasi-steady and far-field limits where the evolution equation reduces in complexity. These limits are explored in appendices B and C, respectively, and identified in the deformation results presented next. Finally, it can also be useful to consider the analogous formulation for an axisymmetric body (see Gavara & Chadwick 2010), and we include this case in appendix A.

3. Results

Of primary interest is the impact of inertia, both in the flow and the compliant wall, on surface deformations. We also assess the sensitivity of predictions to the choice of elasticity law, and confirm that the expected behaviour is observed in the limiting cases discussed in appendices B and C.
Figure 3. Elastic deformation for a body moving close to a compliant surface under prescribed motion \( h_0 = t^3 \) \((q = 4)\). Images on the left \((a,c,e,g,i,k)\) correspond to a Hookean model for the surface’s elastic response, whereas images on the right assume an Euler–Bernoulli beam treatment. In both cases, both wall and fluid inertia are considered important \((\alpha = 1, \delta = 10)\). The material properties are as those given in table 1 \((\beta^{(1,2)} = 10)\).

First we consider the regime detailed in table 1 \((\alpha = 1, \beta^{(1,2)} = \delta = 10)\), chosen to be representative of soft biological tissue under AFM imaging. Deformations of the elastic substrate under monotonic prescribed motion \((h_0 = t^3, 0 \leq t \leq 6)\) are shown in figure 3. It is immediately apparent that the choice of elasticity
model can have a significant bearing on surface deformations. The deformations under the Hookean assumption (figure 3a,c,e,g,i,k) are seen to be considerably more localized, and initially marginally larger in amplitude. At later times (t > 4), however, it is the Euler–Bernoulli beam responses that appear to be slightly larger in magnitude (figure 3l). The Euler–Bernoulli assumption also seems to suggest the existence of elastic waves, which slowly propagate outwards from the central
Figure 5. Elastic deformation for a body moving close to a compliant sample under oscillatory prescribed motion $h_0 = 10 \sin t \ (q = 4)$. Left-hand panels correspond to a Hookean model, whereas panels on the right-hand side assume an Euler–Bernoulli beam treatment. In both cases, both wall and fluid inertia are considered important ($\alpha = 0.1, \beta = 10$). The material properties are as those given in table 1 ($\beta(1,2) = 10$).

lubrication region. Of course, in many microdevices, oscillatory motion is more usual, hence figure 4 presents the predicted deformations when the prescribed driving is sinusoidal ($h_0 = 10 \sin t \ (0 \leq t \leq 2\pi)$). Here, the elastic waves under the Euler–Bernoulli assumption are even more evident, and are seen to become shorter wavelength as inertia is increased (figure 5).

The importance of flow inertia on the elastic deformation of surfaces with different material properties is explored further in figures 6–9. We first note
Figure 6. Elastic deformations under prescribed forcing $h_0 = 10t^3$ of a body ($q = 4$) with various levels of inertia in the flow: $\alpha = 0.1$ (solid), $\alpha = 1$ (dashed) and $\alpha = 1000$ (dotted), and the wall: $(a,b) \delta = 0$, $(c,d) \delta = 0.1$ and $(e,f) \delta = 10$. In each case $\beta^{(1,2)} = 10$ and time $t = 1$. Circular markers correspond to the quasi-steady predictions (B4). Left-hand panels correspond to the Hookean assumption, while the right-hand panels correspond to an Euler–Bernoulli beam.

that, for all cases, in the limit of low inertia ($\alpha = 1000$) we recover existing quasi-steady predictions (Weekley et al. 2006; Balmforth et al. 2010), with solutions of (B4) indicated by circular markers. As flow inertia is increased, we see that deformations extend over a greater horizontal range, and become larger in amplitude. The nature of the surface’s compliant response is also seen to depend heavily upon both wall inertia and the choice of elasticity model. When flow inertia is significant, but wall inertia slight, the Hookean model predicts a region of almost uniform deformation directly beneath the inelastic body, which is connected to monotonic far-field decay (figures 6a and 9). The deformation is also shown in figure 8 to decay algebraically in the far-field, as predicted by
Figure 7. Elastic deformations under prescribed forcing $h_0 = 10^3$ of a body ($q = 4$) with various levels of inertia in the flow: $\alpha = 0.1$ (solid), $\alpha = 1$ (dashed) and $\alpha = 1000$ (dotted), and the wall: $(a,b)$ $\delta = 0$, $(c,d)$ $\delta = 0.1$ and $(e,f)$ $\delta = 10$. In each case, $\beta^{(1,2)} = 100$ and time $t = 1$. Circular markers correspond to the quasi-steady predictions (B4). Left-hand panels correspond to the Hookean assumption, while the right-hand panels correspond to an Euler–Bernoulli beam.

The influence of wall inertia is even more marked under the Euler–Bernoulli elasticity law. In the absence of wall inertia ($\delta = 0$), the elastic response is highly non-local, a standard feature of the quasi-steady Euler–Bernoulli beam model (see Balmforth et al. 2010). There is also a non-monotonic variation in the elastic deformation, which takes its greatest value at some distance from the centre of the lubrication region (figure 6b). The presence of even a small amount of wall inertia ($\delta = 0.1$), however, is seen to dramatically alter the deformation profiles...
Figure 8. Elastic deformation for a body moving close to a compliant sample under prescribed motion \( h_0 = 0.1t^3 \) \((q = 4)\), shown on log–log axes, using the Hookean model. Wall inertia is varied in each figure \((a) \delta = 0\), \((b) \delta = 0.1\) and \((c) \delta = 10\), and in each case, flow inertia is considered important \((\alpha = 1)\). The stiffness of the wall is such that \(\beta^{(1)} = 10\). Deformation profiles are shown at times \(t = 1, 2, 3, 4\) and \(5\), and the dashed lines correspond to the predictions of the far-field analysis of 1. Prescribed motion \(h_0 = 0.1t^3\) leads to \(d \rightarrow 0.3\) \([t/(\beta^{(1)})] - \sqrt{\delta/(\beta^{(1)})^2 \sin(\sqrt{\beta^{(1)}/\delta t})}]a x^2\) as \(x \rightarrow \infty\) (dashed lines) through (C3).

(figure 6d). The deformation becomes more localized and decreases monotonically with distance from the body’s line of symmetry. As wall inertia is further increased \((\delta = 10)\), we note that the differences between the predictions of the Hookean and Euler–Bernoulli models become much less pronounced (figures 6e,f and 7e,f), as expected given the identical inertial terms in both elasticity models (2.16), and the amplitude of the deformations become noticeably reduced.

Altering the geometry of the inelastic body, such that its local shape changes from \(s_1 = x^4\) to \(s_1 = x^6\) (figure 9) results in more localized deformation. This is not surprising given that the undeformed distance between surface and particle diverges more rapidly at large \(x\) when \(s_1 = x^6\). We also note that when wall inertia is low, wall deformations appear to be less sensitive to the degree of inertia in the flow (figure 9a–d).

There are also important occasions where it can be useful to consider an axisymmetric geometry, see, for instance, Gavara & Chadwick (2010), and some deformations computed under such circumstances are presented in figure 10 (see appendix A for the formulation).

The force experienced by the body (which is equivalent to that experienced on the compliant wall, owing to zero vertical pressure gradients in the lubrication region) is understandably an important quantity. Figure 11 considers the influence of inertia on the leading-order hydrodynamic forces exerted on body (2.22), under oscillatory prescribed motion of the compliant wall \((h_0 = 10 \sin(3t))\). Clearly increases in both flow inertia and wall inertia lead to greater hydrodynamic loading. We also note that in the presence of significant flow inertia, but relatively low wall inertia, the Hookean model predicts increased loading with successive oscillations. This effect, however, is not evident under the Euler–Bernoulli model, which also predicts lower overall force. As wall inertia is increased, the predictions of both elasticity models converge, which is somewhat expected given that increasing wall inertia, while maintaining the same wall stiffness (i.e. value of \(\beta^{(1,2)}\)), reduces the influence of bending effects in the pressure–displacement laws (the terms which differentiate between Hookean and Euler–Bernoulli models).

Figure 9. Elastic deformations under prescribed forcing \( h_0 = 10t^3 \) of a body \( (q = 6) \) with various levels of inertia in the flow: \( \alpha = 0.1 \) (solid), \( \alpha = 1 \) (dashed) and \( \alpha = 1000 \) (dotted) and the wall: (a, b) \( \delta = 0 \), (c, d) \( \delta = 0.1 \) and (e, f) \( \delta = 10 \). In each case, \( \beta^{(1,2)} = 10 \) and time \( t = 1 \). Circular markers correspond to the quasi-steady predictions (B4). Left-hand panels correspond to the Hookean assumption, while the right-hand panels correspond to an Euler–Bernoulli beam.

Finally, the compliant surface’s frequency response as a function of driving frequency is explored in figure 12. When examining the frequency response of the compliant surface under oscillatory driving at frequency \( \omega^* \) (equivalent to \( 1/T^*_0 \) when \( T^*_0 \) is taken to be the period of oscillation), it can be useful to note that \( \omega \equiv \omega^*/\omega_R^{(j)} = \alpha^{-1} \tilde{\delta}^{1/2}/(\tilde{\beta}^{(j)})^{1/2} \) where \( \omega_R^{(1)} \equiv \sqrt{c_0E/\rho_eD^2} \) and \( \omega_R^{(2)} \equiv \sqrt{c_1ED^2/\rho_eR^4} \) are the natural frequencies of the compliant surface under the Hookean and Euler–Bernoulli beam laws, respectively. Furthermore, \( \tilde{\delta} = \alpha \delta, \tilde{\beta}^{(j)} = \beta^{(j)}/\alpha \) are convenient quantities that depend upon material properties only, and not driving frequency.
Figure 10. Elastic deformations of an axisymmetric body (see appendix A) under prescribed forcing $h_0 = 10t^4$ of a body ($q = 4$) with $\alpha = 1$, $\delta = 10$ and $\beta^{(1,2)} = 10$, when time $t = 0.5$. (a) Hookean assumption, (b) Euler–Bernoulli beam.

Figure 11. Forces as a function of time when flow inertia is $\alpha = 0.1$ (solid line), $\alpha = 1$ (dashed line) and $\alpha = 1000$ (dotted line). Wall inertia is (a,b) $\delta = 0.1$, and (c,d) $\delta = 10$. In all cases, the material properties of the wall are such that $\beta^{(1,2)} = 10$. Prescribed motion is oscillatory ($h_0 = 10\sin(3t)$) and the body shape is such that $q = 4$. (a,c) Hookean models, (b,d) Euler–Bernoulli model.

This response is taken to be the maximum amplitude at a given point on the surface, over a specified interval of time (note that qualitative behaviour has been observed to be insensitive to the exact location on the surface, as well as the duration of the time interval). There are several important features
Figure 12. Frequency response curve, showing $\omega$ against $\max |d|$. A Hookean elasticity law is assumed with (a) $\tilde{d} = 0.1$, (b) $\tilde{d} = 1$, (c) $\tilde{d} = 10$ and (d) $\tilde{d} = 100$. The shape of the body is such that $q = 4$ and prescribed motion is oscillatory ($h_0 = 10 \sin(t)$, $0 \leq t \leq 4\pi$). Stiffnesses are $\tilde{\beta}^{(1)} = 1$ (dotted line), $\tilde{\beta}^{(1)} = 10$ (dashed line) and $\tilde{\beta}^{(1)} = 100$ (full line).

to note. Firstly, at low values of $\tilde{d}$ the resonant frequency is observed to drop. Recalling that $\tilde{d}$ is inversely proportional to fluid density, this trend is consistent with the notion that increasing the fluid density increases the virtual mass of the fluid, and hence decreases the resonant frequency (see Sader (1998) and Clarke et al. (2008) for similarly observed behaviour when the surface is rigid). At large $\tilde{d}$, the response curves are seen to broaden (i.e. their quality factors decrease). Remembering that flow unsteadiness is quantified by $a^{-1} \propto \tilde{d}^{-1}$, it is clear that increased $\tilde{d}$ is associated with less flow inertia, hence more dominant viscous effects, and consequently response curves with lower quality factors. The impact of changing the value of $\tilde{\beta}^{(1)}$ is seen to be more involved. When the value of $\tilde{d}$ is relatively large, it is seen that increasing $\tilde{\beta}^{(1)}$ reduces the amplitude of the response. This seems logical, given that $\tilde{\beta}^{(1)}$ is directly proportional to Young’s modulus of the surface. However, when the value of $\tilde{d}$ is small, increasing $\tilde{\beta}^{(1)}$ increases the amplitude of the response. At first, this is perhaps counterintuitive, until we take into account the fact that $a^{-1} \propto (\tilde{\beta}^{(1)})^{1/2} / \tilde{d}^{1/2}$. Therefore, when $\tilde{d}$ is sufficiently small we expect $a^{-1}$ to be large, and hence we find ourselves in the inertial regime where $p \propto a^{-2}$, i.e. $p \propto \tilde{\beta}^{(1)}$. Hence despite the fact that the $1/\tilde{\beta}^{(1)}$ is proportional, the degree of deformation at a given pressure, the deformation amplitude will not necessarily decrease as $\tilde{\beta}^{(1)}$ is increased, owing to simultaneously increased hydrodynamic pressures. (Note that in the lower flow inertia limit, appropriate when $\tilde{d}$ is sufficient large, the hydrodynamic forces asymptote to the steady lubrication limit.)
4. Conclusions

We have formulated a simple model that describes the coupled fluid–structure interactions, which occur when an inelastic microscopic body moves at high speed close to a compliant surface. The body was assumed to lie sufficiently near to the surface that a lubrication theory flow approximation was deemed appropriate. Crucially, it was assumed that both the flow and the compliant surface possessed significant inertia. This is a novel regime that is increasingly relevant in a number of different areas, for example, atomic force microscope imaging of biological samples, which at times we used as a case study. Two different elasticity laws connecting flow pressures to surface displacements were considered: an unsteady Hookean model, and an unsteady Euler–Bernoulli beam model. Both led to an evolution equation that is integro-differential in form, which reflects the fact that linearized unsteady Stokes flow, present in the lubrication region, has long been known to possess memory effects (Boussinesq 1885; Basset 1888). In the absence of inertia, the integro-differential equation was seen to reduce down to the linearized form of the well-known partial differential equation that governs the quasi-steady regime (Lighthill 1968; Weekley et al. 2006; Balmforth et al. 2010), as expected.

Increasing flow inertia was shown to significantly increase the amplitudes and lateral extents of elastic surface deformations. Wall inertia and choice of elasticity law were also observed to play important roles. Introducing even mild wall inertia was seen to have a dramatic effect, particularly on the deformations predicted by the Euler–Bernoulli elasticity law. In the presence of wall inertia, the deformations predicted under the beam assumption were seen to exhibit localized surface deformations (as opposed to the non-local deformations standard under a quasi-steady Euler–Bernoulli beam model, see Balmforth et al. 2010), as well as elastic waves.

Given that flow pressure is related to elastic deformation, these findings were shown to have direct implications for the forces exerted in the lubrication region. This, in turn, has implications for the hydrodynamic loading on that body. In the case where the body represents an AFM cantilever, this will effect the deflections of that cantilever (which we are expecting to be much smaller in amplitude than those of the compliant surface, owing to the cantilever’s much larger Young’s modulus—for the regime considered in table 1, this means cantilever deflections that are smaller than 0.1 μm in amplitude). This, of course, has the potential to influence experimental measurements.

Considering other possible useful extensions to the model, it would obviously be valuable to be able to solve the equivalent problem in three dimensions. Also, Balmforth et al. (2010) recently demonstrated how fluid compressibility can introduce some significant effects in the quasi-steady case, and it may be interesting to see how these findings carry over to the unsteady regime (although the flows within the microdevices which directly motivate this study are generally considered to be incompressible (Sader 1998; Paul & Cross 2004; Clarke et al. 2005, 2006)). Note also that we allow for unsteady inertia, while neglecting convective inertia, owing to deformations being much smaller than the separation distance between the wall and the body. This, of course, prevents us from using the model to study an unsteady version of the film-thinning problem that has motivated much of the interest in the quasi-steady regime. While this small

deformation restriction is often perfectly valid in the microdevice context, it may be interesting to see how inertia alters the time scales for film draining between the surface and the body. However, treatment of the flow dynamics would undoubtedly be considerably more challenging, owing to accompanying nonlinearities in the flow equations. One further extension that would be very worthwhile concerns the solid mechanical behaviour of the deformable material. In particular, it would be useful to generalize the model to incorporate viscoelastic or plastic behaviour.

Finally, it would be very interesting to verify these theoretical predictions against physical data in appropriate applications, much in the same spirit as the recent work by Gavara & Chadwick (2010). In particular, it will be very useful to determine whether the predicted trends in the frequency response curves are seen experimentally. For example, to observe whether the quality factors and resonant frequencies of the response curves decrease and increase, respectively, as the density of the compliant surface is increased. Also, whether the relationship between response amplitude and wall stiffness switches between two different modes of behaviour depending upon wall density. Such confirmation is now the focus of a current collaborative study with colleagues in the Department of Chemical and Materials at the University of Auckland, using AFM technology.

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Appendix A. Axisymmetric geometries

It can be useful to consider an axisymmetric version of the problem (see Gavara & Chadwick 2010). Using cylindrical polar coordinates \((r, \theta, z)\), the flow velocity has respective components \(u = (u_r, 0, u_z)\). As before, let \(z = -s_1(r) + \alpha^{1/2}Z\), where \(s_1(r)\) specifies the axisymmetric shape of the particle. Under these circumstances, the unsteady lubrication equations take the form:

\[
\frac{\partial}{\partial t}u_r = -\alpha \partial_r p + \partial_Z u_r \quad \text{and} \quad \partial_Z p = 0, \quad r^{-1} \partial_r (ru_r) + \partial_Z u_z = 0, \tag{A 1}
\]

subject to linearized boundary conditions

\[
\begin{align*}
&u_r(Z = 0) = 0, \quad u_r(Z = \alpha^{-1/2}S(r)) = 0, \\
&u_z(Z = 0) = \alpha^{-1/2} \partial_t h_0, \quad u_z(Z = \alpha^{-1/2}S(r)) = \alpha^{-1/2} \partial_t d
\end{align*} \tag{A 2}
\]

\((S(r) = 1 + s_1(r))\). Following the same analysis as in §2 \((u_r\) is qualitatively the same as in equation (2.10)), we obtain

\[
\partial_t d = \sum_{n=1}^{\text{odd}} \frac{8\alpha}{n^2 \pi^2 r} \partial_r \left( rS(r) \int_0^t \partial_r p(r, \tau) \exp \left[ -\alpha n^2 \pi^2 (t - \tau)/S(r)^2 \right] d\tau \right) + \partial_t h_0, \tag{A 3}
\]

where

\[
\begin{align*}
&\text{I. } p = \beta^{(1)} d + \delta \partial_{tt} d \quad \text{(Hookean)}, \\
&\text{II. } p = \beta^{(2)} L^4 d + \delta \partial_{tt} d \quad \text{(Euler–Bernoulli beam)},
\end{align*} \tag{A 4, 5}
\]

with \( \mathcal{L}^4 \equiv \partial_{rrrr} + 2r^{-1}\partial_{rrr} - r^{-2}\partial_{rr} + r^{-3}\partial_r \). These are subject to

I. \( d \to 0 \) as \( r \to \infty, \partial_r d \to 0 \) as \( r \to 0 \) (Hookean), \( (A\ 6) \)

i.e. pinning in the far-field and regularity at the origin, or

II. \( d, \partial_r d, \mathcal{L}^4 d \to 0 \) as \( r \to \infty, (A\ 7) \)

i.e. clamping and asymptotically small pressures in the far-field, together with

II. \( \partial_r d \to 0, \partial_r(\mathcal{L}^4 d) \to 0, \mathcal{L}^3 d \to 0 \) as \( r \to 0 \) (Euler–Bernoulli), \( (A\ 8) \)

\((\mathcal{L}^3 w = \partial_{rrr} w + r^{-1}\partial_{rr} w - r^{-2}\partial_r w)\) which correspond to regularity of deformations and pressures, as well as no shear, at the origin, respectively.

Under Laplace Transforms

\[
\hat{s}d = \sum_{n} \frac{8\alpha}{n^2\pi^2 r} \partial_r \left( r(\hat{\beta}^{(j)}\partial_r(\hat{\mathcal{L}^{(j)} d}) + \delta s^{2}\partial_r d) \left( \frac{S(r)^3}{S(r)^2 + \alpha n^2 \pi^2} \right) \right) + \hat{\partial}_t h_0. \quad (A\ 9)
\]

where \( \mathcal{L}^{(1)} = 1 \) and \( \mathcal{L}^{(2)} = \mathcal{L}^4 \). Some computed profiles are shown in figure 10.

**Appendix B. Low-inertia limit \((\alpha \to \infty)\)**

In the zero-inertia limit, we expect to recover the linearized version of the partial differential equation previously derived under the quasi-steady assumption. This is most easily demonstrated by integrating equation (2.13) by parts

\[
\partial_t d = \sum_{n} \frac{8}{n^4\pi^4} \partial_x \left( S(x)^3 \left[ \partial_x p(x, \tau) \exp \left( -\alpha n^2 \pi^2 (t - \tau) \right) \right]_{\tau = t}^{\tau = 0} \right)
\]

\[-\sum_{n} \frac{8}{n^4\pi^4} \partial_x \left( S(x)^3 \int_{0}^{t} \partial_{xt} p(x, \tau) \exp \left( -\alpha n^2 \pi^2 (t - \tau) \right) d\tau \right) + \partial_t h_0, \quad (B\ 1)
\]

to obtain

\[
\partial_t d = \frac{1}{12} \partial_x (S(x)^3 \partial_x p(x, t)) + \partial_t h_0
\]

\[-\sum_{n} \frac{8}{n^4\pi^4} \partial_x \left( S(x)^3 \int_{0}^{t} \partial_{xt} p(x, \tau) \exp \left( -\alpha n^2 \pi^2 (t - \tau) \right) d\tau \right), \quad (B\ 2)
\]

using the fact that \( \partial_x p(x, 0) = 0 \) owing to zero initial deformation, and

\[\sum \left( \frac{1}{n^4} \right) = (\pi^4/96).\] In the quasi-steady limit \((\alpha \to \infty)\), the integral term can
be approximated through Watson’s lemma:

\[
\int_0^t \partial_x p(x, \tau) \exp \left( -\frac{\alpha n^2 \pi^2 (t - \tau)}{S(x)^2} \right) d\tau = \frac{S(x)^2}{\pi^2 n^2} \int_0^{n^2 \pi^2 \rho(t)} \partial_x p(x, s) \exp(-\alpha s) ds
\]

\[
\approx \frac{S(x)^2}{\pi^2 n^2} \sum_{m \geq 0} \frac{1}{\alpha^{m+1}} \left( \frac{S(x)^2}{n^2 \pi^2} \right)^m \partial_x \partial_t^{m+1} p(x, t),
\]

(B3)

\[(s = n^2 \pi^2 (t - \tau) S(x)^{-2})\]

which clearly decays to zero as \(\alpha \to \infty\). Hence

\[
\partial_t d = \frac{1}{12} \partial_x (S(x)^3 \partial_x p) + \partial_t h_0
\]

(B4)

in the limit \(\alpha \to \infty\), in agreement with the linearized version of the well-established quasi-steady evolution equation (Weekley et al. 2006; Balmforth et al. 2010). Attention is drawn to this limiting behaviour in the low inertia results presented in §3.

**Appendix C. Far-field limit \(|x| \to \infty\)**

Although the Laplace-transformed linearized integro-differential equation (2.21) is not generally amenable to analytical treatment, under the Hookean elasticity model its limiting form as \(x \to \infty\) (where \(S(x) \to x^q\)) becomes

\[
s \hat{d}(x, s) = \sum_{n}^{\text{odd}} \frac{8 \alpha (\beta(1) + \delta s^2)}{n^2 \pi^2} \partial_x \left( \frac{x^{3q} \partial_x \hat{d}(x, s)}{sx^{2q} + \alpha n^2 \pi^2} \right) + \hat{\partial_t} h_0
\]

\[
\to \frac{\alpha (\beta(1) + \delta s^2)}{s} \partial_x (x^q \partial_x \hat{d}(x, s)) + \hat{h}_0 \quad \text{as } x \to \infty.
\]

(C1)

This does have an analytical solution

\[
\hat{d} \to \hat{h}_0 + A_0 x^{(1-q)/2} I_{(1-q)/(q-2)} \left( \frac{2 \sqrt{sx^{(2-q)/2}}}{\sqrt{k_0(q-2)}} \right)
\]

\[
+ B_0 x^{(1-q)/2} K_{(q-1)/(q-2)} \left( \frac{2 \sqrt{sx^{(2-q)/2}}}{\sqrt{k_0(q-2)}} \right),
\]

(C2)

where \(k_0 = \alpha (\beta(1) + \delta s^2)/s\), \(A_0\) and \(B_0\) are constants, and \(I\) and \(K\) are modified Bessel functions. Hence in the far-field \(x \to \infty\), where we require \(d \to 0\),

\[
\hat{d} \to \frac{-s^2 \hat{h}_0}{(2 - q) \alpha (\beta(1) + \delta s^2)} x^{2-q} + O(x^{1-q}).
\]

(C3)

This predicted far-field algebraic decay is evident in the profiles presented in §3.

References


