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Alternative derivation of the Feigel effect and call for its experimental verification

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A recent theory by Feigel predicts the finite transfer of momentum from a quantum vacuum to a fluid placed in strong perpendicular electrical and magnetic fields. The momentum transfer arises because of the optically anisotropic magnetoelectric response induced in the fluid by the fields. After summarizing Feigel's original assumptions and derivation (corrected of trivial mistakes), we rederive the same result by a simpler route, validating Feigel's semiclassical approach. We then derive the stress exerted by the vacuum on the fluid that, if the Feigel hypothesis is correct, should induce a Poiseuille flow in a tube with maximum speed at $\approx 100\,\mu\mathrm{m\,s^{-1}}$ (2000 times larger than Feigel's original prediction). An experiment is suggested to test this prediction for an organometallic fluid in a tube passing through the bore of a high-strength magnet. The predicted flow can be measured directly by tracking microscopy or indirectly by measuring the flow rate ($\approx 1\,\mathrm{ml\,min^{-1}}$) corresponding to the Poiseuille flow. A second experiment is also proposed, whereby a 'vacuum radiometer' is used to test a recent prediction that the net force on a magnetoelectric slab in the vacuum should be zero.

Keywords: Feigel effect; quantum vacuum; magnetoelectric; strong magnetic fields; particle-tracking velocimetry; vacuum radiometer

1. Introduction

It is well known that quantum vacuum fluctuations can transfer momentum to macroscopic matter. This usually results from a modification of the spectrum of allowed vacuum modes by symmetric boundaries. For example, in the Casimir effect (Casimir 1948; Milonni 1994; Lamoreaux 1997, 2005), two conducting plates are placed parallel to each other in a vacuum. The plates reduce the allowed number of modes between them, which causes the total pressure (owing to the momentum transfer of the modes) to be smaller between the plates than outside, resulting in a net attraction: the Casimir force (Milonni *et al.* 1988).

However, the vacuum does not usually transfer its momentum to matter in the absence of boundaries: because its fluctuation spectrum is isotropic, the expectation value of the momentum density of the vacuum in free space or

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isotropic media is zero (Milonni 1994). In this study, we revisit a theoretical argument recently proposed by Feigel (2004), which claims that it is possible to transfer vacuum momentum to an isolated region of dielectric liquid if it is placed in perpendicularly crossed electrical and magnetic fields. Critical to Feigel's argument is the assumption that vacuum fluctuations see the dielectric in crossed fields as an anisotropic magnetoelectric medium. This implies that vacuum modes will propagate faster (have greater momentum) in one direction than in the opposite direction, causing a net momentum transfer to the dielectric liquid.

In going through Feigel's argument, we have corrected trivial errors in the original derivation. The corrected expressions agree with our own more direct derivation of Feigel's result. In the original study, Feigel proposed a prediction for the speed acquired by an effective magnetoelectric fluid as a result of the momentum transfer from the vacuum. The prediction was, however, predicated on an ideal fluid, and thus not realistically testable. We derive here the flow of a real, viscous fluid driven into motion by the stress on the fluid caused by the vacuum. An experimental test of this improved prediction is proposed. Indeed, this study is written in the hope that our revised predictions will stimulate experiments to try and measure Feigel's original effect and related predictions proposed recently. Since the publication of Feigel's result, several theoretical works have emerged, which have been very enthusiastic about Feigel's idea (van Tiggelen *et al.* 2005, 2006; Shen *et al.* 2006; Birkeland & Brevik 2007; Obukhov & Hehl 2008). Some have, however, questioned the soundness of the original argument stating that no unbounded macroscopic Feigel effect should exist if proper regularization is applied to the momentum integral (van Tiggelen *et al.* 2005, 2006). Microscopically, however, recent work suggests that a properly regularized Feigel effect could exist (Kawka & van Tiggelen 2010). Most alternative macroscopic theories consider the momentum from the vacuum for magnetoelectric fluids (Birkeland & Brevik 2007) and samples of magnetoelectric materials (van Tiggelen *et al.* 2006) confined in parallel plate geometries, like the Casimir effect. The latter work predicts an unmeasurably small linear momentum transfer (van Tiggelen *et al.* 2006). Interestingly, following a semiclassical approach similar to Feigel, Obukhov & Hehl (2008) found that there is no net force on a magnetoelectric slab of finite thickness in the vacuum. Feigel (2009) has also recently considered the interesting possibility of constructing 'quantum wheels' using magnetoelectric nanoparticles. No experiments have been carried out to test Obukhov & Hehl's null prediction, which implies that Feigel's wheels should not function. More surprisingly, Feigel's original theory, which remains the one predicting the largest effect, remains untested. Experimental tests for the effects described are suggested at the end of the study, but we leave the field open to imaginative experimentalists with access to the strong fields or magnetoelectric materials required.

The paper is organized as follows: in §2, we present a summary of Feigel's original derivation and underlying assumptions. In §3, our quicker route to Feigel's result is then presented. In §4, a new expression for vacuum stress is derived and a revised prediction of Feigel's effect is evaluated and discussed, together with Obukhov & Hehl's null prediction. In §5, we make some suggestions for experimentally testing these predictions. Finally, in §6, conclusions are drawn from the preceding analysis and a few final comments are made.

2. Feigel's semiclassical model

We summarize here Feigel's semiclassical derivation and assumptions (highlighted in italics). Slight inaccuracies in the derivation of two key results have been corrected, so these expressions differ from those of Feigel's original paper (in appendix A, the original results are compared with our corrected ones).

(*a*) *Physical system and initial conditions*

Feigel considers the following situation: a region of a dielectric fluid far from the boundaries of its container is initially at rest ($t=0$). Subsequently, strong electrical and magnetic fields crossed at right angles to each other are applied to the region. As the fields reach their constant final values, $\mathbf{E}_{\text{ext}}$ and $\mathbf{B}_{\text{ext}}$ for electrical and magnetic fields, respectively, the fluid is accelerated by the Lorentz forces ($F_{\text{Lorentz}} \propto \partial_t(\mathbf{E}_{\text{ext}} \times \mathbf{B}_{\text{ext}})$) to a final velocity $\mathbf{v}$.

(*b*) *Brief summary of Feigel's derivation and fundamental assumptions*

1. *Feigel assumes that the portion of dielectric fluid under consideration is to a good approximation ideal* (*inviscid*), *incompressible, homogeneous and not acted upon by external stresses or body forces.*
2. As the fields become steady, they have a maximum momentum, the opposite of which is transferred to the fluid by momentum conservation, as the combined system conserves its initial zero net momentum. Feigel derives the conservation law from the relativistically transformed Lagrangian of the moving dielectric. This provides the 'classical' fluid momentum (ignoring terms of order $(v/c)^2$)

$$\rho\mathbf{v} = \frac{\epsilon\mu - 1}{4\pi\mu c}(\mathbf{E}\times\mathbf{B}). \tag{2.1}$$

   It should be noted that *this conservation law holds only if the region under consideration is stress free*, as mentioned in point 1. Otherwise, momentum is not conserved.
3. It has been shown that the optical response of a dielectric in crossed $\mathbf{E}$ and $\mathbf{B}$ fields is the same as that of a magnetoelectric material (Roth & Rikken 2002). *Feigel assumes that electromagnetic modes of the vacuum will also 'see' a magnetoelectric.* The Lagrangian density of a magnetoelectric material is derived by Feigel using relativistic transformation of the Lagrangian of a magnetoelectric (see equation (B 7) of appendix B) in the small speed limit (ignoring, as above, terms of order $(v/c)^2$). The Euler–Lagrange equation of this Lagrangian then provides a momentum conservation law,

$$\rho\mathbf{v} = \frac{1}{4\pi}\left(\frac{\epsilon\mu - 1}{\mu c}\mathbf{E}\times\mathbf{B} + \frac{1}{\mu c}\mathbf{E}\times(\hat{\chi}^{\mathrm{T}}\mathbf{E}) + \frac{1}{\mu c}(\hat{\chi}\mathbf{B})\times\mathbf{B}\right), \tag{2.2}$$

   which allows to evaluate the momentum associated with the vacuum. We note that using the inverted constitutive equations (B 3) and (B 4), we can rewrite equation (2.2) as

$$\rho\mathbf{v} = \frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B} - \mathbf{E}\times\mathbf{H}\right). \tag{2.3}$$

We will consider the appropriateness of using this momentum density to evaluate the fluid momentum in §6.

4. The field modes and refractive indices for electromagnetic waves in a magnetoelectric are given by (taking the optical axis along the z-direction, $\mathbf{e}_3$; see appendix A),

$$[\mathbf{E}_{\pm\mathbf{k}1}, \mathbf{B}_{\pm\mathbf{k}1}] = E_{\pm\mathbf{k}1}[\mathbf{e}_1, n_{\pm\mathbf{k}1}\mathbf{e}_2] \quad \text{and} \quad [\mathbf{E}_{\pm\mathbf{k}2}, \mathbf{B}_{\pm\mathbf{k}2}] = E_{\pm\mathbf{k}2}[\mathbf{e}_2, -n_{\pm\mathbf{k}2}\mathbf{e}_1],$$

where $E_{\pm\mathbf{k}\lambda} = E_{0k}\mathrm{e}^{\mathrm{i}(k_\lambda z-\omega t)}$, for each polarization, $\lambda = 1, 2$, and

$$n_{\pm\mathbf{k},1} = \pm n_0 + \chi_{xy} \quad \text{and} \quad n_{\pm\mathbf{k},2} = \pm n_0 - \chi_{yx},$$

where χ_{xy} and χ_{yx} are the magnetoelectric susceptibilities responsible for the optical anisotropy of the magnetoelectric (see appendix A). Feigel substitutes the above modes and refractive indices into equation (2.2) to evaluate the time-averaged (denoted by an overbar) momentum flux for a mode in the z-direction,

$$\overline{\rho v}_k = 2\Delta\chi\frac{1}{c}\frac{\epsilon E_{0k}^2}{4\pi}, \tag{2.4}$$

where $\Delta\chi \equiv \chi_{xy} - \chi_{yx}$.

5. Feigel then replaces the electrical field with its operator and *evaluates the electromagnetic vacuum energy density expectation value*,

$$\left\langle 0 \left| \frac{\epsilon\hat{E}_{0k}^2}{4\pi} \right| 0 \right\rangle = \frac{1}{V}\frac{1}{2}\hbar\omega,$$

to obtain the vacuum momentum density per mode from equation (2.4),

$$g_{0k} \equiv \langle 0|\overline{\rho v}_k|0\rangle = \frac{1}{V}\Delta\chi\hbar k_0, \tag{2.5}$$

where $k_0 \equiv \omega/\mathrm{c}$.

6. Summing over all modes, the total momentum density in the z-direction is then

$$g_0 \equiv \sum_k g_{0k} = \frac{1}{V}\Delta\chi\hbar\sum_{k_0} k_0 \to \frac{1}{2\pi^2}\Delta\chi\hbar\int_0^\infty k_0^3\,\mathrm{d}k_0, \tag{2.6}$$

where the last step involves the standard replacement $\sum_{k_0} \to V/(8\pi^3)\int \mathrm{d}^3k_0$ (see §3).

7. The integral in equation (2.6) is divergent. Feigel makes the *crucial assumption that vacuum modes with frequency greater than the dielectric's 'cut-off frequency'*, ω_c (*the frequency above which the dielectric's molecular polarizability vanishes*) *do not interact with it.* (Implicit in the derivation is also the assumption that absorption and dispersion are not significant, i.e. that for $\omega < \omega_\mathrm{c}$, the permittivity and magnetoelectric susceptibility of the dielectric as seen by vacuum modes do not change appreciably

with frequency.) This allows us to evaluate a finite value for the momentum density (2.6),

$$g_0 = \rho v_{\mathrm{vac}} = \frac{1}{8\pi^2}\Delta\chi\frac{\hbar\omega_{\mathrm{c}}^4}{c^4}. \tag{2.7}$$

From which, dividing by the fluid density ρ, Feigel obtained an estimate for the vacuum contribution to the fluid speed, v_{vac}. We iterate that the above expression is different from that obtained by Feigel owing to some trivial errors in the original derivation (see appendix B).

8. The magnitude of the corresponding 'classical' contribution of the dielectric fluid's speed is given by equation (2.1),

$$v_{\mathrm{class}} = \frac{1}{\rho}\frac{\epsilon\mu - 1}{4\pi\mu c}\mathbf{E}_{\mathrm{ext}}\mathbf{B}_{\mathrm{ext}}. \tag{2.8}$$

The relative magnitude of quantum vacuum and classical contributions will be discussed in §4.

3. Simpler derivation of the vacuum contribution to magnetoelectric momentum density

Here, we present our alternative route to Feigel's result, equivalent to steps 1–5, but very much quicker. From quantum electrodynamics, the expectation value for the momentum density $\hat{\mathbf{g}}$ of the vacuum is given by Milonni (1994),

$$\mathbf{g}_0 \equiv \langle 0|\hat{\mathbf{g}}|0\rangle = \frac{1}{V}\sum_{\mathbf{k}\lambda}\frac{1}{2}\hbar\mathbf{k}, \tag{3.1}$$

where V is a sample volume of the medium under consideration and $\mathbf{k}$ is the wavevector of each vacuum mode *in this medium* with possible polarization states $\lambda = 1, 2$. Next, we assume with Feigel that the vacuum experiences the same birefringence in a magnetoelectric medium as light does. In the direction $\mathbf{e}_z$ parallel to the optical axis, the medium has the following dispersion relation:

$$\mathbf{k}\cdot\mathbf{e}_z = k_0 n_{\mathbf{k}\lambda}, \tag{3.2}$$

where $n_{\mathbf{k}\lambda}$ (given below) are the refractive indices parallel to the optical axis and we define $k_0 = \omega/c$. On the other hand, in directions perpendicular to the optical axis, the medium is isotropic, so that contributions to equation (3.1) vanish by symmetry and, substituting for equation (3.2) gives

$$\mathbf{g}_0 = \frac{1}{V}\frac{1}{2}\hbar\sum_{k_0}\sum_{\lambda} k_0(n_{+\mathbf{k}\lambda} + n_{-\mathbf{k}\lambda})\mathbf{e}_z, \tag{3.3}$$

where the sum has been expanded in terms of the contributions by modes for each direction ($\pm$) of travel along $\mathbf{e}_z$. The expressions for the anisotropic indices $n_{\pm\mathbf{k}\lambda}$, derived in appendix A, are

$$n_{\pm\mathbf{k},1} = \pm n_0 + \chi_{xy} \quad \text{and} \quad n_{\pm\mathbf{k},2} = \pm n_0 - \chi_{yx}, \tag{3.4}$$

where we recall $|\chi_{ij}| \ll |n_0|$ so, e.g. $n_{\mathbf{k},1} > 0$ and $n_{-\mathbf{k},1} < 0$. Substituting these expressions into equation (3.3) and summing over all polarizations, λ gives (considering only the magnitude of $\mathbf{g}_0$)

$$g_0 = \frac{1}{V} \Delta\chi\hbar \sum_{k_0} k_0, \tag{3.5}$$

where we recall $\Delta\chi = \chi_{xy} - \chi_{yx}$. Next, making the standard replacement $\sum_{k_0} \to V/(8\pi^3) \int \mathrm{d}^3 k_0$, we find

$$g_0 = \frac{1}{2\pi^2} \Delta\chi\hbar \int_0^\infty k_0^3 \mathrm{d}k_0. \tag{3.6}$$

These integration limits assume that modes of all wavelengths contribute to the momentum density, as they would in free space, so g_0 is divergent. However, vacuum electromagnetic modes with very small wavelengths are not expected to interact with the macroscopic electromagnetic properties of a material medium. The choice of a reasonable value for the cut-off will be discussed in §4. Here, with Feigel, we simply assume that it is reasonable to approximate equation (3.6) using an upper cut-off on the wavenumber $k_c = \omega_c / c$ to finally obtain

$$g_0 = \frac{1}{8\pi^2} \Delta\chi \frac{\hbar\omega_c^4}{c^4}. \tag{3.7}$$

Equation (3.7) represents the vacuum contribution to the momentum density of a magnetoelectric. In media without the special symmetry of magnetoelectrics, the refractive indices obey $n_{+\mathbf{k}\lambda} = -n_{-\mathbf{k}\lambda}$, so that $\Delta\chi = 0$, and there is no transfer of momentum, as expected. Our alternative derivation provides a concise route to Feigel's result, which coincides with equation (2.7). It is equivalent and validates his semiclassical approach (once trivial errors in the original derivation are corrected, see appendix A).

4. Vacuum stress and realistic predictions

As mentioned in point 4 of §2, the original prediction of the Feigel effect was obtained assuming conservation of momentum to obtain the vacuum contribution to the speed of a dielectric fluid placed in crossed fields (an effective magnetoelectric) from equation (3.7) (Feigel 2004). Feigel's fluid was ideal and far from boundaries, making the measurement of the vacuum speed as originally predicted a utopian pursuit. We show here how more realistic predictions for the original experiments can be simply obtained and propose a new experimental test. These improved predictions are based on the fact that the transfer of momentum from the vacuum to a magnetoelectric results in a stress.

For the case of an effective magnetoelectric fluid, an expression for the vacuum stress can be derived by applying kinetic theory to a gas of vacuum modes (virtual photons) of momentum $\frac{1}{2}\hbar\mathbf{k}$ travelling in the fluid. Optical anisotropy implies that the net momentum

$$\Delta\mathbf{p}_\mathbf{k} = \frac{1}{2}\hbar k_0 \sum_\lambda (n_{+\mathbf{k}\lambda} + n_{-\mathbf{k}\lambda})\mathbf{e}_z \tag{4.1}$$

is transferred by counter-propagating vacuum modes across a surface in the fluid of area A and normal to $\mathbf{e}_z$. The corresponding stress on the fluid is the time rate of change of this momentum transfer per unit area: $\mathit{\Pi}_{\mathbf{k}} = (1/A)(\Delta \mathbf{p}_{\mathbf{k}}/\Delta t)$. Modes crossing A in the interval Δt are recruited from a slice of fluid of thickness $\Delta z = c[(1/n_{+\mathbf{k}\lambda}) - (1/n_{-\mathbf{k}\lambda})]\Delta t$ (recall $n_{-\mathbf{k}\lambda} < 0$), where c is the speed of light *in vacuo*. Thus, the magnitude of the net stress in the z-direction owing to a mode pair is given by

$$\mathit{\Pi}_{\mathbf{k}} = \frac{1}{V}\frac{1}{2}\hbar k_0 c \sum_{\lambda} \frac{n_{-\mathbf{k}\lambda}^2 - n_{+\mathbf{k}\lambda}^2}{n_{+\mathbf{k}\lambda} n_{-\mathbf{k}\lambda}}, \tag{4.2}$$

where $V = A\Delta z$. Substituting the refractive indices (3.4) into equation (4.2) (neglecting terms of order $\|\hat{\chi}\|^2$) and summing over k_0 gives

$$\mathit{\Pi}_0 = 2\frac{c}{n_0}\frac{1}{V}\Delta\chi\hbar \sum_{k_0} k_0. \tag{4.3}$$

Comparing this expression with equation (3.5), we see that $\mathit{\Pi}_0 = 2g_0 c/n_0$, as one would expect (Loudon *et al.* 2005). We can then use equation (3.7) to write

$$\mathit{\Pi}_0 = 4\pi^2 \Delta\chi\hbar \frac{c}{n_0}\frac{1}{\lambda_c^4}, \tag{4.4}$$

where we recall that $n_0 = \sqrt{\epsilon\mu}$ is the fluid's isotropic refractive index and note that equation (4.4) has been re-written in terms of the cut-off wavelength λ_c in view of the evaluations here below. Equation (4.4) is the magnitude of the stress (acting in the z-direction) exerted by the vacuum on an effective magnetoelectric fluid. In §4*a*, we apply this expression to calculate the speed of a dielectric fluid in a tube with a portion of its length placed in strong crossed fields (a realistic version of Feigel's original scenario). Note that an equivalent result for the vacuum stress could also have been obtained semiclassically by evaluating the contribution from counter-propagating modes to the electromagnetic stress tensor: $\mathit{\Pi}_{\mathbf{k}}^{zz} \sim \epsilon E_0^2 (n_{-\mathbf{k}\lambda}^2 - n_{\mathbf{k}\lambda}^2)$.

Indeed, it was by using such a semiclassical approach applied to a magnetoelectric slab of finite thickness in a vacuum which Obukhov & Hehl (2008) recently predicted that the net stress on the slab should be

$$\mathit{\Pi}_0 = 0. \tag{4.5}$$

That is, the vacuum exerts no net force on a magnetoelectric slab in spite of its anisotropy! In §4*b*, we propose to test this surprising prediction with an experiment where the angular drift speed of a 'vacuum radiometer' with paddles made of magnetoelectric materials is measured.

(*a*) *Dielectric fluid in a tube in crossed fields*

We consider Feigel's original situation of a dielectric fluid placed in perpendicular electrical and magnetic fields. In our case, however, the fluid is realistically contained in a tube, a section of which is exposed to the fields (figure 1*a*). When the fields are switched on, the Lorentz and ponderomotive forces act on the fluid. The fields induce a magnetoelectric susceptibility in the fluid where they act so, according to Feigel's theory, the vacuum exerts a

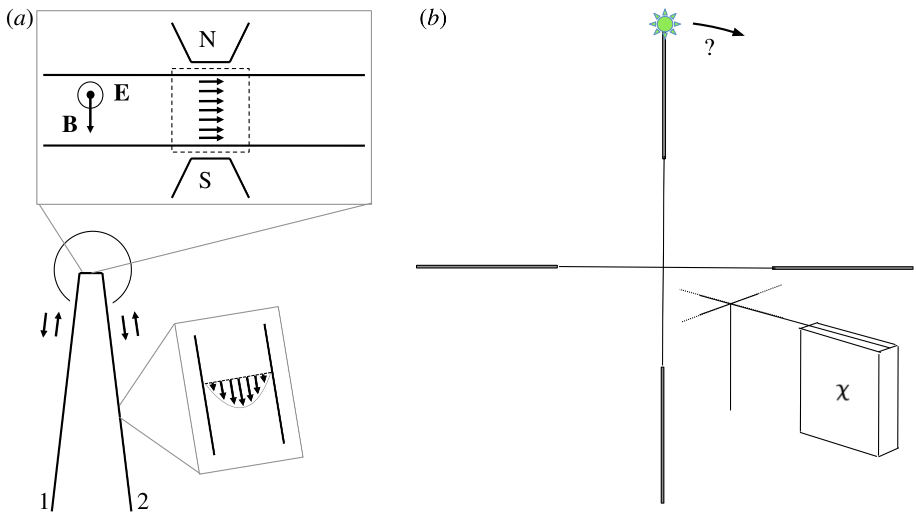

Figure 1. Schematic of the proposed experimental arrangements for testing the Feigel effect. (*a*) A tube is placed in the magnet bore where it is subjected to a strong magnetic field and a perpendicular electrical field. The channel enters and exits the magnet via a conical aperture, and flow is measured down stream by the elution or tracking the drift of colloidal tracers. Reversing the electrical field allows us to test the predicted reversal of the flow and to eliminate unwanted drifts. (*b*) A 'vacuum radiometer' with magnetoelectric panes should rotate if the vacuum exerts a pressure on them. According to a prediction by Obukhov & Hehl (2008), this should not happen. A light-emitting diode laser on the top of the slab allows us to track any displacement of the radiometer in a darkened container (to avoid it being set in motion by ambient light). (Online version in colour.)

stress in that region. The fluid in the tube thus obeys the following Navier–Stokes equation:

$$\rho_{\mathrm{M}} \frac{\partial \mathbf{v}}{\partial t} = (\mathbf{P} \cdot \nabla)\mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \times \mathbf{B} + \eta \nabla^2 \mathbf{v} + \nabla \cdot \boldsymbol{\Pi}_{\mathbf{0}}, \tag{4.6}$$

where $\mathbf{v}$ is the flow speed, $\mathbf{E}$ and $\mathbf{B}$ are the imposed fields, $\mathbf{P} = (\epsilon - 1)\mathbf{E} + \hat{\chi}\mathbf{B}/\mu + o(\|\hat{\chi}\|^2)$ is the electrical polarization of the fluid, ρ_{M} and η are the density and dynamic viscosity of the fluid, respectively, and $\boldsymbol{\Pi}_{\mathbf{0}}$ is the stress on the fluid owing to the vacuum. The fluid is assumed incompressible, so $\nabla \cdot \mathbf{v} = 0$ and we note that the advection term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ vanishes in the cylindrical geometry of a tube. When both fields and the flow are in a steady state, ignoring ponderomotive forces owing to edge effects, equation (4.6) reduces to $\eta \nabla^2 \mathbf{v} + \nabla \cdot \boldsymbol{\Pi}_{\mathbf{0}} = 0$. We thus expect a standard Poiseuille flow solution, with a maximum speed, $U_{\max} \equiv \max |\mathbf{v}|$, at the centre of the pipe given by Brody *et al.* (1996),

$$U_{\max} = \frac{\Pi_0 a^2}{4\eta L}, \tag{4.7}$$

where Π_0 is the magnitude of the vacuum stress, a and L are the tube diameter and length, respectively, and η is the dynamic viscosity of the fluid, as above.

To estimate $U_{\max}$ we consider, like Feigel, the same organometallic liquids and the same magnitudes of crossed $\mathbf{E}$ and $\mathbf{B}$ fields used in the experiments by Roth & Rikken (2002). In particular, we focus on methylcyclopentadienyl

Table 1. Values of the parameters used in the evaluation of the vacuum and classical contributions to the velocity of an effective magnetoelectric organometallic fluid.

Planck's angular constant	$\hbar$	1.05×10^{-34} J s
speed of light	c	2.99×10^{8} m s^{-1}
permittivity of free space	ϵ_0	8.85×10^{-12} F m^{-1}
external electrical field	$\mathbf{E}_{\mathrm{ext}}$	10^5 V m^{-1}
external magnetic field	$\mathbf{B}_{\mathrm{ext}}$	17 T
magnetoelectric susceptibility difference (632.8 nm)	$\Delta\chi$	10^{-11} (Roth & Rikken 2002)
dynamic viscosity (25°C)	η	4.5×10^{-3} Pa s (YX-KR 2011)
density (25°C)	ρ_{M}	1.38 g cm^{-3} (Sigma-Aldrich 2011)
refractive index (589.3 nm, 20°C)	n_0	1.58 (Sigma-Aldrich 2011)
cut-off wavelength (*cis*-Pt-DEBP)	λ_{c}	4 nm (Matassa *et al.* 2010)

manganese tricarbonyl (MMT), whose relevant properties are shown in table 1. The magnitude of the vacuum stress is given by equation (4.4) and is seen to depend sensitively on the cut-off wavelength λ_{c}. Feigel used $\lambda_{\mathrm{c}}\approx 0.1$ nm, taking intermolecular distances as a cut-off. However, at such scales, vacuum modes will not interact with a sufficiently large number of molecules to experience magnetoelectric anisotropy. At larger distances, on the other hand, modes can interact with the collective effect of an assembly of molecules. The radial distribution function (RDF) for a fluid provides a good measure of the distance beyond which modes see a smooth electromagnetic landscape. The RDF for MMT has not been measured to the best of our knowledge, but a recent X-ray diffraction study has measured it for the powder of a similar organometallic compound, *cis*-Pt-DEBP (Matassa *et al.* 2010). In the study, it was found that the RDF becomes flat (uniform density) for distances approximately greater than 4 nm, which we assume as the value of the cut-off λ_{c}. The difference in susceptibilities $\Delta\chi$ is estimated approximately from the value of the birefringence measured by the Roth and Rikken experiment (Roth & Rikken 2002). Using equation (3.4), we see that this birefringence is $\Delta n\equiv\chi_{xy}+\chi_{yx}$, where Roth and Rikken measured $\Delta n\approx10^{-11}$ for the applied fields shown in table 1. With Feigel, we approximate $\Delta\chi\approx\Delta n$. Hence, using the parameters of table 1, equation (4.4) gives a vacuum stress of $\Pi_0=0.03$ Pa, which using equation (4.7), for a tube with $a=1$ mm and $L=2$ m in equation (4.7), implies a flow with maximum speed

$$U_{\mathrm{max}}=100\ \mu\mathrm{m\,s^{-1}}. \tag{4.8}$$

This prediction for the flow speed is 2000 times larger than Feigel's original prediction. The corresponding 'classical' contribution to the velocity is negligibly smaller, 7.5 nm s^{-1}, as can be seen from equation (2.8) using the parameters in table 1. As a matter of interest, we note that equation (2.8) follows from equation (4.6) when ponderomotive, vacuum and viscous stress contributions are neglected, so that the classical momentum is conserved. Inclusion of these realistic contributions, however, shows that the classical contribution, which we recall is due to Lorentz forces caused by polarization currents, is transient, and so it is not only negligible in magnitude, but also vanishes at long times when fields are steady. It will not contribute significantly to the steady flow predicted above.

(*b*) *A vacuum radiometer*

Another way to test the reality of vacuum momentum might be to use naturally magnetoelectric compounds. In the organometallic fluids just discussed, it is the high fields that induce magnetoelectric anisotropy in the fluid microstructure. However, this anisotropy, and the resulting birefringence, can also arise in solids whose structures support both spontaneous polarization and magnetization, breaking both time and space inversion symmetries (Figotin & Vitebsky 2001). By virtue of this birefringence, a slab of such materials in the vacuum should acquire momentum from the vacuum. Now consider an arrangement analogous to a Crookes radiometer (Crookes 1876; Woodruff 1968), but with the vacuum pressure driving rotation as opposed to temperature gradients: a mill consisting of square panes made of thin magnetoelectric slabs joined together with rods hinged on a low-friction axle (figure 1*b*). If each pane has area A and distance l from the centre of rotation, then a vacuum stress of magnitude Π_0 normal to each pane causes a torque $\tau_0 = \Pi_0 A l$. The equation of motion of the radiometer mill is then

$$I\dot{\omega} = -\gamma\omega + \tau_0, \tag{4.9}$$

where γ is the frictional damping constant and $I = \rho_s A \delta l^2$ is the moment of inertia, where ρ_s is the slab density and δ its thickness. Integrating equation (4.9), we find

$$\omega(t) = \omega_\infty(1 - e^{-t/t_c}), \tag{4.10}$$

where $\omega_\infty \equiv \tau_0/\gamma$ is the terminal angular speed and $t_c = I/\gamma$ is the characteristic time for the approach to this speed. Assuming that the prediction of Obukhov & Hehl (2008) also applies to slabs of finite extent (i.e. the contributions of any stresses at the edge of the slab are negligible), we then expect

$$\omega(t) = 0. \tag{4.11}$$

The radiometer should not turn. If in an experiment it did actually turn, then measuring its angular drift (see below) would allow us to estimate the vacuum stress from

$$\Pi_0 = \frac{\gamma\omega_\infty}{Al}. \tag{4.12}$$

We would expect such a stress to scale with $\Delta\chi$, as in equation (4.4).

5. Possible experimental tests

If the Feigel effect has the magnitude we have calculated, then it should be possible to test the prediction for a dielectric fluid's velocity using current experimental techniques. Very recently, a 17 T magnet with temperature control in the range 1.6–300 K and conical (±10°) access to its bore has been built (Holmes *et al.* 2010). The magnet was designed for small-angle neutron or X-ray scattering experiments, but it could also be used to test our prediction (4.8). Organometallic fluids would be placed in a tube arrangement going in and out of the magnet; a short portion of the tube would reside inside the magnet bore and would be fitted with electrodes to generate the required $10^5\,\mathrm{V\,m^{-1}}$ electrical fields. A schematic of the set-up is shown in figure 1*a*. The magnet bore would have to be large enough to accommodate the tube and electrodes. In addition, fields need to be uniform over a channel and kept gradient free to a good approximation. The 17 T magnet described has 0.1 per cent uniformity over 1 cm and a bore diameter approximately less than 4 cm (A. Holmes 2010, personal communication), so it would be reasonable to use tubes with millimetre-sized diameters, as assumed in §4(a). Our prediction is that the vacuum should generate a Poiseuille flow with a maximum speed of $100\,\mu\mathrm{m\,s}^{-1}$, as obtained from equation (4.7). The corresponding flow rate, Φ, is given by

$$\Phi = \frac{\pi a^2 U_{\max}}{2} \approx 1\,\mathrm{ml\ min}^{-1}, \tag{5.1}$$

so that, supposing the fields are as shown in figure 1a, experimenters should be able to collect 1 ml of fluid 1 minute after opening a tap at 2 (provided the fields are in steady state). Reversing one of the fields, the same amount should instead be accumulated at 1 (this should also allow us to subtract out any systematic variations). The MMT is transparent, so, if the estimate (4.7) is incorrect but a finite vacuum-induced flow still exists, it might also be possible to place a stripped down (and minimally magnetic!) microscope with a camera downstream to perform particle velocimetry on colloidal tracers. The small flows inside the tube could then be measured far enough away from the magnet to avoid perturbations by large stray fields on the velocimetry apparatus. The mean $\langle X\rangle$ and mean-squared $\langle X^2\rangle$ displacements of these tracers could be measured as a function of time, T,

$$\langle X\rangle = U_0 T \tag{5.2}$$

and

$$\langle X^2\rangle = 4DT + U_0^2 T^2. \tag{5.3}$$

Integrating measurements over a long enough time and fitting equations (5.2) and (5.3) to the data should allow us to establish whether there is a non-zero vacuum drift U_0 in the fluid and what its magnitude may be. From equation (5.3), any vacuum-induced drift present can be distinguished beyond doubt from the Brownian motion by tracking for $T \gg 4D/U_0^2$. Supposing tracking can be carried out for T_{exp}, then, if no drift is observed, we infer: $0 \leq U_0 \lesssim \sqrt{4D/T_{\mathrm{exp}}}$. Again, the direction of the flow should be reversed by reversing one of the fields to test the symmetry of the Feigel effect and eliminate unwanted systematic drifts.

Turning now to the prediction of Obukhov & Hehl (2008), this implies that a vacuum radiometer, such as the four pane one shown in figure 1b, should not turn. An experimental test would involve a radiometer with thin panes made out of a solid magnetoelectric material (figure 1b). A good candidate for this could be the polar ferrimagnet $GaFeO_3$. Recent studies have characterized the birefringence of this compound in the optical and X-ray ranges (Jung *et al.* 2004; Kubota *et al.* 2004). Such studies suggest values larger than those induced by imposed fields in organometallic liquids: $\Delta n \sim 10^{-4}$ (van Tiggelen *et al.* 2005, 2006). This should amplify any vacuum effects if they are non-zero. We should point out that $\Delta n \approx \Delta\chi$ drops with increasing temperature for $GaFeO_3$, vanishing with ferrimagnetic order at the transition temperature $T_{\mathrm{C}} \approx 225\,\mathrm{K}$; further, small magnetic fields are also required for a non-zero birefringence (Jung *et al.* 2004). This means the radiometer should be placed in an evacuated chamber and maintained at temperatures and fields that maximize $\Delta\chi$. As light could cause the radiometer to turn, the chambers should be light-tight. Light from a light-emitting diode laser placed on the top of each pane (figure 1*b*) would then allow a camera in the chamber to track any rotation of the radiometer. A mirror version of the radiometer should also be constructed and its rotation, or lack thereof, tracked. This allows us to check the prediction that the rotation should reverse when the direction of the panes' optical axes is reversed, as well as allowing us to detect unwanted drifts.

6. Discussion

In this study, we argue for the experimental verification of Feigel's theory that the vacuum can transfer momentum to a fluid placed in strong crossed fields. The momentum transfer occurs because vacuum modes in such a fluid are no longer isotropic as in an ordinary dielectric. The crossed electrical and magnetic fields, $\mathbf{E}_{\text{ext}}$ and $\mathbf{B}_{\text{ext}}$, change the symmetry of the dielectric fluid so that it behaves like a magnetoelectric medium with different refractive indices for waves propagating along or against the direction of the optical axis, defined by $\mathbf{E}_{\text{ext}} \times \mathbf{B}_{\text{ext}}$. Feigel's semiclassical argument allows us to derive the net momentum transfer caused by counter-propagating vacuum modes and the corresponding classical contribution owing to relative motion in crossed fields. Our alternative derivation confirms Feigel's result, once minor inaccuracies in the original paper are corrected. Further, we derive a new expression for the vacuum stress on the fluid, predicting that this will induce Poiseuille flow in a tube, with maximum speed $U_{\max} \approx 100\,\mu\text{m}\,\text{s}^{-1}$ (2000 times larger than Feigel's original estimate of $50\,\text{nm}\,\text{s}^{-1}$). This prediction contrasts with that of Obukhov & Hehl (2008), in which a magnetoelectric slab in the vacuum experiences no net force.

Two experiments are proposed to test the above predictions. Feigel's original scenario can be tested by measuring the flow of an organometallic fluid in a tube placed in the bore of a strong magnet. The predicted flow rate can be measured from the rate of eluted fluid through one of the tube ends. Weaker flows could alternatively be measured from particle-tracking velocimetry with the same set-up. In a second experiment, we propose testing Obukhov and Hehl's null prediction for the force on a slab by measuring the rotation of a vacuum radiometer with panes made of magnetoelectric material.

Like all theories, Feigel's theory makes some assumptions that can and should be questioned. The boldest of these, which we have adopted and are pivotal to our own realistic prediction, are the cut-off frequency assumption and the postulate that vacuum modes see a magnetoelectric (assumptions 4 and 5, respectively, in §2b). The latter assumption amounts to assuming that the interaction of vacuum modes (virtual photons) with media is identical to that of light (photons). This seems reasonable: many quantum electrodynamic effects, both macroscopic (e.g. Casimir–Lifshitz forces) and microscopic (e.g. Lamb and Stark shifts), can be successfully explained by considering the interaction of matter with electromagnetic vacuum modes (Milonni 1994). It has been recently shown that divergences in microscopic Feigel theories can be regularized using a cut-off based on the electron mass, as in the Lamb shift (Kawka & van Tiggelen 2010). However, in Feigel's macroscopic theory, the cut-off assumption has been criticized as an improper regularization of the momentum integral, which causes the momentum density to become Lorentz variant. It has been claimed that when a proper dimensional regularization is applied, the theory predicts a null momentum density, i.e. there should be no Feigel effect (van Tiggelen *et al.* 2005, 2006). We note however that the dimensional regularization employed was pushed beyond the strict limits of its validity, which could yield an erroneous result. The use of a cut-off in a macroscopic quantum electrodynamic theory is not without precedent. When evaluating the surface tension of liquid helium films, Schwinger *et al.* (1978) used a momentum cut-off based on interatomic distances and obtained predictions by a factor of 3 larger than the experiment. They stated that a better agreement would have resulted by employing a microscopic model accounting for the short-distance physics not considered in their continuum model. We believe similar considerations apply to the Feigel effect: an improved model using a microscopically derived complex, frequency-dependent refractive index would dispense of the need for a cut-off (or other regularization).

The same microscopic model allowing us to dispense of the need for a cut-off would also allow us to properly account for the effects of absorption and dispersion in the magnetoelectric media, which Feigel's and subsequent theories (including the present revision) have neglected. We discuss these effects briefly. As mentioned in §4*a*, high-frequency vacuum modes will not sample a large enough number of molecules to experience a magnetoelectric response (which is the statistical product of many molecules). If this is the case, then the vacuum contribution to momentum (pressure) from the absorption of high-frequency modes is isotropic and can be neglected. If not, then the exact consequences of absorption are hard to fathom in the absence of a microscopic model, but one would intuitively expect absorption to cause a smaller net transfer of momentum from the vacuum (owing to inelastic collisions between the virtual photons and the medium, roughly speaking). If the broadband spectrum of magnetoelectric (or effectively magnetoelectric) materials is similar to most dielectric materials, then the effect of absorption will be less important for smaller frequencies, which have smaller weight in the momentum density integral. By a similar argument, the dispersion of the real part of the refractive index will also matter more at high frequencies than lower ones.

We should also comment on the agreement between Feigel's semiclassical derivation and our own. The latter is also semiclassical in the sense that the form of the expectation value of the momentum density (our starting point) is implicitly given by quantization of the classical momentum density. The question then arises as to what the appropriate momentum density should be. This is the so-called Abraham–Minkowski problem, which, as many researchers have pointed out, is not a problem at all, provided one is consistent about conserving total momentum (see Loudon *et al.* 2005). As can be seen from equation (2.2), Feigel uses the classical pseudo-momentum $\mathbf{g} = (\mathbf{D}\times\mathbf{B} - \mathbf{E}\times\mathbf{H})/4\pi$. We directly quantize this, and evaluate $\langle 0|\hat{\mathbf{g}}|0\rangle$ using

$$\hat{\mathbf{g}} = \hat{\mathbf{g}}_{\mathrm{M}} - \hat{\mathbf{g}}_{\mathrm{A}}, \tag{6.1}$$

where $\hat{\mathbf{g}}_{\mathrm{M}} = (\hat{\mathbf{D}}\times\hat{\mathbf{B}} + \hat{\mathbf{B}}\times\hat{\mathbf{D}})/4\pi$ is the Minkowski contribution, where photon momentum is proportional to refractive index, and $\hat{\mathbf{g}}_{\mathrm{A}} = (\hat{\mathbf{E}}\times\hat{\mathbf{H}} + \hat{\mathbf{H}}\times\hat{\mathbf{E}})/4\pi$, is the Abraham contribution, for which momentum is instead inversely proportional to refractive index. Feigel, on the other hand, derives the difference in momentum density for counter-propagating modes classically, and quantizes the energy term that emerges. The two derivations thus differ only in the timing of the application of quantization. Our $\langle 0|\hat{\mathbf{g}}|0\rangle$ is taken to be proportional to $n_{\mathbf{k}\lambda}$, which would suggest that we are using the Minkowski term only and not the expectation value of equation (6.1). However, considering the expectation value of the Abraham form vanishes in a vacuum $\langle 0|\hat{\mathbf{g}}_{\mathrm{A}}|0\rangle = 0$ (van Tiggelen *et al.* 2005), we see that $\langle 0|\hat{\mathbf{g}}|0\rangle = \langle 0|\hat{\mathbf{g}}_{\mathrm{M}}|0\rangle$, i.e. the momentum and its density scale like the refractive index, as we have assumed in our derivations. This Minkoswki pseudo-momentum has been shown to be the appropriate expression for calculating the momentum transferred by light to matter (Peierls 1991).

A comparison between the prediction by Obukhov & Hehl (2008) and that of Feigel's theory is also in order. The two scenarios differ in geometry. In Feigel's scenario, the fluid is unbounded; in our revision, it is bounded by a tube, but still unbounded along the optical axis. In Obukhov & Hehl's case, the magnetoelectric slab is bounded by its surfaces and it is the surface contributions that cause the net vacuum stress to vanish. In our case, there are no surfaces and so the vacuum stress is non-zero. In reality, there will be gradients on each side of the region of the fluid where fields are applied, and it is conceivable that these also might cause the vacuum stress to vanish. The calculation of these contributions is beyond the scope of this study, but would represent an interesting matter to pursue theoretically. In their study, Obukhov & Hehl state that the force on the slab by real photons (e.g. from counter-propagating laser beams) should be non-zero. Intuitively, this seems to contradict their vacuum result and they do not make clear why real photons should behave differently from virtual ones. Indeed, both the experiments suggested in this study could be carried out with light: the expectation is that the fluid should move and the radiometer should turn when they are placed in counter-propagating laser beams. If there is a difference between real and virtual photons, then it may be that the latter are already involved in providing the magnetoelectric matter with its properties (e.g. fine structure, etc.), so in some loose sense there are not enough virtual photons to cause a Feigel effect. Another objection might be that, if the Feigel effect was real, we would be able to extract small amounts of energy from the vacuum (e.g. using the radiometer or the quantum wheels of Feigel (2009)). To some, this might constitute a violation of the second law of thermodynamics. On the other hand, a system macroscopically coupled to the vacuum, a reservoir of virtual photons at zero temperature, is arguably not a conventional thermodynamic system.

The Feigel effect has inspired many alternative theories, most adopting a regularizable Casimir geometry (van Tiggelen *et al.* 2006; Birkeland & Brevik 2007). Any experimental predictions of these theories are beyond the reach of current instrumentation with Feigel's original prediction recognized to be the best candidate for an experimental test (van Tiggelen *et al.* 2006). Our realistic prediction is 2000 times larger than the original, making the case for an experimental test even stronger. Testing Obukhov & Hehl's null prediction is also compelling. Both predictions involve a qualitative expectation that will hopefully be easy to detect: in one case, the fluid moves (or not), in the other the radiometer stays still (or turns). Theories can be challenged theoretically, but the final arbiter of a theory in physics is experiment. This author is surprised that, some 7 years since it was proposed, an experiment to test Feigel's theory is yet to be carried out. We hope this work will stimulate experimentalists to find out if vacuum momentum can be asymmetrically transferred to matter.

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Appendix A. Plane wave modes and refractive indices in a magnetoelectric with orthogonal symmetry

In this appendix, we derive expressions for the amplitudes and refractive indices of electromagnetic modes propagating in a magnetoelectric. The approach is very similar to the appendix of Figotin & Vitebsky (2001). When describing time-dependent fields in media, the following Maxwell equations are sufficient (see Schwinger *et al.* 1998, ch. 41):

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{A 1}$$

and

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}. \tag{A 2}$$

In magnetoelectric materials with isotropic permittivity and permeability tensors ($\hat{\epsilon} = \epsilon\hat{I}$; $\hat{\mu} = \mu\hat{I}$), equations (A 1) and (A 2) are supplemented by the constitutive relations (O'Dell 1970; Figotin & Vitebsky 2001)

$$\mathbf{D} = \epsilon\mathbf{E} + \hat{\chi}\mathbf{H} \tag{A 3}$$

and

$$\mathbf{B} = \mu\mathbf{H} + \hat{\chi}^{\mathrm{T}}\mathbf{E}, \tag{A 4}$$

where the $\hat{\chi}$ is the magnetoelectric susceptibility tensor, defined, in matrix form, by

$$\hat{\chi} \equiv \begin{pmatrix} 0 & \chi_{xy} & 0 \\ \chi_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A 5}$$

In equation (A 15), χ_{xy} and χ_{yx} are magnetoelectric susceptibilities. They are a measure of the electrical polarization caused by a magnetic field and the magnetization caused by an electrical field applied along the x- and y-axes in the magnetoelectric. We assume plane-wave solutions propagating in the z-direction

$$\mathbf{E}_{\mathbf{k}\lambda}(z,t) = E_{0\mathbf{k}}\mathrm{e}^{\mathrm{i}(k_\lambda z - \omega t)}\mathbf{e}_{\mathbf{k}\lambda} \equiv E_{\mathbf{k}\lambda}\mathbf{e}_{\mathbf{k}\lambda}, \tag{A 6}$$

with wavevector $\mathbf{k}_\lambda$, frequency ω and polarization vector $\mathbf{e}_{\mathbf{k}\lambda}$. For such waves (removing the suffices for clarity),

$$\nabla \times \mathbf{E} = \partial_z \begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix} = \partial_z \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix}. \tag{A 7}$$

Thus, defining

$$\hat{\sigma} \equiv \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A 8}$$

we can write

$$\nabla \times \mathbf{E} = \hat{\sigma}\partial_z \mathbf{E}. \tag{A 9}$$

Analogous considerations apply to $\nabla \times \mathbf{H}$. Thus, as $\partial_z \mathbf{E} = \mathrm{i}k\mathbf{E}$ and $\partial_t \mathbf{B} = -\mathrm{i}\omega \mathbf{B}$ and similar to $\mathbf{H}$ and $\mathbf{D}$, equations (A 1) and (A 2) become

$$n\hat{\sigma}\mathbf{E} = \mathbf{B} \tag{A 10}$$

and

$$n\hat{\sigma}\mathbf{H} = -\mathbf{D}, \tag{A 11}$$

where we have defined $n \equiv k(c/\omega)$. We aim to find the values of n for which the plane waves are solutions of the Maxwell's equations. Substituting equations (A 3) and (A 4) into equations (A 10) and (A 11) and rearranging, we find

$$\mathbf{H} = \frac{1}{\mu}(n\hat{\sigma} - \hat{\chi}^{\mathrm{T}})\mathbf{E} \tag{A 12}$$

and

$$(n\hat{\sigma} + \hat{\chi})\mathbf{H} = -\epsilon \mathbf{E}. \tag{A 13}$$

Substituting for $\mathbf{H}$ from equation (A 12) into equation (A 13), the following eigenvalue equation is obtained:

$$\hat{N}\mathbf{E} = \epsilon\mu\mathbf{E}, \tag{A 14}$$

where

$$\hat{N} \equiv -(n\hat{\sigma} + \hat{\chi})(n\hat{\sigma} - \hat{\chi}^{\mathrm{T}}) = \begin{pmatrix} (n-\chi_{xy})^2 & 0 & 0 \\ 0 & (n+\chi_{xy})^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A 15}$$

(a) *Mode amplitudes and indices of refraction*

We look for linearly polarized plane-wave solutions to the Maxwell's equations. There are two independent polarizations. Consider first a plane wave with $\mathbf{E}$ along $\mathbf{e}_1 = (1, 0, 0)$. The eigenvalue equation (A 14) then entails

$$\begin{pmatrix} (n_1-\chi_{xy})^2 & 0 & 0 \\ 0 & (n_1+\chi_{xy})^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \epsilon\mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{A 16}$$

so that the refractive indices for waves propagating in the $\mathbf{k}$ and $-\mathbf{k}$ directions with polarization 1 are given by

$$n_{\mathbf{k},1} = \sqrt{\epsilon\mu} + \chi_{xy} \tag{A 17}$$

and

$$n_{-\mathbf{k},1} = -\sqrt{\epsilon\mu} + \chi_{xy}. \tag{A 18}$$

Table 2. Trivially incorrect quantities in Feigel's derivation (Feigel 2004) and the correct values derived in this paper.

quantity	Feigel's value	correct value
magnetoelectric modes	$(1,0,0,\sqrt{\epsilon\mu})$ $(1,0,0,-\sqrt{\epsilon\mu})$ $(0,1,-\sqrt{\epsilon\mu},0)$ $(0,1,\sqrt{\epsilon\mu},0)$	$(1,0,0,\sqrt{\epsilon\mu}+\chi_{xy})$ $(1,0,0,-\sqrt{\epsilon\mu}+\chi_{xy})$ $(0,1,-\sqrt{\epsilon\mu}+\chi_{yx},0)$ $(0,1,\sqrt{\epsilon\mu}+\chi_{yx},0)$
momentum density of mode $\mathbf{k}$	$2\Delta\chi\frac{1}{c}\left(\epsilon+\frac{1}{\mu}\right)\frac{E_{0k}^2}{4\pi}$	$2\Delta\chi\frac{1}{c}\frac{\epsilon E_{0k}^2}{4\pi}$
vacuum momentum density	$\frac{1}{32\pi^3}\Delta\chi\frac{1+\epsilon\mu}{\mu}\frac{\hbar\omega_{\max}^4}{c^4}$	$\frac{1}{8\pi^2}\Delta\chi\frac{\hbar\omega_{\max}^4}{c^4}$

Analogously, for waves with $\mathbf{E}$ along $\mathbf{e}_2=(0,1,0)$, the eigenvalue equation (A 14) requires

$$n_{\mathbf{k},2}=\sqrt{\epsilon\mu}-\chi_{yx} \tag{A 19}$$

and

$$n_{-\mathbf{k},2}=-\sqrt{\epsilon\mu}-\chi_{yx}. \tag{A 20}$$

The magnetic field components corresponding to the above electrical field modes are found using the Maxwell equation (A 10). For polarization $\mathbf{e}_1$,

$$\mathbf{B}_1=n_1\hat{\sigma}\mathbf{E}_1=\begin{pmatrix}0&-n_1&0\\n_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}E_1=\begin{pmatrix}0\\n_1\\0\end{pmatrix}E_1, \tag{A 21}$$

and similarly for $\mathbf{e}_2$,

$$\mathbf{B}_2=n_2\hat{\sigma}\mathbf{E}_2=\begin{pmatrix}-n_2\\0\\0\end{pmatrix}E_2. \tag{A 22}$$

So finally, the four possible modes of propagation in a magnetoelectric are

$$\left.\begin{aligned}[\mathbf{E}_{\mathbf{k}1},\mathbf{B}_{\mathbf{k}1}]=E_{\mathbf{k}1}[\mathbf{e}_1,n_{\mathbf{k}1}\mathbf{e}_2],\quad [\mathbf{E}_{-\mathbf{k}1},\mathbf{B}_{-\mathbf{k}1}]=E_{-\mathbf{k}1}[\mathbf{e}_1,n_{-\mathbf{k}1}\mathbf{e}_2].\\ [\mathbf{E}_{\mathbf{k}2},\mathbf{B}_{\mathbf{k}2}]=E_{\mathbf{k}2}[\mathbf{e}_2,-n_{\mathbf{k}2}\mathbf{e}_1]\quad\text{and}\quad [\mathbf{E}_{-\mathbf{k}2},\mathbf{B}_{-\mathbf{k}2}]=E_{-\mathbf{k}2}[\mathbf{e}_2,-n_{-\mathbf{k}2}\mathbf{e}_1],\end{aligned}\right\} \tag{A 23}$$

or, in Feigel's notation $(1/E_{\mathbf{k}\lambda})(E_x,E_y,B_x,B_y)$,

$$(1,0,0,n_{\mathbf{k}1}),\quad (0,1,-n_{\mathbf{k}2},0)$$

and

$$(1,0,0,n_{-\mathbf{k}1}),\quad (0,1,-n_{-\mathbf{k}2},0).$$

The modes (A 23) just calculated differ from those presented in the original paper of Feigel (2004). As shown in table 2, the modes Feigel quoted are those of an isotropic dielectric material, not of a magnetoelectric (this is probably a trivial omission, as Feigel does indeed quote the correct magnetoelectric refractive indices in his paper). Table 2 also displays the other minor errors in Feigel's results that we have corrected in this study.

Appendix B. Lagrangian and Hamiltonian densities for magnetoelectric media

In general, the electromagnetic Lagrangian in ponderable media is given by

$$L = \iint \mathcal{L} \mathrm{d}\mathbf{r}\, \mathrm{d}t, \tag{B 1}$$

where the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{1}{2}\mathbf{E}\cdot\mathbf{D} - \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right]. \tag{B 2}$$

A magnetoelectric material satisfies the constitutive relations (A 3) and (A 4). These give $\mathbf{D}=\mathbf{D}(\mathbf{E},\mathbf{H})$ and $\mathbf{B}=\mathbf{B}(\mathbf{E},\mathbf{H})$. We can invert equation (A 3) to find $\mathbf{H}=\mathbf{H}(\mathbf{E},\mathbf{B})$ and so $\mathbf{D}=\mathbf{D}(\mathbf{E},\mathbf{B})$ from equation (A 4),

$$\mathbf{H} = \frac{1}{\mu}\left(\mathbf{B} - \hat{\chi}^{\mathrm{T}}\mathbf{E}\right) \tag{B 3}$$

and

$$\mathbf{D} = \epsilon\mathbf{E} + \frac{1}{\mu}\hat{\chi}\left(\mathbf{B} - \hat{\chi}^{\mathrm{T}}\mathbf{E}\right). \tag{B 4}$$

Substituting equations (B 3) and (B 4) into the Lagrangian density (B 2), we find

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{1}{2}\mathbf{E}\cdot\left(\epsilon\mathbf{E} + \frac{1}{\mu}\hat{\chi}\mathbf{B} - \frac{1}{\mu}\hat{\chi}\hat{\chi}^{\mathrm{T}}\mathbf{E}\right) - \frac{1}{2}\mathbf{B}\cdot\left(\frac{1}{\mu}\mathbf{B} - \frac{1}{\mu}\hat{\chi}^{\mathrm{T}}\mathbf{E}\right)\right] \tag{B 5}$$

or

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{1}{2}\epsilon\mathbf{E}^2 - \frac{1}{2\mu}\mathbf{B}^2 + \frac{1}{2\mu}\mathbf{E}\cdot(\hat{\chi}\mathbf{B}) + \frac{1}{2\mu}\mathbf{B}\cdot(\hat{\chi}^{\mathrm{T}}\mathbf{E}) - \frac{1}{2\mu}\mathbf{E}\cdot(\hat{\chi}\hat{\chi}^{\mathrm{T}}\mathbf{E})\right]. \tag{B 6}$$

Letting $Q=\mathbf{E}\cdot(\hat{\chi}\mathbf{B})$, we see Q is a scalar ($Q=Q^{\mathrm{T}}$) so that $\mathbf{E}\cdot(\hat{\chi}\mathbf{B})=\mathbf{B}\cdot(\hat{\chi}^{\mathrm{T}}\mathbf{E})$ and, neglecting terms of order χ_{xy}^2, χ_{yx}^2,

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{1}{2}\epsilon\mathbf{E}^2 - \frac{1}{2\mu}\mathbf{B}^2 + \frac{1}{\mu}\mathbf{B}\cdot(\hat{\chi}^{\mathrm{T}}\mathbf{E})\right] + o(\chi^2). \tag{B 7}$$

Similarly, the Hamiltonian is given by

$$H = \frac{1}{4\pi}\left[\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right] \approx \frac{1}{4\pi}\left[\frac{1}{2}\epsilon\mathbf{E}^2 + \frac{1}{2\mu}\mathbf{B}^2\right] + o(\chi^2). \tag{B 8}$$

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