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# Revealing competitive behaviours in music by means of the multifractal detrended fluctuation analysis: application to Bach's Sinfonias

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The one-, two- and three-dimensional multifractal detrended fluctuation analysis (MF-DFA) was applied to Bach's Sinfonias, which are characterized by the superposition of three different voices. Each voice, represented as a time series, can be considered as a component of a one-, two- or three-dimensional vector. The one-dimensional MF-DFA was applied to any single voice, while the two- and three-dimensional MF-DFA was applied to the couples of voices and to the triple, respectively. Each voice is characterized by a multifractal degree (MD), indicated by the range of the generalized Hurst exponents; the higher the MD, the larger the amount of heterogeneity and irregularity. Competitive scaling multifractal behaviours in Bach's Sinfonias were revealed; although one (or two) voices showed a relatively high MD, the other two voices, or voice, are characterized by a low MD. Nevertheless, the overall effect of the Sinfonia, measured by the MD of the triple, tends towards homogeneity, or at least to an average between the different competitive scaling behaviour shown by the different voices.

**Keywords: interdisciplinary applications; music; multifractal detrended fluctuation analysis; detrended cross-correlation**

## 1. Introduction

Music can be qualified as a highly organized system, owing to the complexity of the features that characterize it, such as network structures, recursiveness and long-range correlation. Therefore, several mathematical and statistical methodologies, well suited to investigating organized systems, have been applied to music in order to disclose its inner properties. Voss & Clarke (1975), while analysing amplitude spectra of audio signals, found a characteristic frequency $f_c$ separating white noise behaviour (flatness in power spectrum) at frequencies much lower than $f_c$, from very correlated behaviour ($S(f) \sim 1/f^2$) at frequencies much higher than $f_c$; also a flicker noise ($1/f$) behaviour was detected between 1 and 10 Hz. The music, whose spectrum is shaped like $1/f$ noise, seems the most pleasant to the human ear, because it does not have the randomness and unpredictability of the white noise, nor the boringness and predictability of the

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Brownian noise (Klimontovich & Boon 1987). Boon & Decroly (1995) applied fractal tools to the pitch variations and revealed regularities in scaling behaviour and long-range characteristics. Similar analysis was performed by Hsü & Hsü (1990, 1991). Jafari *et al.* (2007) analysed power-law behaviour in the frequency series of Bach's pitches, finding that fat tails in the probability density function has more effect than long-range correlations in the multifractality. Long-range correlations in Mozart's music scores were investigated by Dagdug *et al.* (2007), revealing the presence of correlations for relatively small note distances, with a tendency towards non-correlated behaviour for long note distances.

## 2. Methods

The main feature of multifractals is to be characterized by high variability on a wide range of temporal scales, associated to intermittent fluctuations and long-range power-law correlations.

The multifractal detrended fluctuation analysis (MF-DFA; Kantelhardt *et al.* 2002) derives from the detrended fluctuation analysis (DFA; Peng *et al.* 1994; Buldyrev *et al.* 1995) and operates on the time series $x(i)$, where $i = 1, 2, \ldots, N$ and $N$ is the length of the series. With $x_{\mathrm{ave}}$, we indicate the mean value

$$x_{\mathrm{ave}} = \frac{1}{N}\sum_{k=1}^{N} x(k). \tag{2.1}$$

We assume that $x(i)$ are increments of a random walk process around the average $x_{\mathrm{ave}}$, thus the 'trajectory' or 'profile' is given by the integration of the signal

$$y(i) = \sum_{k=1}^{i}[x(k) - x_{\mathrm{ave}}]. \tag{2.2}$$

Furthermore, the integration will reduce the level of measurement noise present in observational and finite records. Next, the integrated time series is divided into $N_{\mathrm{S}} = \mathrm{int}(N/s)$ non-overlapping segments of equal length $s$. Since the length $N$ of the series is often not a multiple of the considered time scale $s$, a short part at the end of the profile $y(i)$ may remain. In order to avoid disregarding this part of the series, the same procedure is repeated starting from the opposite end. Thereby, $2N_{\mathrm{S}}$ segments are obtained altogether. Then, we calculate the local linear trend (unitary order of detrending polynomial) for each of the $2N_{\mathrm{S}}$ segments by a least-square fit of the series. Then, we determine the variance

$$F^2(s, \nu) = \frac{1}{s}\sum_{i=1}^{s}\{y[(\nu - 1)s + i] - y_\nu(i)\}^2 \tag{2.3}$$

for each segment $\nu$, $\nu = 1, \ldots, N_{\mathrm{S}}$ and

$$F^2(s, \nu) = \frac{1}{s}\sum_{i=1}^{s}\{y[N - (\nu - N_{\mathrm{S}})s + i] - y_\nu(i)\}^2 \tag{2.4}$$

for $\nu = N_S + 1, \ldots, 2N_S$. Here, $y_\nu(i)$ is the fitting line in segment $\nu$. Then, after detrending the series, we average over all segments to obtain the $q$th order fluctuation function

$$F_q(s) = \left\{ \frac{1}{2N_S} \sum_{\nu=1}^{2N_S} [F^2(s,\nu)]^{q/2} \right\}^{1/q}, \tag{2.5}$$

where, in general, the index variable $q$ can take any real value except zero.

Repeating the procedure described above, for several time scales $s$, $F_q(s)$ will increase with increasing $s$. Then, by analysing log–log plots $F_q(s)$ versus $s$ for each value of $q$, we determine the scaling behaviour of the fluctuation functions. If the series $x_i$ is long-range power-law correlated, $F_q(s)$ increases for large values of $s$ as a power-law

$$F_q(s) \propto s^{h(q)}. \tag{2.6}$$

In general, the exponent $h(q)$ will depend on $q$. For a stationary time series, $h(2)$ is the well-defined Hurst exponent H. Thus, we call $h(q)$ the generalized Hurst exponent. Monofractal time series with compact support are characterized by $h(q)$ independent of $q$. The different scaling of small and large fluctuations will yield a significant dependence of $h(q)$ on $q$. For positive $q$, the segments $\nu$ with large variance (i.e. large deviation from the corresponding fit) will dominate the average $F_q(s)$. Therefore, if $q$ is positive, $h(q)$ describes the scaling behaviour of the segments with large fluctuations; and generally, large fluctuations are characterized by a smaller scaling exponent $h(q)$ for a multifractal time series. For negative $q$, the segments $\nu$ with small variance will dominate the average $F_q(s)$. Thus, for negative $q$ values, the scaling exponent $h(q)$ describes the scaling behaviour of segments with small fluctuations, usually characterized by larger scaling exponents.

The value $h(0)$ corresponds to the limit $h(q)$ for $q \to 0$, and cannot be determined directly using the averaging procedure of equation (2.5) because of the diverging exponent. Instead, a logarithmic averaging procedure has to be employed,

$$F_0(s) \equiv \exp\left\{ \frac{1}{4N_S} \sum_{\nu=1}^{2N_S} \ln[F^2(s,\nu)] \right\} \approx s^{h(0)}. \tag{2.7}$$

The generalization of the MF-DFA to two- or three-dimensional records is straightforward (Rosas *et al.* 2002; Telesca *et al.* 2007) as one needs only to change the basic equations (2.3) and (2.4), which are rewritten as a square fit of the series. Then, we determine the variance

$$F^2(s,\nu) = \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{D} \{y_j[(\nu-1)s+i] - y_{\nu,j}(i)\}^2 \tag{2.8}$$

and

$$F^2(s,\nu) = \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{D} \{y_j[N-(\nu-N_S)s+i] - y_{\nu,j}(i)\}^2, \tag{2.9}$$

where $D$ is the dimension of the vector space.

Figure 1. Excerpt of Sinfonia N. 787.

In order to evaluate the cross-correlation between any two voices of each Sinfonia, we applied the detrended cross-correlation method (DXA), which is developed similarly to the DFA. If the two voices are cross-correlated, the detrended covariance $F_{\mathrm{DXA}}$ scales with the scale $s$, $F_{\mathrm{DXA}} \approx s^{\lambda}$, where $\lambda$ is approximately the average between the scaling exponents of the two voices. Generally, if the detrended covariance oscillates around zero, there are no power-law cross-correlations with an unique exponent, but either no cross-correlations or only short-range cross-correlations exist between the two voices (Podobnik & Stanley 2008; Zhou 2008; Shadkhoo & Jafari 2009).

## 3. Data analysis

We analysed three Bach's Sinfonias: N. 787 (Sinfonia 1 in C major), N. 790 (Sinfonia 4 in D minor) and N. 793 (Sinfonia 7 in E minor; the data were extracted from the public website http://www.musedata.org). The data are given by a sequence of pitches and durations of notes, which are included within measures. Pitches are symbolized by the first seven letters of the Latin alphabet (A, B, C, D, E, F and G). These pitches indicate a frequency, which is measured in hertz (Hz), they are related to each other and are defined around a central note pitch, with a frequency of 440 Hz. The frequency of any other note pitch is given by $\nu = 440 \times 2^{n/12}$ Hz, where $n$ indicates the integer number of half-steps away from the central note pitch (Jafari *et al.* 2007). Each Sinfonia is characterized by three voices. Figure 1 shows, as an example, the first six measures of Sinfonia

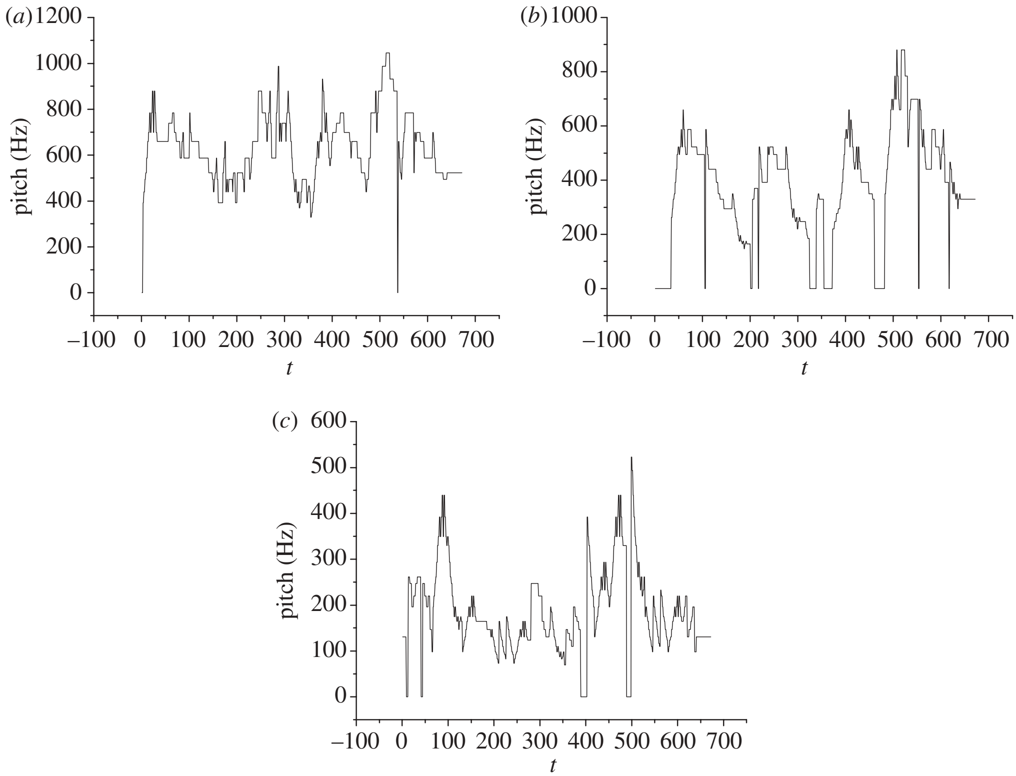


Figure 2. Time series representation of Sinfonia N. 787. (*a*) voice 1; (*b*) voice 2; (*c*) voice 3.

N. 787: three voices are present, two on the top staff and one on the bottom staff. In particular, in the first measure of the top staff the second voice is just a rest.

In order to analyse the temporal fluctuations in Bach's Sinfonias, the pitch sequence was transformed into a time series, where the sampling time was given by the minimum pitch duration in the whole music composition. Figure 2 shows the time series associated with Sinfonia N. 787. The zero values correspond to rests. Using this representation, the voices of each Sinfonia were analysed as time series.

Firstly, all data were normalized in order that their range of variability is between 0 and 1. The MF-DFA was applied to the single voices. The two-dimensional MF-DFA was applied to the couples (voices 1 and 2; voices 1 and 3; and voices 2 and 3). The three-dimensional MF-DFA was applied to the triple (voices 1, 2 and 3).

Figure 3 shows, as an example, the $F_q \sim q$ curves for voice 1 of Sinfonia N. 787 for $q = -5$, 0 and 5. The different generalized Hurst exponents $h_q$ of the fluctuation curves indicate that small and large fluctuations scale differently. We calculated the fluctuation functions $F_q(s)$ for $q \in [-5, 5]$, with 0.25 step.

Figure 4*a* shows the $h_q \sim q$ relationships for the single voices of Sinfonia N. 787 as well as for the couples and the triple. The $q$-dependence of the generalized Hurst exponent $h_q$ determined by fits in the regime between 50 and $N/4$, where

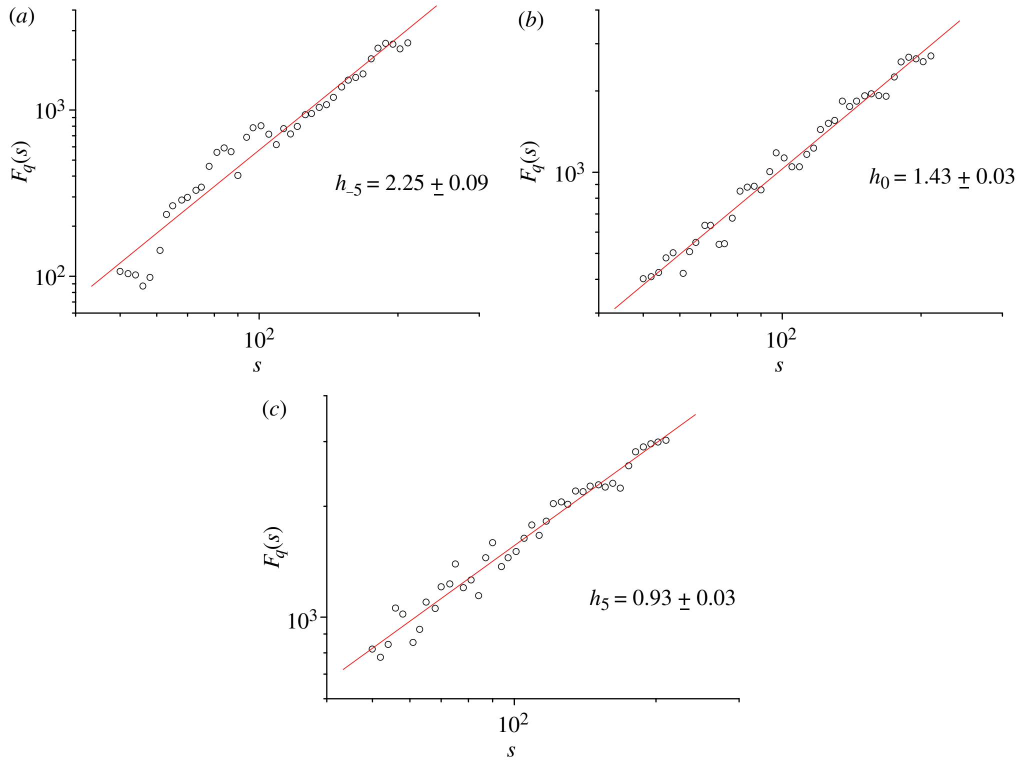


Figure 3. $F_q \sim q$ curves for the voice 1 of Sinfonia N. 787 for (*a*) $q = -5$, (*b*) $q = 0$ and (*c*) $q = 5$. (Online version in colour.)

$N$ is the length of the series, is characterized by the typical multifractal form, monotonically decreasing with an increase of $q$. The multifractal degree (MD) can be estimated by the $h_q$ range (maximum–minimum). Figure 4*b* shows the MD for the single voices, couples and the triple for Sinfonia N. 787. Considering the single voices, it is clear that voices 1 and 2 have a higher MD (MD$\sim$1.33 and MD$\sim$1.50, respectively), while voice 3 is characterized by a lower MD (MD$\sim$0.55). This indicates that the first two voices are characterized by larger heterogeneity and higher complexity, while the third voice reveals a more regular behaviour with a relatively higher homogeneity. It is also visible that the larger departure between voice 3 and the other two voices is for negative $q$, indicating that such a departure is mainly relative to the scaling properties of the small fluctuations. Analysing the couples and the triple, the role played by voice 3 is clear, which lowers the MD of voices 1 and 2, producing a value of MD$\sim$0.55 for the couple (voices 1 and 3), MD$\sim$0.98 for the couple (voices 2 and 3) and MD$\sim$0.89 for the triple (voices 1, 2 and 3); while the couple (voices 1 and 2) has MD$\sim$1.4 which is an intermediate value between those of the single voices 1 and 2, but higher than that produced by the combination with voice 3. Therefore, Sinfonia N. 787 shows two different competitive effects: one more heterogeneous, owing to the first two voices and one more regular given by the third voice. The overall effect, given by the triple, is a kind of average between the two tendencies.

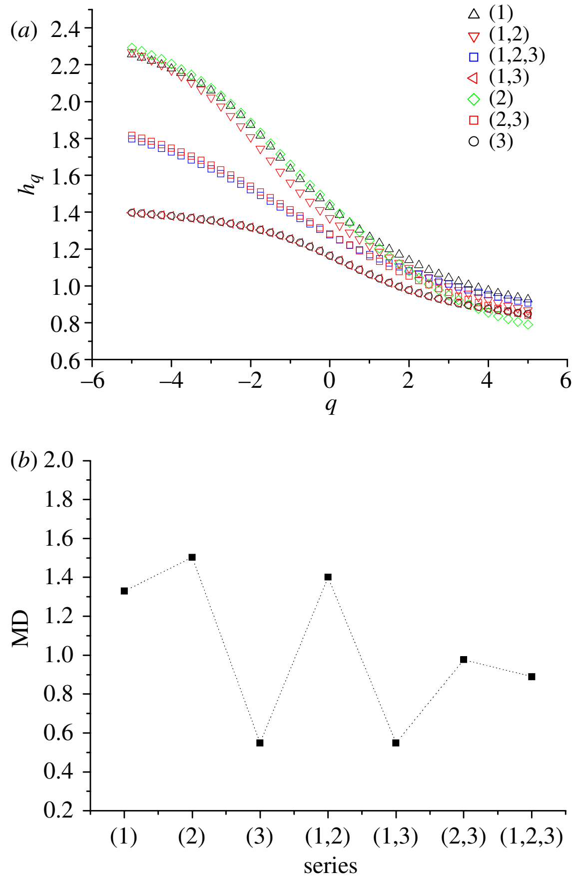


Figure 4. (*a*) Relationships of $h_q \sim q$ for the single voices of Sinfonia N. 787, as well as for the couples and the triple. (*b*) MD for the single voices, couples and the triple for Sinfonia N. 787. (Online version in colour.)

Figure 5 shows the results for Sinfonia N. 790. In this case, voice 1 is characterized by a more regular and relatively homogeneous behaviour with respect to voices 2 and 3 (figure 5*a*); their MDs are approximately 0.53, 1.03 and 0.94, respectively. It is striking that analysing the couples, the effect of voice 1 is predominant on voice 2, because the MD of the couple (voice 1 and 2; MD∼0.5) is almost identical to that of the single voice 1; while the effect of voice 3 is predominant on voice 1, because the MD of the couple (voices 1 and 3; MD∼0.94) is identical to that of the single voice 3. Anyway, the overall effect shown by the triple indicates the predominance of voice 1 (MD∼0.5), which lowers the MD of the other two voices, characterizing the Sinfonia (figure 5*b*).

Figure 6 shows the results for Sinfonia N. 793. In this case, voice 2 is characterized by a heterogeneity (MD∼0.91) larger than that of voice 1 (MD∼0.33) and voice 3 (MD∼0.31). Analysing the combined multifractal effect, we can observe that the multifractal effect of voice 2 is completely hidden by that of voice 1 (the MD of the couple (voices 1 and 2) is MD∼0.34), while it enhances that of voice 3, leading to an almost average value of the MD for the couple

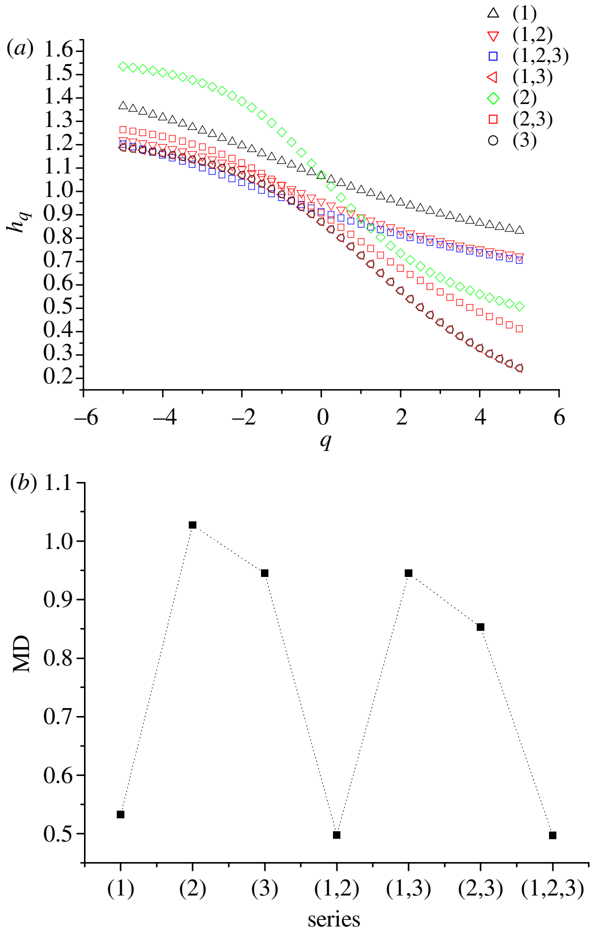


Figure 5. As in figure 4 but for Sinfonia N. 790. (Online version in colour.)

(voices 2 and 3), MD$\sim$0.52. The triple is characterized by MD$\sim$0.26, which again indicates the predominance of the multifractal behaviour of voice 1, which in this case becomes the leading voice of the Sinfonia.

In order to evaluate the cross-correlation between any two voices of a Sinfonia, the DXA method was applied. Figure 7*a* shows the results of the DXA performed on voices 1 and 2 of Sinfonia N. 787. The curve $F_{\mathrm{DXA}}(s)$ scales with $s$ with a power-law with exponent $\lambda \sim 1.15$, which is approximately the average of the scaling exponents $h_2$ for single voices 1 and 2. Figure 7*b* shows the scaling exponent $\lambda$ of the couples of voices for each analysed Sinfonia.

## 4. Conclusions

A multi-layered composition is given by two or more voices playing together. A Sinfonia is a particular case of multi-layered composition constituted by voices playing different notes with a different rhythm.

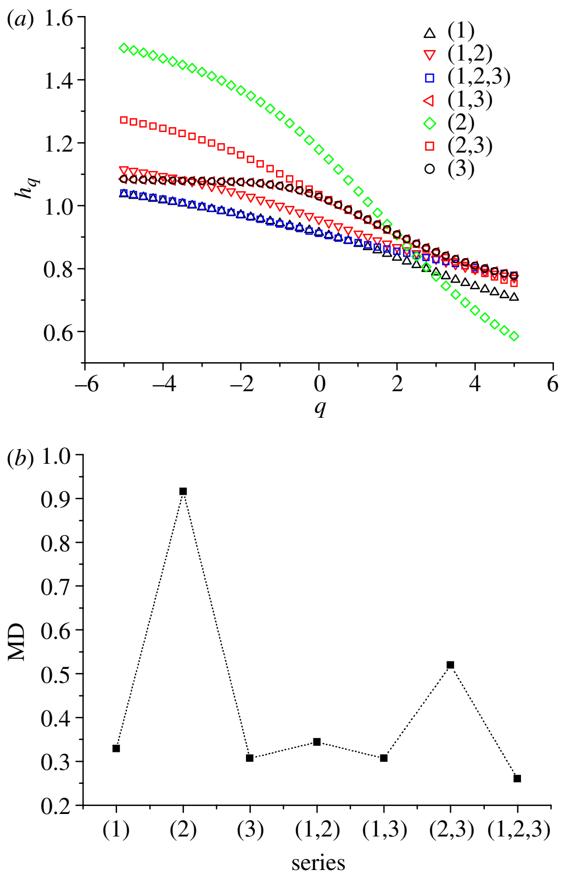


Figure 6. As in figure 4 but for Sinfonia N. 793. (Online version in colour.)

Bach's Sinfonias are characterized by the superposition of three different voices. By representing each voice as a time series, the MF-DFA was applied to the single voice separately. The two- and three-dimensional MF-DFA was applied to the couples of voices and to the triple, where each voice represented a component of the two- or three-dimensional vector, respectively. Each voice is featured by a MD, whose value quantifies its degree of heterogeneity and irregularity; the higher the MD, the larger the amount of heterogeneity. The MD can be related to the width of the multifractal spectrum which physically suggests that a relatively large opening in the multifractal spectrum of melody reveals that music features a more drastic fluctuation in pitch (Su *et al.* 2008).

Competitive scaling behaviours in Bach's Sinfonias were revealed. In fact, although one or two voices show relatively high MD while the remaining voice(s) are characterized by a low MD, the overall effect of the Sinfonia, measured by the MD obtained applying the three-dimensional MF-DFA, is a tendency towards homogeneity, or at least to an average between the different competitive scaling behaviour shown by the different voices. Therefore, a Sinfonia can be interpreted as a sort of 'competition' between different behaviours or trends, signalled by

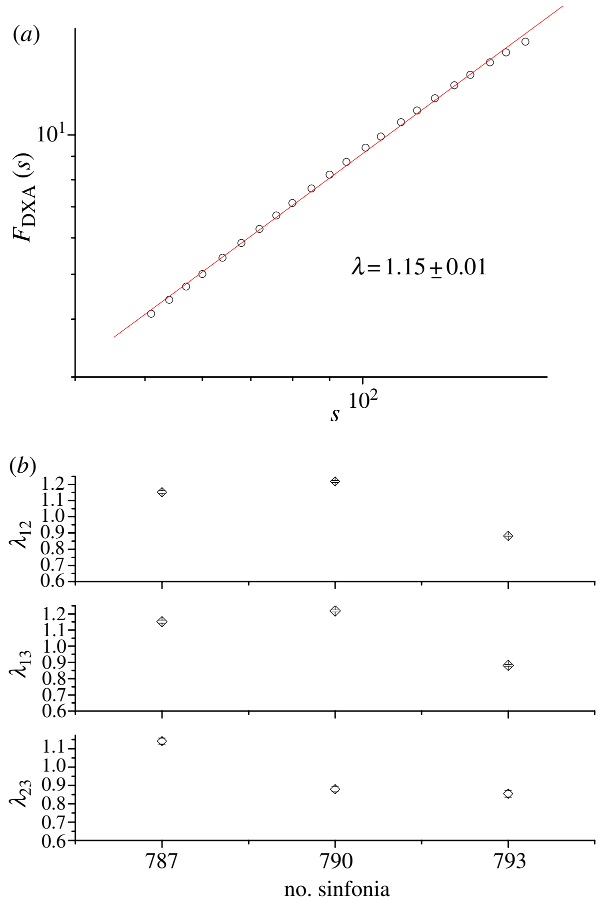


Figure 7. (*a*) DXA on the couple of voices 1 and 2 for Sinfonia N. 787. (*b*) Scaling exponents $\lambda$ obtained applying the DXA method on the couples of voices for the analysed Sinfonias. (Online version in colour.)

the different constituent voices, which are combinations of melody, given by successive changes in pitch and rhythm, given by successive changes in tone duration (Su & Wu 2006). Each voice follows its own fluctuating pitch scheme modulated by the corresponding tone duration scheme. Furthermore, any voice seems correlated with the other voices of the same Sinfonia, as the DXA results reveal. The particular multifractal behaviour of a single voice does not depend on the pitch variability, which is on average lowest for the third voice and highest for the first voice; this indicates that the multifractality is not a property of the range of variability of the voice but resides in the particular combination of melody and rhythm.

Even if one or two voices are characterized by higher multifractality, and so by higher irregularity (where the scaling behaviour of the large fluctuations is significantly different from that of the low fluctuations), the whole Sinfonia arranges all the voices in the order that the final effect would be tending to a more homogeneous feature, in which the scaling behaviour of the large fluctuations is similar to that of the low fluctuations. This would confer an undeniably pleasing effect to the ear.

The particular multifractal behaviour shown by the whole Sinfonia would reflect the aesthetic essence of the music score, which is not conveyed by the single voices that are structurally different and characterized by competitive or antagonist behaviours, but by their combination.

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