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Effect of surface elasticity on an interface crack in plane deformations

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We consider the effect of surface elasticity on an interface crack between two dissimilar linearly elastic isotropic homogeneous materials undergoing plane deformations. The bi-material is subjected to either remote tension (mode-I) or in-plane shear (mode-II) with the faces of the (interface) crack assumed to be traction-free. We incorporate surface mechanics into the model of deformation by employing a version of the continuum-based surface/interface theory of Gurtin & Murdoch. Using complex variable methods, we obtain a semi-analytical solution valid throughout the entire domain of interest (including at the crack tips) by reducing the problem to a system of coupled Cauchy singular integro-differential equations, which is solved numerically using Chebychev polynomials and a collocation method. It is shown that, among other interesting phenomena, our model predicts finite stress at the (sharp) crack tips and the corresponding stress field to be size-dependent. In particular, we note that, in contrast to the results from linear elastic fracture mechanics, when the bi-material is subjected to uniform far-field stresses (either tension or in-plane shear), the incorporation of surface effects effectively eliminates the oscillatory behaviour of the solution so that the resulting stress fields no longer suffer from oscillatory singularities at the crack tips.

Keywords: interface crack; plane deformations; surface elasticity; oscillatory singularities; Cauchy singular integro-differential equations

1. Introduction

The analysis of the plane-strain deformations of an elastic solid incorporating a crack is of fundamental importance in the understanding of the failure mechanism and in the general stress analysis of engineering materials. In particular, problems involving cracks arising specifically in the interfacial region between two dissimilar materials have drawn considerable attention from both theoreticians and practitioners in the fields of composites engineering and fracture mechanics. Traditional attempts at the modelling and solution of such problems have incorporated the classical assumptions of linear elastic fracture mechanics (LEFM) which, unfortunately have led to various inconsistencies, for example, the rapid oscillation in both stress and displacement fields in the vicinity of the crack, singularities at a crack tip and the possibility of material interpenetration (see England 1965; Rice & Sih 1965).

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It is well-known that the region near the surface of a solid material experiences a local environment different from the bulk material. Consequently, in many cases of practical interest, a more accurate model of the deformation of an elastic solid with one or more surfaces can be achieved by incorporating a description of the separate surface mechanics near the surface of the solid. In the case of a solid with a crack (interfacial or otherwise), a comprehensive model would include the surface effects corresponding to the two surfaces (faces) of the crack. This would allow for a more accurate representation of local stress distributions near, for example, a crack-tip or across a material interface. In this paper, we refine the classical LEFM model to incorporate surface mechanics using a version of the surface model proposed by Gurtin & Murdoch (1975) (see Ru 2010). The latter has been used extensively in a number of studies including several problems dealing with fracture mechanics (see Wang *et al.* 2008; Kim *et al.* 2010; Ru 2010 and references therein). The model developed in this paper is sufficiently general to incorporate the most general cases, where the upper and lower surfaces of the crack as well as the separate materials comprising the bulk have distinct material properties (see Sharma & Ganti 2004). We show that the addition of surface mechanics on the faces of the crack actually eliminates the aforementioned inconsistencies of the classical model including stress singularities at the crack tips and the oscillating singularity in the vicinity of the crack.

Throughout the paper, we make use of a number of well-established symbols and conventions. Thus, unless otherwise stated, Greek and Latin subscripts take the values 1, 3 and 1, 2, 3, respectively, summation over repeated subscripts is understood and (x, z) and (x, y, z) are generic points in the (x, z)-plane and $\mathbb{R}^3$ respectively. Finally, we note that the notation (x, z) and (x, y, z) may also be replaced by (x_1, x_3) and (x_1, x_2, x_3), respectively, when reference is made to $\{\underline{e}_i\}_{i=1}^3$, the standard basis for $\mathbb{R}^3$.

2. Preliminaries and governing equations

In the absence of body forces, the equilibrium equations and constitutive relations describing the deformation of a linearly elastic, homogeneous and isotropic (bulk) solid are given by:

$$\sigma_{ij,j} = 0, \quad \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \tag{2.1a}$$

and

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3. \tag{2.1b}$$

Here λ, μ are the Lame constants of the material, σ_{ij} and ε_{ij} the components of the stress and strain tensors, respectively and u_i denotes the i-th component of the displacement vector $\boldsymbol{u}$ in $\mathbb{R}^3$. In addition, $()\sigma_j$ denotes differentiation with respect to x_j and δ_{ij} is the kronecker delta. When the separate mechanics of the (crack) surfaces are incorporated, the equilibrium condition on the crack can be described by the equations (see Gurtin & Murdoch 1975; Gurtin *et al.* 1998; Ru 2010, for detailed derivations):

$$\sigma^s_{\alpha\beta,\beta}\underline{e}_\alpha + [\sigma_{ij}n_j\underline{e}_i] = 0, \quad \text{(tangential-direction)}, \tag{2.2a}$$

$$k_{\alpha\beta}\sigma^s_{\alpha\beta} = [\sigma_{ij}n_in_j], \quad \text{(normal-direction)} \tag{2.2b}$$

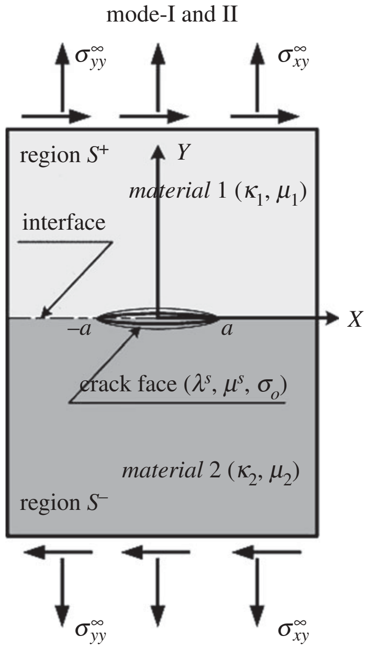

Figure 1. Schematic of the problem.

and

$$\sigma_{\alpha\beta}^{s} = \sigma_o\delta_{\alpha\beta} + 2(\mu^s - \sigma_o)\varepsilon_{\alpha\beta}^{s} + (\lambda^s + \sigma_o)\varepsilon_{\gamma\gamma}^{s}\delta_{\alpha\beta} + \sigma_o\nabla_s u. \tag{2.2c}$$

Here, the suffix 's' denotes the corresponding quantity on the crack face as a result of surface elasticity, $[*] = (*)^{\text{in}} - (*)^{\text{out}}$ denotes the jump of the quantity '$*$' across the surface (here 'in' and 'out' refer, respectively, to the inside and outside of the body) and σ_o is the surface tension. The principal curvature components $k_{\alpha\beta}$ of the undeformed surface (Ru 2010) are defined in such a way to be positive if the centre of curvature is within the '$-$' side (here, the '$-$' and '$+$' sides are designated the lower $(y < 0)$ and upper $(y > 0)$ half-planes as illustrated in figure 1). Finally, $\boldsymbol{n} = (n_1, n_2, n_3)$ is the unit normal vector of the surface pointing from the '$-$' side to the '$+$' side.

(*a*) *Complex variable formulation*

In plane-strain deformations of a linearly isotropic elastic medium, we assume that the displacement vector $\boldsymbol{u}$ with components now denoted by (u, v, ω) satisfies

$$u = u(x, y), \quad v = v(x, y), \quad \omega = 0. \tag{2.3}$$

In the absence of body forces, the corresponding governing equations of two-dimensional elasticity are described by (see England 1971)

$$2\mu(u + \mathrm{i}v) = \kappa\Omega(z) - z\overline{\Omega'(z)} - \overline{w(z)} \tag{2.4a}$$

and

$$\sigma_{yy} - \mathrm{i}\sigma_{xy} = \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{w'(z)}, \tag{2.4b}$$

Here $\Omega(z)$ and $w(z)$ are analytical functions of the complex variable $z = x + \mathrm{i}y$ and κ are defined as:

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\upsilon \quad \text{(for plane-strain)},$$

where υ is Poisson's ratio taking values in the range $0 < \upsilon < 1/2$. Thus, κ satisfies the following inequality

$$1 < \kappa < 3. \tag{2.5}$$

Noting that, in our case, the normal to the crack face is aligned with the $\underline{e}_2$ or y- direction (figure 1), the equilibrium conditions on the (crack) surface are now obtained from equations (2.2*a*) and (2.2*b*) via equations (2.1*a*), (2.1*b*) and (2.3) as:

$$\sigma^s_{xx,x} + [\sigma_{xy}] = 0 \tag{2.6a}$$

and

$$[\sigma_{yy}] = -\sigma_o \frac{\partial^2 \nu}{\partial x^2}. \tag{2.6b}$$

This leads to

$$[\sigma_{yy} - \mathrm{i}\sigma_{xy}] = \mathrm{i}\sigma^s_{xx,x} - \sigma_o \frac{\partial^2 \nu}{\partial x^2}. \tag{2.7}$$

Using the assumption of a coherent interface ($\varepsilon^s_{xx} = \varepsilon_{xx}$), with the aid of equations (2.1*b*) and (2.3), equation (2.2*c*) can be re-written as:

$$\sigma^s_{xx} = \sigma_o + 2(\mu^s - \sigma_o)\varepsilon_{xx} + (\lambda^s + \sigma_o)\varepsilon_{xx}.$$

Using equation (2.1*b*), the surface stress now can be expressed explicitly in terms of displacements:

$$\sigma^s_{xx,x} = \frac{\partial(\sigma_o + (2\mu^s - \sigma_o + \lambda^s)\partial u/\partial x)}{\partial x} = (2\mu^s - \sigma_o + \lambda^s)\frac{\partial^2 u}{\partial x^2}. \tag{2.8}$$

Finally, using equation (2.8) to replace $\sigma^s_{xx,x}$ in the right-hand side of equation (2.7), we obtain the equilibrium condition on the surface as:

$$[\sigma_{yy} - \mathrm{i}\sigma_{xy}] = \mathrm{i}J_o \frac{\partial^2 u}{\partial x^2} - \sigma_o \frac{\partial^2 \nu}{\partial x^2}, \quad \text{where } J_o \equiv 2\mu^s - \sigma_o + \lambda^s. \tag{2.9}$$

(*b*) *Traction-free interface crack with surface stress*

Let the upper half-plane ($y > 0$, occupied by *material* '1') and the lower half-plane ($y < 0$, occupied by *material* '2') be designated the '+' and '−' sides of the crack, respectively. We note that the elastic properties of *materials* '1' and '2' are, in general, different. We consider plane deformations of two-bonded dissimilar half planes incorporating a single traction-free (tractions on the crack faces are zero. i.e. $\sigma_{yy} = \sigma_{xy} = 0$ on $-a \le x \le a$, $y = \pm 0$.) interface crack subjected to uniform remote tension ($\sigma_{yy} = \sigma^\infty_{yy}$, mode-I) and shear ($\sigma_{xy} = \sigma^\infty_{xy}$, mode-II) stresses (figure 1). Then, from equations (2.4a) and (2.4b), the displacements and stresses on the upper (S^+) and lower (S^-) regions can be written, respectively, as:

$$\left.\begin{aligned}2\mu_1(u+\mathrm{i}v)^{S^+} &= \kappa_1\Omega_1(z) - z\overline{\Omega_1'(z)} - \overline{w_1(z)},\\(\sigma_{yy}-\mathrm{i}\sigma_{xy})^{S^+} &= \Omega_1'(z) + \overline{\Omega_1'(z)} + z\overline{\Omega_1''(z)} + \overline{w_1'(z)},\quad \text{for upper region } (y>0)\end{aligned}\right\} \tag{2.10a}$$

and

$$\left.\begin{aligned}2\mu_2(u+\mathrm{i}v)^{S^-} &= \kappa_2\Omega_2(z) - z\overline{\Omega_2'(z)} - \overline{w_2(z)},\\(\sigma_{yy}-\mathrm{i}\sigma_{xy})^{S^-} &= \Omega_2'(z) + \overline{\Omega_2'(z)} + z\overline{\Omega_2''(z)} + \overline{w_2'(z)},\quad \text{for lower region } (y<0),\end{aligned}\right\} \tag{2.10b}$$

where, again, subscripts '1' and '2' denote the corresponding quantities from the upper and lower regions, respectively. Here, we confine our interest to the situation where the interface ($y=0, |x|>a$) under consideration is perfectly bonded, across which the tractions (σ_{yy}, σ_{xy}) and displacements (u, v) are continuous. Then, from equations (2.10a) and (2.10b), the stress continuity condition yields

$$[\Omega_1(z) + z\overline{\Omega_1'(z)} + \overline{w_1(z)}]^+ = [\Omega_2(z) + z\overline{\Omega_2'(z)} + \overline{w_2(z)}]^-.$$

Noting that $\overline{\Omega_1'(z)}^+ = \overline{\Omega_1}(z)^-$, on $y=\pm 0$, we have

$$\Omega_1(z)^+ - z\overline{\Omega_2'}(z)^+ - \overline{w_2}(z)^+ = \Omega_2(z)^- - z\overline{\Omega_1'}(z)^- - \overline{w_1}(z)^-. \tag{2.11}$$

In view of equation (2.11), we define an analytical function $\theta(z)$ in the entire plane ($S^+\cup S^- = S$) outside the cut $L = -a \le x \le a$, $y=0$ as:

$$\Omega_1(z) - z\overline{\Omega_2'}(z) - \overline{w_2}(z) = \Omega_2(z) - z\overline{\Omega_1'}(z) - \overline{w_1}(z) \equiv \theta(z). \tag{2.12}$$

Therefore, we obtain

$$w_1(z) = \overline{\Omega_2}(z) - z\Omega_1'(z) - \overline{\theta}(z) \tag{2.13a}$$

and

$$w_2(z) = \overline{\Omega_1}(z) - z\Omega_2'(z) - \overline{\theta}(z). \tag{2.13b}$$

By substituting equations (2.13a) and (2.13b) back into equations (2.10a) and (2.10b), the displacements from the upper and lower regions can be re-written as:

$$2\mu_1(u+\mathrm{i}v)^{S^+} = \kappa_1\Omega_1(z) - \Omega_2(\bar{z}) - (z-\bar{z})\overline{\Omega_1'(z)} + \theta(\bar{z}) \tag{2.14a}$$

and

$$2\mu_2(u+\mathrm{i}v)^{S^-} = \kappa_2\Omega_2(z) - \Omega_1(\bar{z}) - (z-\bar{z})\overline{\Omega_2'(z)} + \theta(\bar{z}). \tag{2.14b}$$

Next, applying the displacement continuity condition, equations (2.14a) and (2.14b) yield

$$\frac{1}{\mu_1}[\kappa_1\Omega_1(z) - \Omega_2(\bar{z}) + \theta(\bar{z})]^+ = \frac{1}{\mu_2}[\kappa_2\Omega_2(z) - \Omega_1(\bar{z}) + \theta(\bar{z})]^-.$$

Since $\Omega_1(\bar{z})^+ = \Omega_1(z)^-$ on $y=\pm 0$, we have

$$(\mu_1 + \mu_2\kappa_1)\Omega_1(z)^+ - \mu_1\theta(z)^+ = (\mu_2+\mu_1\kappa_2)\Omega_2(z)^- - \mu_2\theta(z)^-. \tag{2.15}$$

From equation (2.15), we again define an analytical function $\phi(z)$ in the entire plane $(S^+ \cup S^- = S)$ outside the cut region $(L = -a \le x \le a,\ y = 0)$ by

$$(\mu_1 + \mu_2\kappa_1)\Omega_1(z) - \mu_1\theta(z) = (\mu_2 + \mu_1\kappa_2)\Omega_2(z) - \mu_2\theta(z) \equiv \phi(z). \tag{2.16}$$

Thus, we find that

$$\Omega_1(z) = \frac{\phi(z) + \mu_1\theta(z)}{(\mu_1 + \mu_2\kappa_1)} \quad \text{and} \quad \Omega_2(z) = \frac{\phi(z) + \mu_2\theta(z)}{(\mu_2 + \mu_1\kappa_2)}. \tag{2.17}$$

In addition, from equations (2.10*a*), (2.10*b*) and (2.13*a*), (2.13*b*), we derive expressions for the stresses

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^{S^+} = \Omega_1'(z) + \Omega_2'(\bar{z}) + (z - \bar{z})\overline{\Omega_1''(z)} - \theta'(\bar{z}),$$
$$\text{for upper region } (S^+,\ y > 0) \tag{2.18a}$$

and

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^{S^-} = \Omega_2'(z) + \Omega_1'(\bar{z}) + (z - \bar{z})\overline{\Omega_2''(z)} - \theta'(\bar{z}),$$
$$\text{for lower region } (S^-, y < 0). \tag{2.18b}$$

Consequently, the stress and displacement fields for the upper and lower regions can now be completely described by two complex potentials $(\phi(z), \theta(z))$ which are analytical in the entire plane $(S^+ \cup S^- = S)$ outside the cut $(L = -a \le x \le a,\ y = 0)$.

Now, from equation (2.9), the boundary conditions on the crack faces $(-a < x < a, y = \pm 0)$ can be written as (see Kim *et al.* 2010):

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^+ = \mathrm{i}J_o\frac{\partial^2 u^+}{\partial x^2} - \sigma_o\frac{\partial^2 v^+}{\partial x^2}$$

and

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^- = -\mathrm{i}J_o\frac{\partial^2 u^-}{\partial x^2} + \sigma_o\frac{\partial^2 v^-}{\partial x^2}.$$

In view of equations (2.18*a*) and (2.18*b*), we obtain from the above that

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^+ = \Omega_1'(z)^+ + \Omega_2'(z)^- - \theta'(z)^- = \mathrm{i}J_o\frac{\partial^2 u^+}{\partial x^2} - \sigma_o\frac{\partial^2 v^+}{\partial x^2},$$
$$\text{on the upper face, } (y = +0,\ -a < x < a) \tag{2.19a}$$

and

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy})^- = \Omega_2'(z)^- + \Omega_1'(z)^+ - \theta'(z)^+ = -\mathrm{i}J_o\frac{\partial^2 u^-}{\partial x^2} + \sigma_o\frac{\partial^2 v^-}{\partial x^2},$$
$$\text{on the lower face, } (y = -0,\ -a < x < a). \tag{2.19b}$$

Adding and subtracting equations (2.19*a*) and (2.19*b*) yields

$$\theta'(z)^+ - \theta'(z)^- = \mathrm{i}J_o\left(\frac{\partial^2 u^+}{\partial x^2} + \frac{\partial^2 u^-}{\partial x^2}\right) - \sigma_o\left(\frac{\partial^2 v^+}{\partial x^2} + \frac{\partial^2 v^-}{\partial x^2}\right) \tag{2.20a}$$

and

$$2(\Omega_1'(z)^+ + \Omega_2'(z)^-) - (\theta'(z)^+ + \theta'(z)^-) = \mathrm{i}J_o\left(\frac{\partial^2 u^+}{\partial x^2} - \frac{\partial^2 u^-}{\partial x^2}\right) - \sigma_o\left(\frac{\partial^2 v^+}{\partial x^2} - \frac{\partial^2 v^-}{\partial x^2}\right). \tag{2.20b}$$

From equation (2.17), equation (2.20*b*) can be re-written as:

$$\frac{2\phi'(z)^+}{\mu_1+\mu_2\kappa_1} + \frac{2\phi'(z)^-}{\mu_2+\mu_1\kappa_2} + \left(\frac{\mu_1-\mu_2\kappa_1}{\mu_1+\mu_2\kappa_1}\right)\theta'(z)^+ + \left(\frac{\mu_2-\mu_1\kappa_2}{\mu_2+\mu_1\kappa_2}\right)\theta'(z)^- = \mathrm{i}J_o\left(\frac{\partial^2 u^+}{\partial x^2} - \frac{\partial^2 u^-}{\partial x^2}\right) - \sigma_o\left(\frac{\partial^2 v^+}{\partial x^2} - \frac{\partial^2 v^-}{\partial x^2}\right). \tag{2.21}$$

Now, from equations (2.14*a*) and (2.14*b*), we have that

$$\frac{\partial^2}{\partial x^2}(u^+ + \mathrm{i}v^+) = \frac{1}{2\mu_1}(\kappa_1\Omega_1''(z)^+ - \Omega_2''(z)^- + \theta''(z)^-) \tag{2.22a}$$

and

$$\frac{\partial^2}{\partial x^2}(u^- + \mathrm{i}v^-) = \frac{1}{2\mu_2}(\kappa_2\Omega_2''(z)^- - \Omega_1''(z)^+ + \theta''(z)^+). \tag{2.22b}$$

Adding and subtracting equations (2.22*a*) and (2.22*b*) with the further use of equation (2.17) gives

$$\frac{\partial^2}{\partial x^2}(u^+ + u^- + \mathrm{i}(v^+ + v^-)) = \frac{(\mu_2\kappa_1-\mu_1)\phi''(z)^+}{2\mu_1\mu_2(\mu_1+\mu_2\kappa_1)} + \frac{(\mu_1\kappa_2-\mu_2)\phi''(z)^-}{2\mu_1\mu_2(\mu_2+\mu_1\kappa_2)} + \frac{\kappa_1}{\mu_1+\mu_2\kappa_1}\theta''(z)^+ + \frac{\kappa_2}{\mu_2+\mu_1\kappa_2}\theta''(z)^- \tag{2.23a}$$

and

$$\frac{\partial^2}{\partial x^2}(u^+ - u^- + \mathrm{i}(v^+ - v^-)) = \frac{1}{2\mu_1\mu_2}(\phi''(z)^+ - \phi''(z)^-). \tag{2.23b}$$

Consequently, from equations (2.23*a*), (2.23*b*), equations (2.20*a*) and (2.21) take the following forms:

$$\theta'(z)^+ - \theta'(z)^- = \mathrm{i}J_o\ \mathrm{Re}\ \mathcal{P} - \sigma_o\ \mathrm{Im}\ \mathcal{P} \tag{2.24a}$$

and

$$\frac{2\phi'(z)^+}{\mu_1+\mu_2\kappa_1} + \frac{2\phi'(z)^-}{\mu_2+\mu_1\kappa_2} + \left(\frac{\mu_1-\mu_2\kappa_1}{\mu_1+\mu_2\kappa_1}\right)\theta'(z)^+ + (\frac{\mu_2-\mu_1\kappa_2}{\mu_2+\mu_1\kappa_2})\theta'(z)^- = \frac{\mathrm{i}J_o}{2\mu_1\mu_2}\mathrm{Re}[\phi''(z)^+ - \phi''(z)^-] - \frac{\sigma_o}{2\mu_1\mu_2}\mathrm{Im}[\phi''(z)^+ - \phi''(z)^-]. \tag{2.24b}$$

where

$$\mathcal{P} = \left[\frac{(\mu_2\kappa_1-\mu_1)\phi''(z)^+}{2\mu_1\mu_2(\mu_1+\mu_2\kappa_1)} + \frac{(\mu_1\kappa_2-\mu_2)\phi''(z)^-}{2\mu_1\mu_2(\mu_2+\mu_1\kappa_2)} + \frac{\kappa_1}{\mu_1+\mu_2\kappa_1}\theta''(z)^+ + \frac{\kappa_2}{\mu_2+\mu_1\kappa_2}\theta''(z)^-\right].$$

As in Schiavone & Ru (2009), a unique solution of the boundary value problem under consideration is achieved if we impose additional (endpoint) conditions at the crack tips. In this scenario, these (endpoint) conditions are chosen to ensure that we seek only solutions with bounded stresses at the crack tips. Consequently, if we express the unknowns $\phi'(z)$ and $\theta'(z)$ as Cauchy integrals (Muskhelishvili 1953), we have that

$$\left.\begin{aligned} \phi'(z) &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f(t)+\mathrm{i}g(t)}{t-z}\,\mathrm{d}t + \frac{1}{M}(\sigma_{yy}^{\infty} - \mathrm{i}\sigma_{xy}^{\infty}), \\ \text{where}\quad M &= \left(\frac{1}{\mu_1+\mu_2\kappa_1} + \frac{1}{\mu_2+\mu_1\kappa_2}\right), \\ \phi''(z) &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f(t)+\mathrm{i}g(t)}{(t-z)^2}\,\mathrm{d}t \\ &= -\left[\frac{f(t)+\mathrm{i}g(t)}{t-z}\right]_{-a}^{+a} + \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f'(t)+\mathrm{i}g'(t)}{t-z}\,\mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f'(t)+\mathrm{i}g'(t)}{t-z}\,\mathrm{d}t, \end{aligned}\right\} \tag{2.25}$$

where,

$$f(a) = f(-a) = g(a) = g(-a) = 0.$$

$$\begin{aligned} \theta'(z) &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{\alpha(t)+\mathrm{i}\beta(t)}{t-z}\,\mathrm{d}t, \\ \theta''(z) &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{\alpha(t)+\mathrm{i}\beta(t)}{(t-z)^2}\,\mathrm{d}t \\ &= -\left[\frac{\alpha(t)+\mathrm{i}\beta(t)}{t-z}\right]_{-a}^{+a} + \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{\alpha'(t)+\mathrm{i}\beta'(t)}{t-z}\,\mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{\alpha'(t)+\mathrm{i}\beta'(t)}{t-z}\,\mathrm{d}t \end{aligned}$$

and $$\alpha(a) = \alpha(-a) = \beta(a) = \beta(-a) = 0. \tag{2.26}$$

In addition, the boundary values of $\phi'(z)$ and $\theta'(z)$ on the crack faces ($-a < t_o < a$, $y = \pm 0$) can be found as (see Muskhelishvili 1953):

$$\phi'(z)^+ = \frac{1}{2}(f(t_o) + \mathrm{i}g(t_o)) + \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f(t)+\mathrm{i}g(t)}{t-t_o}\,\mathrm{d}t + \frac{1}{M}(\sigma_{yy}^{\infty} - \mathrm{i}\sigma_{xy}^{\infty}), \tag{2.27a}$$

$$\phi'(z)^- = -\frac{1}{2}(f(t_o) + \mathrm{i}g(t_o)) + \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{f(t)+\mathrm{i}g(t)}{t-t_o}\,\mathrm{d}t + \frac{1}{M}(\sigma_{yy}^{\infty} - \mathrm{i}\sigma_{xy}^{\infty}), \tag{2.27b}$$

$$\theta'(z)^+ = \frac{1}{2}(\alpha(t_o) + \mathrm{i}\beta(t_o)) + \frac{1}{2\pi \mathrm{i}}\int_{-a}^{+a}\frac{\alpha(t)+\mathrm{i}\beta(t)}{t-t_o}\,\mathrm{d}t \tag{2.27c}$$

and $$\theta'(z)^{-} = -\frac{1}{2}(\alpha(t_o) + \mathrm{i}\beta(t_o)) + \frac{1}{2\pi\mathrm{i}}\int_{-a}^{+a}\frac{\alpha(t)+\mathrm{i}\beta(t)}{t-t_o}\,\mathrm{d}t. \tag{2.27d}$$

Thus, in view of equations (2.25)–(2.27), equations (2.24a)–(2.24b) can be re-written as:

$$\alpha(t_o) + \mathrm{i}\beta(t_o) = \mathrm{i}J_o \,\mathrm{Re}\, \mathcal{F} - \sigma_o \,\mathrm{Im}\, \mathcal{F} \tag{2.28a}$$

and

$$\begin{aligned}
&(f(t_o) + \mathrm{i}g(t_o))\left(\frac{1}{\mu_1+\mu_2\kappa_1} - \frac{1}{\mu_2+\mu_1\kappa_2}\right) \\
&\quad + \left(\frac{1}{\mu_1+\mu_2\kappa_1} + \frac{1}{\mu_2+\mu_1\kappa_2}\right)\frac{1}{\pi\mathrm{i}}\int_{-a}^{+a}\frac{f(t)+\mathrm{i}g(t)}{t-t_o}\,\mathrm{d}t + 2(\sigma_{yy}^{\infty} - \mathrm{i}\sigma_{xy}^{\infty}) \\
&\quad + \frac{(\alpha(t_o)+\mathrm{i}\beta(t_o))(\kappa_2\mu_1^2 - \kappa_1\mu_2^2)}{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)} \\
&\quad - \frac{(\kappa_1\kappa_2 - 1)\mu_1\mu_2}{\pi\mathrm{i}(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\int_{-a}^{+a}\frac{\alpha(t)+\mathrm{i}\beta(t)}{t-t_o}\,\mathrm{d}t \\
&= \frac{\mathrm{i}J_o}{2\mu_1\mu_2}\mathrm{Re}[f'(t_o) + \mathrm{i}g'(t_o)] - \frac{\sigma_o}{2\mu_1\mu_2}\,\mathrm{Im}\,[f'(t_o) + \mathrm{i}g'(t_o)].
\end{aligned} \tag{2.28b}$$

Here,

$$\begin{aligned}
\mathcal{F} &= \frac{(f'(t_o)+\mathrm{i}g'(t_o))(\kappa_1\mu_2^2 - \kappa_2\mu_1^2)}{2\mu_1\mu_2(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)} \\
&\quad + \frac{(\kappa_1\kappa_2 - 1)}{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\left(\frac{1}{2\pi\mathrm{i}}\right)\int_{-a}^{+a}\frac{f'(t)+\mathrm{i}g'(t)}{t-t_o}\,\mathrm{d}t \\
&\quad + \frac{1}{2}(\alpha'(t_o)+\mathrm{i}\beta'(t_o))\left\{\frac{\kappa_1}{\mu_1+\mu_2\kappa_1} - \frac{\kappa_2}{\mu_2+\mu_1\kappa_2}\right\} \\
&\quad + \left\{\frac{\kappa_1}{\mu_1+\mu_2\kappa_1} + \frac{\kappa_2}{\mu_2+\mu_1\kappa_2}\right\}\frac{1}{2\pi\mathrm{i}}\int_{-a}^{+a}\frac{\alpha'(t)+\mathrm{i}\beta'(t)}{t-t_o}\,\mathrm{d}t.
\end{aligned}$$

Finally, by separating the real and imaginary parts of equations (2.28a) and (2.28b), we obtain the following coupled system of first-order Cauchy singular integro-differential equations for the unknowns $f(t), g(t), \alpha(t)$ and $\beta(t)$:

$$\begin{aligned}
\alpha(t_o) &= -g'(t_o)\frac{\sigma_o(\kappa_1\mu_2^2 - \kappa_2\mu_1^2)}{2\mu_1\mu_2(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)} \\
&\quad + \frac{\sigma_o(\kappa_1\kappa_2 - 1)}{2\pi(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\int_{-a}^{+a}\frac{f'(t)}{t-t_o}\,\mathrm{d}t \\
&\quad - \beta'(t_o)\frac{\sigma_o}{2}\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2} - \frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right) \\
&\quad + \frac{\sigma_o}{2\pi}\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2} + \frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right)\int_{-a}^{+a}\frac{\alpha'(t)}{t-t_o}\,\mathrm{d}t,
\end{aligned} \tag{2.29a}$$

$$
\begin{aligned}
&- g(t_o)\left(\frac{1}{\mu_1+\mu_2\kappa_1}-\frac{1}{\mu_2+\mu_1\kappa_2}\right)\\
&+\left(\frac{1}{\mu_1+\mu_2\kappa_1}+\frac{1}{\mu_2+\mu_1\kappa_2}\right)\frac{1}{\pi}\int_{-a}^{+a}\frac{f(t)}{t-t_o}\,\mathrm{d}t\\
&-\frac{\beta(t_o)(\kappa_2\mu_1^2-\kappa_1\mu_2^2)}{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}-\frac{(\kappa_1\kappa_2-1)\mu_1\mu_2}{\pi(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\int_{-a}^{+a}\frac{\alpha(t)}{t-t_o}\,\mathrm{d}t\\
&=-\frac{J_o}{2\mu_1\mu_2}f'(t_o)-2\sigma_{xy}^{\infty},
\end{aligned}
\tag{2.29b}
$$

$$
\begin{aligned}
\beta(t_o)=&f'(t_o)\frac{J_o(\kappa_1\mu_2^2-\kappa_2\mu_1^2)}{2\mu_1\mu_2(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\\
&+\frac{J_o(\kappa_1\kappa_2-1)}{2\pi(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\int_{-a}^{+a}\frac{g'(t)}{t-t_o}\,\mathrm{d}t\\
&+\alpha'(t_o)\frac{J_o}{2}\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2}-\frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right)\\
&+\frac{J_o}{2\pi}\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2}+\frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right)\int_{-a}^{+a}\frac{\beta'(t)}{t-t_o}\,\mathrm{d}t
\end{aligned}
\tag{2.29c}
$$

and

$$
\begin{aligned}
&f(t_o)\left(\frac{1}{\mu_1+\mu_2\kappa_1}-\frac{1}{\mu_2+\mu_1\kappa_2}\right)\\
&+\left(\frac{1}{\mu_1+\mu_2\kappa_1}+\frac{1}{\mu_2+\mu_1\kappa_2}\right)\frac{1}{\pi}\int_{-a}^{+a}\frac{g(t)}{t-t_o}\,\mathrm{d}t\\
&+\frac{\alpha(t_o)(\kappa_2\mu_1^2-\kappa_1\mu_2^2)}{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}-\frac{(\kappa_1\kappa_2-1)\mu_1\mu_2}{\pi(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}\int_{-a}^{+a}\frac{\beta(t)}{t-t_o}\,\mathrm{d}t\\
&=-\frac{\sigma_o}{2\mu_1\mu_2}g'(t_o)-2\sigma_{yy}^{\infty}.
\end{aligned}
\tag{2.29d}
$$

3. Coupled singular integro-differential equations

The equations in equations (2.29a)–(2.29d) comprise a coupled system of first-order Cauchy singular-integro differential equations. Solutions of the corresponding simplified version (non-coupled) of such equations have been well established in the literature (see Kaya & Erdogan 1987; Frankel 1995; Chakrabarti & Hamsapriye 1999). In Kim *et al.* (2010), the authors extended the methods used by the aforementioned authors to the case of a coupled system (two unknowns with two coupled equations) and provided complete semi-analytical solutions. However, the methods used involved substantial 'pre-treatment' of the equations via either regularization (see Frankel 1995) or the so-called 'T^{-1} transformation' (Chakrabarti & Hamsapriye 1999), resulting in a highly complicated system of equations. Unfortunately, given the additional complications presented by equations (2.29a)–(2.29d) (four unknowns with four coupled equations), neither the existing methods nor their extensions can be applied in this case. Instead, we propose a novel approach, which does not involve any 'pre-treatment' of the equations yet provides complete semi-analytical solutions of equations (2.29*a*) and (2.29*b*).

(*a*) *Mode-II* ($\sigma_{yy}^{\infty}=0$, $\sigma_{xy}^{\infty}\neq 0$) *case*

By setting $t/a=t$ in equations (2.29*a*)–(2.29*d*) and further defining the unknown functions as (see Kim *et al.* 2010 for more details)

$$f(at)=f(t),\quad g(at)=g(t),\quad \alpha(at)=\alpha(t)\quad \text{and}\quad \beta(at)=\beta(t),$$

we obtain from equations (2.29*a*) and (2.29*b*) that

$$\begin{aligned}\alpha(t_o)=g'(t_o)\left(\frac{C_s\sigma_o}{2a\mu_1\mu_2}\right)+\frac{D_s\sigma_o}{2a\pi}\int_{-1}^{+1}\frac{f'(t)}{t-t_o}\,\mathrm{d}t\\ -\beta'(t_o)\left(\frac{F_s\sigma_o}{2a}\right)+\left(\frac{E_s\sigma_o}{2a\pi}\right)\int_{-1}^{+1}\frac{\alpha'(t)}{t-t_o}\,\mathrm{d}t,\end{aligned}\tag{3.1a}$$

$$\begin{aligned}-A_s g(t_o)+\frac{B_s}{\pi}\int_{-1}^{+1}\frac{f(t)}{t-t_o}\,\mathrm{d}t-C_s\beta(t_o)-\frac{D_s\mu_1\mu_2}{\pi}\int_{-1}^{+1}\frac{\alpha(t)}{t-t_o}\,\mathrm{d}t\\ =-\left(\frac{J_o}{2a\mu_1\mu_2}\right)f'(t_o)-2\sigma_{xy}^{\infty},\end{aligned}\tag{3.1b}$$

$$\begin{aligned}\beta(t_o)=-\left(\frac{C_sJ_o}{2a\mu_1\mu_2}\right)f'(t_o)+\frac{D_sJ_o}{2a\pi}\int_{-1}^{+1}\frac{g'(t)}{t-t_o}\,\mathrm{d}t\\ +\alpha'(t_o)\left(\frac{F_sJ_o}{2a}\right)+\left(\frac{E_sJ_o}{2a\pi}\right)\int_{-1}^{+1}\frac{\beta'(t)}{t-t_o}\,\mathrm{d}t\end{aligned}\tag{3.1c}$$

and

$$\begin{aligned}A_sf(t_o)+\frac{B_s}{\pi}\int_{-1}^{+1}\frac{g(t)}{t-t_o}\,\mathrm{d}t+C_s\alpha(t_o)-\frac{D_s\mu_1\mu_2}{\pi}\int_{-1}^{+1}\frac{\beta(t)}{t-t_o}\,\mathrm{d}t\\ =-\left(\frac{\sigma_o}{2a\mu_1\mu_2}\right)g'(t_o),\end{aligned}\tag{3.1d}$$

where,

$$\left(\frac{1}{\mu_1+\mu_2\kappa_1}-\frac{1}{\mu_2+\mu_1\kappa_2}\right)=A_s,\left(\frac{1}{\mu_1+\mu_2\kappa_1}+\frac{1}{\mu_2+\mu_1\kappa_2}\right)=B_s,$$
$$\frac{(\kappa_2\mu_1^2-\kappa_1\mu_2^2)}{(\mu_1+\kappa_1\mu_2)(\mu_2+\kappa_2\mu_1)}=C_s,\frac{(\kappa_1\kappa_2-1)}{(\mu_1+\kappa_1\mu_2)(\mu_2+\kappa_2\mu_1)}=D_s$$

and

$$\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2}+\frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right)=E_s,\left(\frac{\kappa_1}{\mu_1+\kappa_1\mu_2}-\frac{\kappa_2}{\mu_2+\kappa_2\mu_1}\right)=F_s.$$

We assume that surface tension (σ_o) is equal to zero (so that the residual stress field vanishes) based on the fact that its contribution to the system is practically negligible (see Kim *et al.* 2010), equations (3.1*a*)–(3.1*d*) now reduce to:

$$\alpha(t_o) = 0 \tag{3.2a}$$

$$-A_s g(t_o) + \frac{B_s}{\pi}\int_{-1}^{+1}\frac{f(t)}{t-t_o}\,\mathrm{d}t + 2\sigma_{xy}^{\infty} - C_s\beta(t_o) = -\left(\frac{J_o}{2a\mu_1\mu_2}\right)f'(t_o), \tag{3.2b}$$

$$\beta(t_o) = -\left(\frac{C_s J_o}{2a\mu_1\mu_2}\right)f'(t_o) + \frac{D_s J_o}{2a\pi}\int_{-1}^{+1}\frac{g'(t)}{t-t_o}\,\mathrm{d}t + \left(\frac{E_s J_o}{2a\pi}\right)\int_{-1}^{+1}\frac{\beta'(t)}{t-t_o}\,\mathrm{d}t \tag{3.2c}$$

$$\text{and}\quad A_s f(t_o) + \frac{B_s}{\pi}\int_{-1}^{+1}\frac{g(t)}{t-t_o}\,\mathrm{d}t - \frac{D_s\mu_1\mu_2}{\pi}\int_{-1}^{+1}\frac{\beta(t)}{t-t_o}\,\mathrm{d}t = 0. \tag{3.2d}$$

By taking the derivative with respect to 't_o', equations (3.2*d*) becomes

$$f'(t_o) = -\frac{B_s}{A_s\pi}\int_{-1}^{+1}\frac{g'(t)}{t-t_o}\,\mathrm{d}t + \frac{D_s\mu_1\mu_2}{A_s\pi}\int_{-1}^{+1}\frac{\beta'(t)}{t-t_o}\,\mathrm{d}t. \tag{3.3}$$

Substituting equation (3.3) into equation (3.2*c*) yields

$$\begin{aligned}\beta(t_o) = {} & \frac{J_o}{2a\pi}\int_{-1}^{+1}\frac{(B_sC_s/A_s\mu_1\mu_2 + D_s)g'(t)}{t-t_o}\,\mathrm{d}t \\ & + \frac{J_o}{2a\pi}\int_{-1}^{+1}\frac{(-C_sD_s/A_s + E_s)\beta'(t)}{t-t_o}\,\mathrm{d}t.\end{aligned} \tag{3.4}$$

Now, motivated by the statement in Ilyin (1985), we propose approximations for solutions in the form:

$$\left.\begin{aligned} & f(t) \approx f_N(t) = \sum_{m=0}^{N}\frac{1}{\sqrt{1-t^2}}a_m T_m(t) \\ & g(t) \approx g_N(t) = \sum_{m=0}^{N}\sqrt{1-t^2}\,b_m U_{m-1}(t) \\ \text{and}\quad & \beta(t) \approx \beta_N(t) = \sum_{m=0}^{N}\sqrt{1-t^2}\,C_m U_{m-1}(t), \quad m = 0, 1, 2, \ldots \end{aligned}\right\}, \tag{3.5}$$

where $T_m(t)$ and $U_m(t)$ represent the mth Chebychev polynomial of the first and second kind, respectively. Using the properties of Chebychev polynomials, it is rather tedious but not difficult to obtain the following system of linear equations for the unknown coefficients a_m, b_m and C_m.

$$\begin{aligned}&\sum_{m=0}^{N}\left[-a_m\sin\left(\frac{mi\pi}{N}\right)\left(B_s + \frac{J_o m}{2a\mu_1\mu_2\sin(i\pi/N)}\right)\right. \\ &\qquad\left. - a_m\frac{J_o\cos(i\pi/N)\cos(im\pi/N)}{2a\mu_1\mu_2\sin^2(i\pi/N)} + a_m(K_m^{II}A_s + H_m^{II}C_s)\sin\left(\frac{i\pi}{N}\right)\sin\left(\frac{mi\pi}{N}\right)\right] \\ &= 2\sigma_{xy}^{\infty}\sin\left(\frac{i\pi}{N}\right).\end{aligned}$$

where

$$\left.\begin{aligned} b_m &= a_m K_m^{II}, \quad C_m = a_m H_m^{II} \\ K_m^{II} &\equiv \frac{A_s(2aA_s + 1/\sin(i\pi/N)(-C_sD_s + A_sE_s)mJ_o)}{\begin{array}{c}B_s\{2aA_s\sin(i\pi/N) + mJ_o(-C_sD_s + A_sE_s)\}\\ +mJ_o(D_s\mu_1\mu_2)(B_sC_s/\mu_1\mu_2 + A_sD_s)\end{array}} \\ \text{and}\quad H_m^{II} &\equiv \frac{-1/\sin(i\pi/N)(B_sC_s/\mu_1\mu_2 + A_sD_s)mJ_oA_s}{\begin{array}{c}B_s\{2aA_s\sin(i\pi/N) + mJ_o(-C_sD_s + A_sE_s)\}\\ +mJ_oD_s\mu_1\mu_2(B_sC_s/\mu_1\mu_2 + A_sD_s)\end{array}}. \end{aligned}\right\} \tag{3.6}$$

In addition, recalling the end conditions in equation (2.26) together with equation (3.5), we have that

$$\left.\begin{aligned} &\sum_{m=0}^{N} a_m T_m(1) = \sum_{m=0}^{N} a_m = 0, \quad \text{for } i = 0 \\ \text{and}\quad &\sum_{m=0}^{N} a_m T_m(-1) = \sum_{m=0}^{N} a_m(-1)^n = 0, \quad \text{for} i = N. \end{aligned}\right\} \tag{3.7}$$

Consequently, the solution of the coupled equations (3.2*b*)–(3.2*d*) is now reduced to the solution of the two systems of equations (3.6) and (3.7) for the unknown constants a_m, b_m and C_m. The latter can be obtained by using any of the existing numerical software packages (e.g. MATLAB, MAPLE, NAG, etc.). The mode-I case $(\sigma_{yy}^{\infty} \neq 0, \sigma_{xy}^{\infty} = 0)$ is treated similarly.

4. Results and discussion

In this section, the numerical solution of equations (3.6) and (3.7) is derived for the case when a crack is subjected to a uniform remote in-plane shear. We also provide results from the solution of the corresponding equations in the case of a uniform remote tension (mode-I crack problem). Throughout the analysis, we adopt the following range of surface parameters. The listed values are estimated properties of 'GaN' obtained from the work of Sharma & Ganti (2004). GaN is composed of a mixture of nitrified aluminium (AI), gallium (Ga) and indium (In) and used in the manufacture of a semiconductor.

$$\left.\begin{aligned} &Se = \frac{J_o}{a} : 16\times 10^8 \le Se \le 16\times 10^{10} \\ &\mu^s = 161.73(\mathrm{J/m^2}), \quad \sigma_o = 1.3(\mathrm{J/m^2}), \quad J_o = 400(\mathrm{J/m^2}) \\ \text{and}\quad &168\times 10^7(Pa) \le \mu_1, \mu_2 \le 168\times 10^9(Pa),\ 2.3 \le \kappa_1, \kappa_2 \le 2.7. \end{aligned}\right\} \tag{4.1}$$

It is found that the numerical method overall performs well and ensures rapid convergence. For practical purposes, we do not present numerical results here for general cases when the surface material properties corresponding to the upper and lower faces of the crack are distinct (although the model fully incorporates this

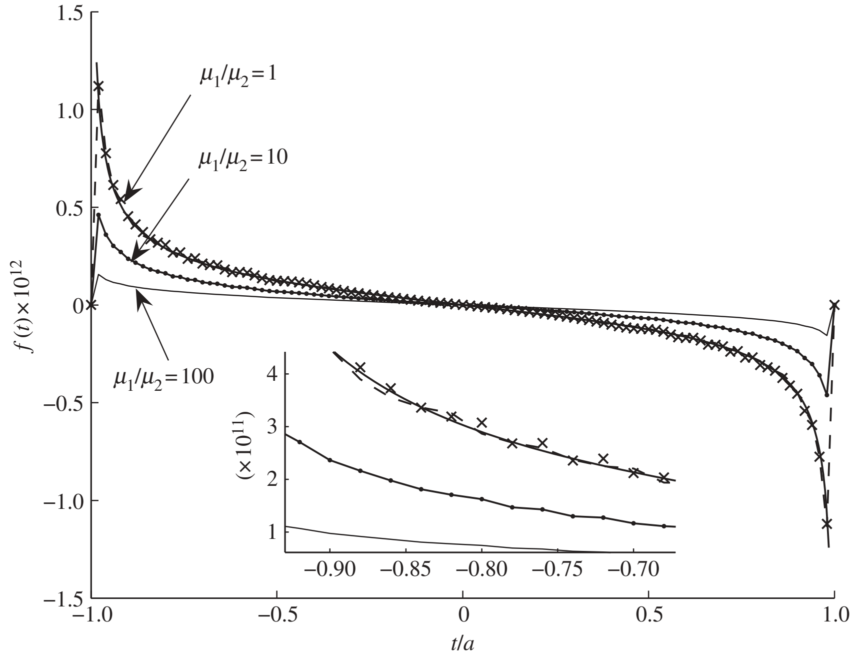

Figure 2. The solutions of $f(t)$ (mode-II). Thick line, classical case (single material); dashed line, with surface effects (single material); crosses, $\mu_1/\mu_2 = 1$; dotted with solid line, $\mu_1/\mu_2 = 10$; thin line, $\mu_1/\mu_2 = 100$.

general case for a range of surface parameters). Instead, we simplify the numerical calculations by assuming that the upper and lower crack faces have identical surface properties, but the bulk materials remain distinct. This is certainly sufficient to illustrate the effect of surface elasticity on the behaviour of this bi-material particularly in the vicinity of the crack.

(*a*) *Functions* $f(t)$, $g(t)$ *and* $\beta(t)$

To verify our model, we consider the special case of a homogeneous material (material properties from the upper and bottom regions coincide) containing a crack and reproduce the classical solutions from LEFM (see Muskhelishvili 1953 or England 1971) followed by the solutions to the corresponding problems which incorporate surface effects (see Kim *et al.* 2010). For a single material crack, we find that

$$f(t) = \Omega'(z)^+ - \Omega'(z)^- = -\frac{\sigma_{xy}^{\infty} t}{\sqrt{t^2 - a^2}}, \quad g(t) = \beta(t) = 0, \ -a < t < a,$$
$$\text{(classical LEFM, mode-II)}$$

and

$$f(t) = \text{non-zero bounded solution (figure 2)}, \ g(t) = \beta(t) = 0,$$
$$\text{(single material crack with surface effects, mode-II).}$$

In the present case, the unknown functions can be obtained subsequently as:

$$f_N(t) = \sum_{m=0}^{N} \frac{1}{\sqrt{1-t^2}} a_m T_m(t),$$

$$g_N(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, b_m U_{m-1}(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, a_m K_m^{II} U_{m-1}(t)$$

and
$$\beta_N(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, C_m U_{m-1}(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, a_m H_m^{II} U_{m-1}(t),$$

and the unknown constants a_m can be determined from equations (3.6) and (3.7). The corresponding solutions for each case are plotted in figures 2–4. The results clearly illustrate that our solutions reduce to the classical results obtained from the literature, as the material properties of the two regions coincide. More specifically, the estimated value of $f(t)$ increases until it reaches the value obtained from the cases corresponding to a crack in a homogeneous material. The solutions of $g(t)$ and $\beta(t)$ decrease to zero, which is compatible with existing results. This also can be directly shown from equations (3.1) and (3.6), i.e. by setting $\mu_1 = \mu_2 = \mu$ and $\kappa_1 = \kappa_2 = \kappa$, we have

$$\left(\frac{1}{\mu + \mu\kappa} - \frac{1}{\mu + \mu\kappa}\right) = A_s = 0, \quad \frac{(\kappa\mu^2 - \kappa\mu^2)}{(\mu + \kappa\mu)(\mu + \kappa\mu)} = C_s = 0,$$

which yields $K_m^{II} = H_m^{II} = 0$. Therefore, we find

$$f_N(t) = \sum_{m=0}^{N} \frac{1}{\sqrt{1-t^2}} a_m T_m(t) \quad \text{(non-zero)},$$

$$g_N(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, b_m U_{m-1}(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, a_m \boldsymbol{K}_m^{II} U_{m-1}(t) = 0$$

and
$$\beta_N(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, C_m U_{m-1}(t) = \sum_{m=0}^{N} \sqrt{1-t^2}\, a_m \boldsymbol{H}_m^{II} U_{m-1}(t) = 0.$$

Similar analyses can be performed for the mode-I case illustrating analogous results. In this case, we have non-zero $g(t)$ and $f(t) = \beta(t) = 0$. In addition, our solutions in figures 2–4 also demonstrate the apparent presence of mixed mode crack tip fields regardless of the type of applied loading.

(b) *Oscillatory singularity*

The solutions of interface crack problems in the context of LEFM show rapid oscillation in both the stress and displacement fields, and indicate the possibility of material interpenetration which is, of course, physically inadmissible (see England 1965; Rice & Sih 1965). The oscillatory behaviour can be avoided either by imposing mathematical restrictions at the crack tips (Atkinson 1977; Comninou 1977) or by considering some special classes of nonlinear materials (see

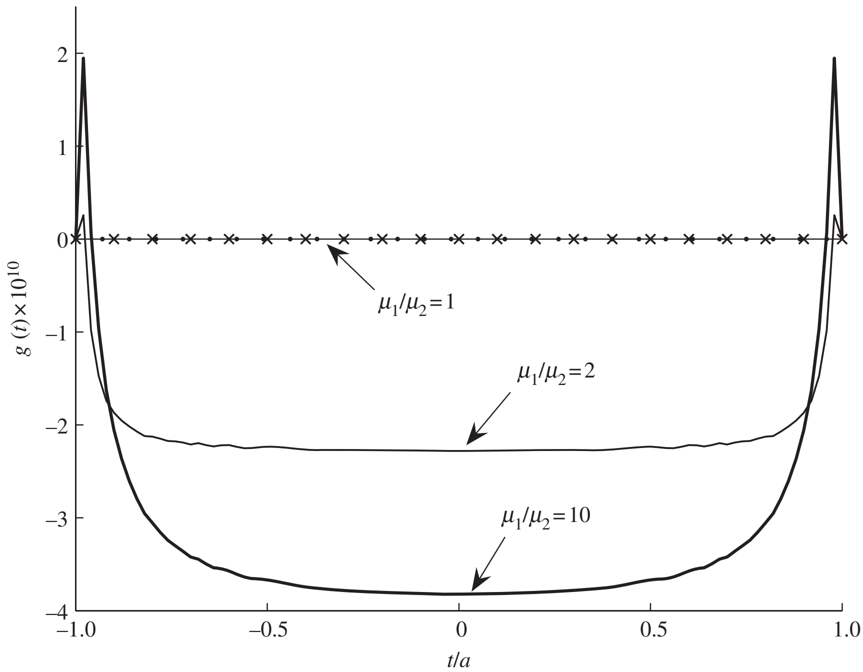

Figure 3. The solutions of $g(t)$ (mode-II). Crosses, classical case (single material); dotted, with surface effects (single material); thin line, $\mu_1/\mu_2 = 1$; thick line, $\mu_1/\mu_2 = 2$; thickest line, $\mu_1/\mu_2 = 10$.

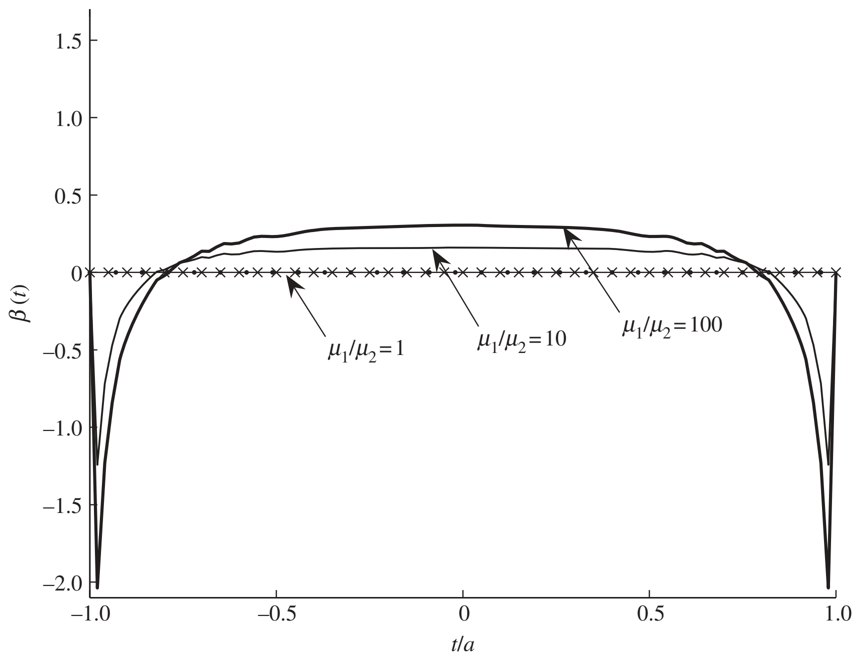

Figure 4. The solutions of $\beta(t)$ (mode-II). Crosses, classical case (single material); circles, with surface effects (single material); thin line, $\mu_1/\mu_2 = 1$; thick line, $\mu_1/\mu_2 = 10$; thickest line, $\mu_1/\mu_2 = 100$.

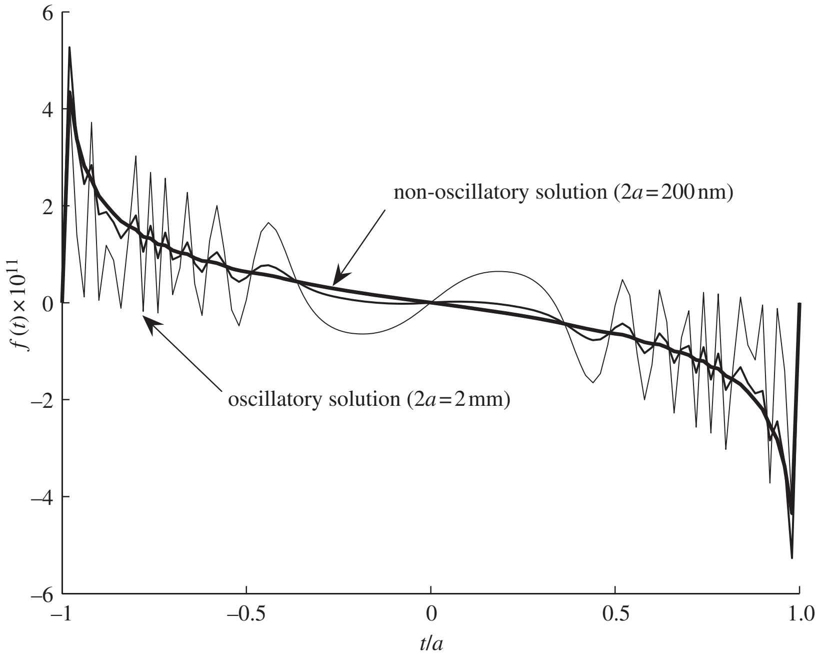

Figure 5. Decaying of oscillatory singularity $f(t)$ (mode-II). Thin line, crack length $2a = 2$ mm; thick line, crack length $2a = 20$ µm; thickest line, crack length, $2a = 200$ nm.

Knowles & Sternberg 1983). The aforementioned results have been the subject of intense discussion in the literature.

In the present study, we have found that the oscillatory singularities can also be eliminated with the incorporation of surface effects. The graphs in figures 5 and 6 clearly indicate that the oscillatory phenomena gradually decays as the effect of surface elasticity increases. In other words, the stress fields obtained from the present solution no longer oscillates. The results hold true for both mode-I and mode-II cases.

Finally, based on the numerical estimations of $g(t)$, $f(t)$, $\alpha(t)$ and $\beta(t)$, the stresses on the real axis can be estimated. More precisely, we have from equations (2.17) and (2.18a) that

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy}) = \left(\frac{1}{\mu_1 + \mu_2\kappa_1} + \frac{1}{\mu_2 + \mu_1\kappa_2}\right)\phi'(z) + \frac{\mu_1 + \mu_2 - \mu_1\mu_2\kappa_1\kappa_2}{(\mu_1 + \mu_2\kappa_1)(\mu_2 + \mu_1\kappa_2)}\theta'(z) \equiv B_s\phi'(z) + X_s\theta'(z). \tag{4.2}$$

In view of equations (2.25) and (2.26), equation (4.2) can be re-written as:

$$(\sigma_{yy} - \mathrm{i}\sigma_{xy}) = \frac{B_s}{2\pi\mathrm{i}}\int_{-1}^{+1}\frac{f(t) + \mathrm{i}g(t)}{t - z}\mathrm{d}t + (\sigma_{yy}^{\infty} - \mathrm{i}\sigma_{xy}^{\infty}) + \frac{X_s}{2\pi\mathrm{i}}\int_{-1}^{+1}\frac{\alpha(t) + \mathrm{i}\beta(t)}{t - z}\,\mathrm{d}t,$$

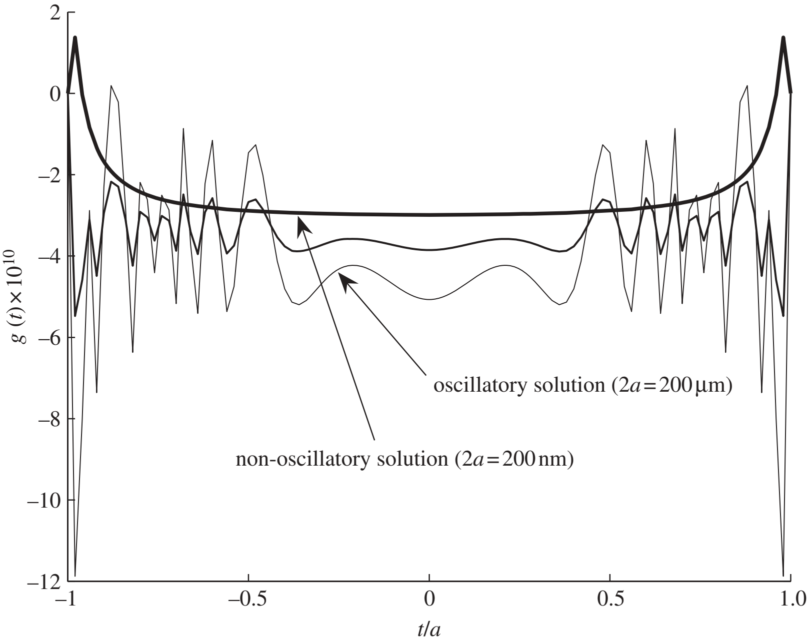

Figure 6. Decaying of oscillatory singularity $g(t)$ (mode-II). Thin line, crack length $2a = 200\,\mu\text{m}$; thick line, crack length $2a = 20\,\mu\text{m}$; thickest line, crack length, $2a = 200\,\text{nm}$.

By separating real and imaginary parts, the above yields

$$\sigma_{yy} = \frac{B_s}{2\pi}\int_{-1}^{+1}\frac{g(t)}{t-z}\,\mathrm{d}t + \frac{X_s}{2\pi}\int_{-1}^{+1}\frac{\beta(t)}{t-z}\,\mathrm{d}t + \sigma_{yy}^{\infty} \tag{4.3a}$$

and

$$\sigma_{xy} = \frac{B_s}{2\pi}\int_{-1}^{+1}\frac{f(t)}{t-z}\,\mathrm{d}t + \frac{X_s}{2\pi}\int_{-1}^{+1}\frac{\alpha(t)}{t-z}\,\mathrm{d}t + \sigma_{xy}^{\infty}. \tag{4.3b}$$

Therefore, the stress field can be completely determined with given $g(t)$, $f(t)$, $\alpha(t)$ and $\beta(t)$. For example, in the mode-II case, we find from equations (4.3*a*) and (4.3*b*) that

$$\sigma_{yy} = \frac{B_s}{2\pi}\int_{-1}^{+1}\frac{g(t)}{t-z}\,\mathrm{d}t + \frac{X_s}{2\pi}\int_{-1}^{+1}\frac{\beta(t)}{t-z}\,\mathrm{d}t, \sigma_{xy} = \frac{B_s}{2\pi}\int_{-1}^{+1}\frac{f(t)}{t-z}\,\mathrm{d}t + \sigma_{xy}^{\infty}.$$

Examples of estimated stresses are illustrated in figure 7 with respect to varying surface parameters. The results recognize the fact that surface effects, acting basically in the same way as a reinforcement, can effectively reduce the corresponding stress fields (materials get stiffer as surface effects increase). In addition, since the surface parameters *Se* are controlled by variations of the crack size, our results also indicate that the corresponding stresses are strongly dependent on crack size. In fact, size dependency of stress fields in materials is

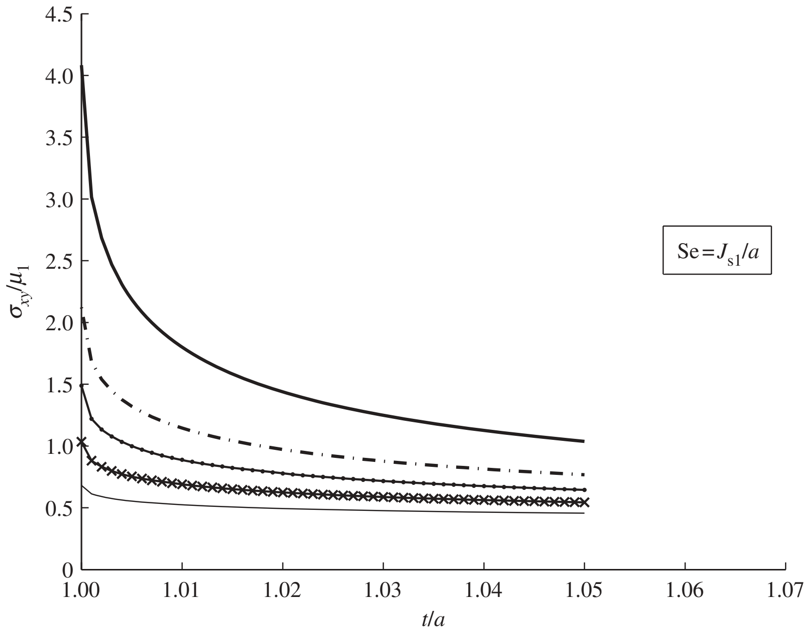

Figure 7. Stress distribution with respect to surface parameter (Se) (mode-II), when ($\sigma_{xy}^{\infty}\mu_1 = 0.37$). Thick line, Se = 16×10^8; dashed-dotted line, Se = 8×10^9; dotted with solid line, Se = 16×10^9; crosses with solid line, Se = 32×10^9; thin line, Se = 8×10^{10}.

the dominant phenomena in the deformations of small-scale structures such as the nano-inhomogeneity, nano-composite materials and dislocations in small-scale structures (see Sharma & Ganti 2004; Tian & Rajapakse 2007).

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