Vibration and buckling analysis of a piezoelectric nanoplate considering surface effects and in-plane constraints

BY Z. YAN AND L. Y. JIANG*

Department of Mechanical and Materials Engineering, University of Western Ontario, London, Ontario, Canada N6A 5B9

This work investigates the surface effects on the vibration and buckling behaviour of a simply supported piezoelectric nanoplate (PNP) by using a modified Kirchhoff plate model. Two kinds of in-plane constraints are defined for the PNP, and the surface effects are accounted in the modified plate theory through the surface piezoelectricity model and the generalized Young–Laplace equations. Simulation results show that the influence of surface effects on the plate resonant frequency depends on the in-plane constraints significantly. For the PNP with different in-plane constraints, the effects of the applied electric potential, the mode number, the plate aspect ratio and the plate thickness on the resonant frequency are examined with consideration of the surface effects. The possible mechanical buckling of the PNP is also studied, and it is found that the surface effects on the critical electric voltage for buckling are sensitive to the plate thickness and aspect ratio. Our results also reveal that there exists a critical transition point at which the combined surface effects on the critical electric voltage may vanish under certain conditions.

Keywords: surface piezoelectricity; surface effects; piezoelectric nanoplate; vibration; buckling

1. Introduction

Since the first prototyping of a nanogenerator by means of piezoelectric nanowire arrays (Wang & Song 2006), piezoelectric nanostructured materials have attracted tremendous interest in the research community for potential applications of various devices in nanotechnology, such as nanosensors, nanoresonators, nanogenerators and nanotransistors (Lao et al. 2007; Su et al. 2007; Tanner et al. 2007; Fei et al. 2009). Most recently, piezoelectric thin films or nanoribbons (lead zirconate titanate (PZT) and BaTiO3) have been successfully transferred onto flexible substrates for stretchable energy harvesting, which suggests new possible applications of piezoelectric nanomaterials (Park et al. 2010; Feng et al. 2011; Qi et al. 2011). Among these novel nanodevices, nanoscale piezoelectric beam or plate structures are the key components.

*Author for correspondence (lyjiang@eng.uwo.ca).
Therefore, understanding the mechanical and physical behaviour of piezoelectric nanostructures with these configurations is essential for their design and applications.

‘Small is different’, the mechanical properties of piezoelectric nanostructures can differ markedly from their macroscopic counterparts. Owing to the increasing aspect ratio of surface area to volume at the nanoscale, it is believed that size dependence of the mechanical properties and piezoelectricity due to surface effects will arise. Existing experiments and atomistic simulations have confirmed that the elastic and piezoelectric coefficients of piezoelectric nanostructures vary with their dimensions (Zhao et al. 2004; Chen et al. 2006a; Zhang & Huang 2006; Stan et al. 2007; Agrawal et al. 2008; Zhang et al. 2010a). Owing to the extreme difficulty in conducting experiments and computational expensiveness of atomistic studies, modified continuum theories incorporating the surface effects have been naturally pursued as alternative and effective tools in mechanical and physical property characterization of nanostructured materials.

For elastic nanostructures, the size-dependent properties have been well studied using modified continuum theories based on a well-known surface elasticity model developed by Gurtin & Murdoch (1975). In addition to the studies on the static and dynamic behaviours of nanobeams (Wang & Feng 2007, 2009; He & Lilley 2008a,b; Liu & Rajapakse 2010), this surface elasticity model has also been adopted for modelling the elastic nanoplates. For example, Lim & He (2004) investigated surface effects on the large deflection of an ultra-thin film using von Karman plate theory. Lu et al. (2006) used modified Kirchoff and Mindlin plate models to characterize the bending, vibration and buckling behaviour of nanoscale plates with surface effects. The transverse vibration of a rectangular nanoplate was investigated by Assadi et al. (2010), considering the influence of surface properties and temperature. The free vibration of a circular nanoplate, including the surface effects, was also investigated using a modified laminated plate theory (Assadi & Farshi 2010). However, the investigation on the size-dependent properties of piezoelectric nanostructures using continuum modelling approaches is still very limited, especially for piezoelectric nanoplates (PNPs). The surface elasticity model was used by Wang & Feng (2010) to study the vibration and buckling of a piezoelectric nanobeam, while the surface piezoelectricity was ignored. As an extension of the surface elasticity model, Huang & Yu (2006) carried out pioneering work in proposing a surface piezoelectricity model to study the effect of piezoelectric surface layers on the static deformation of a piezoelectric nanoring. This surface piezoelectricity model has been further applied in our previous work (Yan & Jiang 2011a,b,c) to study the surface effects on static electroelastic responses and vibrational behaviour of flat and curved piezoelectric nanobeams. Li et al. (2011) studied surface effects on the wrinkling of a piezoelectric nanofilm on a compliant substrate by modelling the film structure as a von Karman beam. Recently, a comprehensive model has been developed for dielectric nanomaterials by Shen & Hu (2010) with consideration of surface effects, flexoelectricity and electrostatic forces. It should be mentioned that another type of modified continuum model has also been explored by researchers to investigate size effects. For example, the vibration of piezoelectric nanobeams was investigated recently either based on a linear or a von Karman
strain–displacement relation (Ke & Wang 2012; Ke et al. 2012). They discussed the influence of the non-local parameter, temperature change and external electric voltage on the thermo-electro-mechanical vibration characteristics of piezoelectric nanobeams. These studies have demonstrated the significance of considering size effects in studying the mechanical and physical properties of piezoelectric nanostructures.

To the authors’ best knowledge, the influence of surface effects on the vibrational behaviour of PNPs has not been studied thus far. This work, therefore, will carry out an investigation for this purpose. Owing to the intrinsic electromechanical coupling of piezoelectric materials and the existence of surface stresses in surface layers, either in-plane forces or in-plane displacements may develop in the PNP depending on the in-plane constraints prescribed. It should be mentioned that such in-plane relaxation strains of elastic nanowires owing to the surface stresses have been discussed in the literature by using atomistic or atomistic-based theories (Park & Klein 2007; Zhang et al. 2010b; Park 2012). In addition, by considering a relaxation process before bending deformation, Song et al. (2011) studied the mechanical behaviour of nanowires by using a continuum model. The results in these studies suggest that accounting for axial strain relaxation may be necessary to improve the accuracy and predictive capability of analytical surface elastic theories. However, this surface-stress-induced relaxation phenomenon has not been accounted for in previous investigations of nanoplates with surface effects (Lim & He 2004; Lu et al. 2006; Assadi & Farshi 2010; Assadi et al. 2010), owing to their particular prescribed in-plane boundary conditions. Therefore, different in-plane constraints will be defined in this work in order to catch all the possible phenomena induced by the surface effects. As a result, distinct vibration behaviour and in-plane motions of the PNPs will be observed under different in-plane boundary conditions. The layout of this study is as follows. The general formulations and solutions with different in-plane constraints are derived in §2. Simulation results and discussions on the vibration and buckling behaviour of PNPs are presented in §3. Finally, we summarize the major results of this work in §4.

2. Formulation

The vibration analysis of a rectangular PNP with length $a$, width $b$ and thickness $h$ as illustrated in figure 1a is conducted in the current work. A Cartesian coordinate system ($x$, $y$, $z$) is used to describe the plate with $z$ along the plate thickness direction and the $x$–$y$ plane sitting on the midplane of the undeformed plate. The piezoelectric body is poled along the $z$-direction and subjected to an electric potential $V$ between the upper and lower surfaces of the plate. For a piezoelectric nanobeam subjected to an electric potential across its thickness (Wang & Feng 2010), the authors argued that the electric field component in the length direction is negligible compared with that in the thickness direction according to the available numerical simulation results (Gao & Wang 2007). Therefore, for a thin piezoelectric plate with large in-plane dimension to thickness ratio, it is reasonable to neglect the in-plane electric field components when the plate is subjected to an electric potential across its thickness, which has been adopted by Zhao et al. (2007). Therefore, in this work, the electric field is assumed...
Figure 1. (a) A piezoelectric nanoplate with both bulk and surface parts. (b) Schematic of a differential element of the piezoelectric nanoplate. (Online version in colour.)

to exist only along the z-direction and can be expressed in terms of the electric potential $\Phi$,

$$E_z = -\frac{\partial \Phi}{\partial z}. \quad (2.1)$$

The electric boundary conditions are prescribed as $\Phi(h/2) = V$ and $\Phi(-h/2) = 0$ according to figure 1a.

To account for the surface effects, a surface piezoelectricity model (Huang & Yu 2006; Yan & Jiang 2011a, b) is adopted here. According to this model, the PNP is considered as being composed of a bulk part and the upper and lower surface layers with negligible thickness. For the bulk part, the material obeys the same constitutive relations as the conventional piezoelectric materials. With the
plane stress assumption that the stress component in the $z$-direction is negligible (Zhao et al. 2007), the linear constitutive equations for the bulk part are written as

$$
\begin{align*}
\sigma_{xx} &= c_{11}\varepsilon_{xx} + c_{12}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{yy} &= c_{12}\varepsilon_{xx} + c_{11}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{xy} &= c_{66}\gamma_{xy}, \\
D_z &= e_{31}\varepsilon_{xx} + e_{31}\varepsilon_{yy} + \kappa_{33}E_z,
\end{align*}
$$

(2.2)

and

$$
\begin{align*}
\sigma_{xx} &= c_{11}\varepsilon_{xx} + c_{12}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{yy} &= c_{12}\varepsilon_{xx} + c_{11}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{xy} &= c_{66}\gamma_{xy}, \\
D_z &= e_{31}\varepsilon_{xx} + e_{31}\varepsilon_{yy} + \kappa_{33}E_z,
\end{align*}
$$

(2.3)

where $c_{11}$, $c_{12}$ and $c_{66}$ are bulk elastic constants; $e_{31}$ and $\kappa_{33}$ are bulk piezoelectric and dielectric constants, respectively.

For the surface layers, the constitutive equations are different from those of the bulk, which can be expressed according to the surface piezoelectricity model (Huang & Yu 2006)

$$
\begin{align*}
\sigma^s_{xx} &= \sigma^0_{xx} + c^s_{11}\varepsilon_{xx} + c^s_{12}\varepsilon_{yy} - e^s_{31}E_z, \\
\sigma^s_{yy} &= \sigma^0_{yy} + c^s_{12}\varepsilon_{xx} + c^s_{11}\varepsilon_{yy} - e^s_{31}E_z, \\
\sigma^s_{xy} &= \sigma^0_{xy} + c^s_{66}\gamma_{xy} \\
D^s_x &= D^0_x, \quad D^s_y = D^0_y,
\end{align*}
$$

(2.4)

where $\sigma^s_{ab}(\alpha, \beta = x, y)$ and $D^s_a(\alpha = x, y)$ are surface stresses and surface electric displacements; $c^s_{11}$, $c^s_{12}$, $c^s_{66}$ are surface elastic constants; $e^s_{31}$ is the surface piezoelectric constant; $\sigma^0_{ab}(\alpha, \beta = x, y)$ and $D^0_a(\alpha = x, y)$ are residual surface stress and residual surface electric displacement without applied strain and electric field.

The existence of the surface stresses of the PNP induces traction jumps exerted on the bulk of the plate, which has been commonly adopted in the surface elasticity model and surface piezoelectric model for nanostructures with a variety of configurations (Huang & Yu 2006; Lu et al. 2006; Wang & Feng 2007, 2009; He & Lilley 2008a,b; Yan & Jiang 2011a,b,c; Li et al. 2011). According to the generalized Young–Laplace equations (Chen et al. 2006b), these traction jumps $T_x$, $T_y$ and $T_z$ with the consideration of plate deformation can be expressed as

$$
\begin{align*}
T_x &= \frac{\partial \sigma^s_{xx}}{\partial x} + \frac{\partial \sigma^s_{yx}}{\partial y}, \\
T_y &= \frac{\partial \sigma^s_{xy}}{\partial x} + \frac{\partial \sigma^s_{yy}}{\partial y}, \\
T^u_z &= \sigma^s_{xx} \frac{\partial^2 w}{\partial x^2} + 2\sigma^s_{xy} \frac{\partial^2 w}{\partial x \partial y} + \sigma^s_{yy} \frac{\partial^2 w}{\partial y^2} \\
T^l_z &= -\sigma^s_{xx} \frac{\partial^2 w}{\partial x^2} - 2\sigma^s_{xy} \frac{\partial^2 w}{\partial x \partial y} - \sigma^s_{yy} \frac{\partial^2 w}{\partial y^2},
\end{align*}
$$

(2.4)

where the superscripts ‘u’ and ‘l’ represent the upper and lower surfaces of the plate.
In order to make a vibration analysis for the PNP, a Kirchhoff plate model is used for modelling purposes. According to Kirchhoff’s hypotheses, the displacements of the plate can be expressed as

\[
\begin{align*}
    u(x, y, z, t) &= u^0(x, y, t) - z \frac{\partial w(x, y, t)}{\partial x}, \\
    v(x, y, z, t) &= v^0(x, y, t) - z \frac{\partial w(x, y, t)}{\partial y}
\end{align*}
\]

(2.5)

and

\[
    w(x, y, z, t) = w(x, y, t),
\]

where \( w(x, y, t) \) is the transverse displacement; \( u^0(x, y, t) \) and \( v^0(x, y, t) \) are the in-plane displacements of the midplane describing the membrane deformations. For an elastic bulk plate, these in-plane displacements are assumed as zero, according to conventional Kirchhoff plate theory. However, they may not be zero for the PNP, for example, the applied electrical load induces the in-plane displacements of the midplane owing to the electromechanical coupling. In addition, the existence of surface stresses may also cause in-plane relaxation displacements, as discussed in the literature (Park & Klein 2007; Zhang et al. 2010b; Park 2012), i.e. for a PNP that is allowed to have free in-plane movement, when it is at equilibrium after relaxation without any applied external loads, the bulk part usually presents initial in-plane deformations owing to the residual surface stresses. According to the displacement fields of equation (2.5), the in-plane strain fields can be obtained from

\[
\begin{align*}
    \varepsilon_{xx} &= \frac{\partial u^0}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \\
    \varepsilon_{yy} &= \frac{\partial v^0}{\partial y} - z \frac{\partial^2 w}{\partial y^2}, \\
    \gamma_{xy} &= \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]

(2.6)

For a differential element of the plate composed of the surface layers and the bulk part as shown in figure 1b, the motion equations are derived as

\[
\begin{align*}
    \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + T^u_x + T^l_z &= \rho h \frac{\partial^2 u^0}{\partial t^2}, \\
    \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + T^u_y + T^l_y &= \rho h \frac{\partial^2 v^0}{\partial t^2}, \\
    \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + T^u_z - T^l_z + q_z &= \rho h \frac{\partial^2 w}{\partial t^2}, \\
    \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_x - (T^u_x - T^l_x) \frac{h}{2} &= \frac{\rho h^3}{12 \partial t^2} \left( \frac{\partial w}{\partial x} \right), \\
    \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_y - (T^u_y - T^l_y) \frac{h}{2} &= \frac{\rho h^3}{12 \partial t^2} \left( \frac{\partial w}{\partial y} \right)
\end{align*}
\]

(2.7)

where \( N_{\alpha\beta} \) and \( Q_\alpha \) are axial and shear forces with dimension of force per unit length, and \( M_{\alpha\beta} \) is the bending moment with dimension of moment per unit length. The axial forces and bending moments are related to stresses by
\[ N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \, dz \quad \text{and} \quad M_{\alpha\beta} = -\int_{-h/2}^{h/2} \sigma_{\alpha\beta} z \, dz, \] respectively. \( \rho \) is the mass density of the material. The transverse load \( q_z \) in the third equation is induced by the in-plane loads \( N_{\alpha\beta} \) and the traction jumps, which can be derived as
\[ q_z = \frac{N_{xx}}{2} \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \frac{N_{yy}}{2} \frac{\partial^2 w}{\partial y^2}, \]
with the consideration of the first two equations of equation (2.7).

In the absence of free electric charges, the electric displacement should satisfy Gauss’s law,
\[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0. \quad (2.8) \]
Under the assumption that the electric field exists only in the \( z \)-direction, \( D_x \) and \( D_y \) are equal to zero and \( D_z \) is given in equations (2.2). Solving the above equation with the applied electric boundary conditions results in the electric potential and electric field
\[
\begin{align*}
\Phi &= -\frac{e_{31}}{2\kappa_{33}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \left( z^3 - \frac{h^2}{4} \right) + \frac{V}{h} z + \frac{V}{2} \\
E_z &= \frac{e_{31}}{\kappa_{33}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) z - \frac{V}{h}.
\end{align*}
\]
After the manipulation of the last three equations of (2.7) with the consideration of \( q_z \) defined earlier, the motion equation of the PNP for the transverse vibration can be derived as
\[
\begin{align*}
\frac{\partial^2 M^*_{xx}}{\partial x^2} + 2 \frac{\partial^2 M^*_{xy}}{\partial x \partial y} + \frac{\partial^2 M^*_{yy}}{\partial y^2} + \rho h \frac{\partial^2 w}{\partial t^2} &= \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\
&= N^*_x \frac{\partial^2 w}{\partial x^2} + 2 N^*_y \frac{\partial^2 w}{\partial x \partial y} + N^*_y \frac{\partial^2 w}{\partial y^2},
\end{align*}
\]
where \( N^*_{\alpha\beta} \) and \( M^*_{\alpha\beta} \) are the generalized resultant forces and moments with consideration of surface effects, which are defined by Lu et al. (2006),
\[
N^*_{\alpha\beta} = N_{\alpha\beta} + (\sigma^s_{\alpha\beta})^u + (\sigma^s_{\alpha\beta})^l \quad \text{and} \quad M^*_{\alpha\beta} = M_{\alpha\beta} - \frac{h}{2} (\sigma^s_{\alpha\beta})^u - (\sigma^s_{\alpha\beta})^l. \quad (2.11)
\]
It is obvious that without considering the surface effects, i.e. \( N^*_{\alpha\beta} = N_{\alpha\beta} \) and \( M^*_{\alpha\beta} = M_{\alpha\beta} \), the motion equation (2.10) is reduced to that for a conventional Kirchhoff plate (Reddy 2007).

For a case study, the vibrational behaviour of a simply supported PNP is investigated with boundary conditions described by the out-plane displacement and the generalized resultant moments
\[
\begin{align*}
w &= 0, \quad M^*_x = 0 \quad \text{at} \quad x = 0, \quad x = a, \\
w &= 0, \quad M^*_y = 0 \quad \text{at} \quad y = 0, \quad y = b.
\end{align*}
\]
In addition to these boundary conditions, the in-plane boundary conditions for the PNP must also be prescribed in order to solve the motion equation (2.10).
These in-plane boundary conditions depend on the in-plane constraints, which are listed as the following two cases.

1. \( N^*_{xx} = N^*_{xy} = 0 \) at \( x = 0, x = a \); \( N^*_{yy} = N^*_{xy} = 0 \) at \( y = 0, y = b \).
   This is a traction-free boundary condition on all the edges of the PNP. Assuming \( \sigma^0_x = \sigma^0_y = \sigma^0 \) and \( \sigma^0_{xy} = 0 \), then equation (2.10) is rewritten in terms of the transverse displacement \( w \) as
   \[
   D_{11} \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \rho h \frac{\partial^2 w}{\partial t^2} \\
   - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0, \tag{2.13}
   \]
   with
   \[
   D_{11} = \left( c_{11} + \frac{e_{31}^2}{\kappa_{33}} \right) \frac{h^3}{12} + \left( c_{11}^s + \frac{e_{31}^s e_{31}}{\kappa_{33}} \right) \frac{h^2}{2}, \\
   D_{12} = \left( c_{12} + \frac{e_{31}^2}{\kappa_{33}} \right) \frac{h^3}{12} + \left( c_{12}^s + \frac{e_{31}^s e_{31}}{\kappa_{33}} \right) \frac{h^2}{2}, \\
   D_{66} = \frac{c_{66} h^3}{12} + \frac{c_{66}^s h^2}{2}. \tag{2.14}
   \]
   It should be mentioned that with these in-plane constraints, in-plane strains will be induced owing to the inherent electromechanical coupling of piezoelectric materials and the surface effects, which are derived from the first two equations of equation (2.7) as
   \[
   \varepsilon = - \frac{e_{31} V + 2(\sigma^0 + e_{31}^s (V/h))}{(c_{11} + c_{11})h + 2(c_{11}^s + c_{12}^s)h}, \tag{2.15}
   \]
   It is obvious that without the applied electrical load (i.e. \( V = 0 \)), the residual surface stress will still induce a relaxation strain for elastic nanomaterials, i.e. \( \varepsilon_{\text{relax}} = -2\sigma^0/[(c_{11} + c_{11})h + 2(c_{11}^s + c_{12}^s)] \). However, this relaxation was not considered in the previous studies on the elastic nanoplate (Lim & He 2004; Lu et al. 2006; Assadi & Farshi 2010; Assadi et al. 2010) using modified continuum mechanics models, while atomistic or atomistic-based studies have confirmed and discussed this phenomenon (Park & Klein 2007; Zhang et al. 2010b; Park 2012).

2. \( u^0(x, y) = v^0(x, y) = 0 \).
   This can be realized by clamping the edges of the PNP without in-plane movement. In this case, the in-plane displacements are assumed to be trivial compared with the transverse deflection as defined by conventional Kirchhoff plate theory. It should be noted that this boundary condition was adopted by Zhao et al. (2007) when conducting electro-elastic analysis of a conventional piezoelectric plate. The equation governing the transverse
vibration of the PNP is then simplified as
\[
D_{11} \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + 2(D_{12} + 2D_{b6}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\
= \left[ 2 \left( \sigma^0 + e_{31}^s \frac{V}{h} \right) + e_{31} V \right] \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right). \tag{2.16}
\]

We can see that under this condition, the electric potential and residual surface-stress-induced axial force \( P = 2(\sigma^0 + e_{31}^s V/h) + e_{31} V \) will influence the transverse vibration of the PNP. Once this force becomes compressive, it may cause the mechanical buckling of the plate, as observed for the piezoelectric nanobeams in the literature (Wang & Feng 2010; Yan & Jiang 2011b).

According to the boundary conditions of equation (2.12), the harmonic solution of equations (2.13) and (2.16) can be expressed as
\[
w = W_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} e^{i\omega t}, \tag{2.17}
\]
where \( W_{mn} \) is a constant representing the mode shape amplitude, \( m \) and \( n \) are the half wavenumbers, and \( \omega \) is the resonant frequency.

Substituting equation (2.17) into equations (2.13) and (2.16), respectively, the resonant frequency can be obtained for case 1 as
\[
(\omega_{mn}^{(1)})^2 = \frac{D_{11}(m^4 \pi^4/a^4 + n^4 \pi^4/b^4) + 2(D_{12} + 2D_{b6})(m^2n^2 \pi^4/a^2b^2)}{\rho h + (\rho h^3/12)(m^2n^2 \pi^2/a^2 + n^2 \pi^2/b^2)}, \tag{2.18}
\]
and for case 2 as
\[
(\omega_{mn}^{(2)})^2 = \frac{1}{\rho h + (\rho h^3/12)(m^2n^2 \pi^2/a^2 + n^2 \pi^2/b^2)} \times \left[ D_{11} \left( \frac{m^4 \pi^4}{a^4} + \frac{n^4 \pi^4}{b^4} \right) + 2(D_{12} + 2D_{b6}) \frac{m^2n^2 \pi^4}{a^2b^2} \right. \\
- \left[ 2 \left( \sigma^0 + e_{31}^s \frac{V}{h} \right) + e_{31} V \right] \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \right]. \tag{2.19}
\]

The mechanical buckling of the PNP is also an interesting phenomenon that requires further investigation. By letting \( \omega_{mn}^{(2)} = 0 \) in equation (2.19), the electric voltage \( V_{mn} \) corresponding to the buckling of the PNP with clamped in-plane constraints can be obtained in terms of \( (m,n) \) as
\[
V_{mn} = \frac{-[D_{11}(m^4 \pi^4/a^4 + n^4 \pi^4/b^4) + 2(D_{12} + 2D_{b6})m^2n^2 \pi^4/a^2b^2] - 2\sigma^0(m^2 \pi^2/a^2 + n^2 \pi^2/b^2)}{(2e_{31}^s/h + e_{31})(m^2 \pi^2/a^2 + n^2 \pi^2/b^2)}. \tag{2.20}
\]

The lowest value of \( V_{mn} \) and associated \( (m,n) \) represents the critical electric voltage for buckling and the buckling mode, respectively.

3. Results and discussion

The formulation developed based on the modified plate theory will be employed to conduct a quantitative analysis on the vibrational behaviour of a simply supported PNP with the different in-plane constraints described in §2. PZT-5H is selected as an example material with macroscopic material constants $c_{11} = 102 \text{ GPa}$, $c_{12} = 31 \text{ GPa}$, $c_{66} = 35.5 \text{ GPa}$, $e_{31} = -17.05 \text{ C m}^{-2}$ and $k_{33} = 1.76 \times 10^{-8} \text{ CV}^{-1} \text{ m}^{-1}$ for the bulk part. For the surface layers, the material constants that can be determined from atomic calculations (Dai et al. 2011) or experiments are not completely available in the literature for PZT-5H owing to the lack of such work. The estimated values of the surface material constants in the literature (Huang & Yu 2006; Yan & Jiang 2011a, b) are taken as $c_{s11} = 7.56 \text{ N m}^{-1}$, $e_{s31} = -3 \times 10^{-8} \text{ C m}^{-1}$, $c_{s12} = 3.3 \text{ N m}^{-1}$ and $c_{s66} = 2.13 \text{ N m}^{-1}$. In addition, the residual surface stress $\sigma^0$ is assumed as $1.0 \text{ N m}^{-1}$. As suggested by Yao et al. (2009), an aspect ratio of the plate between $1/80$ and $1/5$ is adopted for a Kirchhoff plate. In the current simulation, the plate aspect ratio is set within such a range.

Firstly, we consider the separate influence of the surface elasticity, residual surface stress and surface piezoelectricity upon the vibrational behaviour of the PNP. The normalized mode $(1,1)$ resonant frequency $\omega_{11}/\omega_{11}^0$ for the free vibration of a square PNP ($a = b = 20h$) with variation of the plate thickness $h$ is shown in figure 2, in which $\omega_{11}^0$ is the resonant frequency without considering the surface effects. Without the applied electrical load, both surface elasticity and surface piezoelectricity have the same effect on the resonant frequency of the PNP with different in-plane constraints for case 1 and case 2, as indicated in equations (2.18) and (2.19). However, the residual surface stress has no effect on the transverse vibration of the PNP with case 1 in-plane constraints, while it

Figure 2. Separate surface effects on the free vibration of the PNP with different in-plane constraints ($a$ and $b = 20h$). Solid line, surface elasticity (case 1 and case 2); dashed line, surface piezoelectricity (case 1 and case 2); dotted line, residual surface stress (case 2).
will induce in-plane relaxation, as shown in equation (2.15). For the PNP with case 2 in-plane constraints, it is observed in this figure that the residual surface stress has the most significant influence within the considered values of the surface material constants. It is also found in this figure that the surface piezoelectricity has a more prominent effect compared with the surface elasticity, which means the necessity of using this surface piezoelectricity model in the vibration analysis of the piezoelectric nanoplates. The individual influence of these surface effects is more significant for the thinner plate, and will eventually become negligible with increase in the plate thickness.

**Figure 3** plots the normalized resonant frequency $\omega_{mn}/\omega_{0mn}$ of a square PNP ($a = b = 20h$) against the plate thickness $h$ when it is subjected to an electric voltage $V$, in which $\omega_{0mn}$ is the resonant frequency for the PNP without considering surface effects and the applied electric voltage. For the PNP with different in-plane constraints as described by case 1 and case 2 in §2, the surface effects are found to be more pronounced for the PNP with a smaller thickness, while they diminish with increasing plate thickness, as expected. It is also demonstrated in this figure that the in-plane constraints have a significant effect on the vibrational behaviour of the PNP, i.e. the influence of the surface effects on the resonant frequency of the PNP does not change with variation of the applied electrical load and the mode numbers $(m, n)$ for the PNP with in-plane traction-free conditions (case 1), while it is significantly altered by these factors for the PNP with in-plane clamped constraints (case 2). For example, the discrepancy between the curves for the PNP under the same electrical load for different mode numbers $(m, n)$ indicates that the contribution of the surface effects to the resonant frequency of the PNP varies with the vibration modes. It should be mentioned that the applied electric potential induces the in-plane strain, as shown in equation (2.15) for the PNP with in-pane traction-free conditions. This figure also reveals how the applied electrical load influences the vibrational behaviour.

![Figure 3](http://rspa.royalsocietypublishing.org/)

**Figure 3.** Normalized resonant frequency versus plate thickness for the PNP with different in-plane constraints ($a$ and $b = 20h$). (Online version in colour.)
Vibration of piezoelectric nanoplates

Figure 4. Normalized resonant frequency versus plate thickness for the PNP with different aspect ratios ($V = 0$ V). (Online version in colour.)

of the PNP with clamped in-plane constraints. As observed, for a lower vibration mode ($m = n = 1$ for example), the influence of the surface effects is significantly affected by the applied electrical load, which is similar to the results obtained for a piezoelectric nanobeam (Yan & Jiang 2011b). However, for a higher vibration mode (e.g. $m = n = 5$), the electrical load will not influence the surface effect contribution that much. Such variation of the resonant frequency of the PNP with the applied electric voltage at lower modes proposes a possible avenue for frequency tuning of the PNP-based nanodevices by applying electrical load, which may either stiffen or soften the PNP, depending on the direction and amplitude of the electric potential. For example, for mode $(1,1)$, the PNP with thickness 10 nm is stiffened with $V = -0.2$ V and its resonant frequency is increased by approximately 60 per cent, while it is softened with $V = 0.2$ V and its resonant frequency is decreased by approximately 20 per cent. With a sufficiently large electric voltage (e.g. $V = 0.2$ V), the drop down of the resonant frequency of the PNP with thickness $h$ in this figure indicates a possible mechanical buckling of the PNP, which will be discussed later. The results in this figure conclude that the in-plane constraints must be prescribed for the transverse vibration of the PNP; otherwise they may lead to substantial errors in prediction and characterization of the dynamic performance of the nanoplate.

The variation of the surface effects on the mode $(1,1)$ resonant frequency of the PNP with its thickness is demonstrated in figure 4 for the PNP with different aspect ratios. The surface effects on the resonant frequency of the PNP do not change with aspect ratio of the PNP when the in-plane constraints are described by case 1. However, for the PNP with clamped in-plane constraints (case 2), with a given value of $b/a$, the normalized resonant frequency increases when the aspect ratio $a/h$ increases (i.e. the PNP becomes thinner). For a fixed value of $a/h$, the resonant frequency increases with the increase in aspect ratio $b/a$ (i.e. the PNP has a larger surface area). Therefore, it is concluded that the surface
effects on the resonant frequency of the PNP are more prominent for the thinner plate with larger surface area. The variation of the normalized resonant frequency ($\omega_{11}/\omega_{01}^{0}$) with aspect ratio $a/h$ is plotted in figure 5 for a square PNP ($a = b$). The straight lines in this figure indicate the surface effects on the resonant frequency of the PNP when case 1 in-plane conditions are independent of $a/h$. However, the normalized resonant frequency is significantly influenced by $a/h$ for case 2 in-plane conditions. For example, for the PNP under the same electrical load, the resonant frequency increases with an increase in $a/h$ owing to the larger surface effects. Moreover, the resonant frequency will be further increased by the applied electric potential, as indicated by the difference between the curves for $V = -0.2$ V and $V = 0$ V. It is also seen that for both in-plane constraints of the PNP, the surface effects are more significant when the plate thickness $h$ gets smaller.

As mentioned earlier, the normalized resonant frequency does not vary with the mode numbers for the PNP with case 1 in-plane constraints. Therefore, we only plot $\omega_{mn}/\omega_{0mn}^{0}$ ($m = n$) for a square PNP ($a = b = 20h$) with case 2 in-plane constraints in figure 6. As expected, the surface effects are more significant for the PNP with smaller thickness $h$. It is observed that $\omega_{mn}/\omega_{0mn}^{0}$ is larger in lower modes, while it tends to approach a constant value as the mode number increases. This indicates that the surface effects are more prominent in the lower vibration modes, while such effects will not change much for higher mode vibration, which is similar to the results obtained for the vibration of an elastic nanoplate with the consideration of surface effects (Assadi et al. 2010). It is also observed in this figure that the influence of the applied electric potential on the resonant frequency of the PNP decreases with an increase in the mode number and becomes negligible when the mode number becomes sufficient large, for example $m = n = 8$. Therefore, the resonant frequency tuning concept by applying an electrical load for the PNP is applicable only for the lower vibration modes.

As discussed in §2, for the PNP with case 2 in-plane constraints, the applied electric potential may induce a compressive force. When the compressive force reaches the critical value, it may cause mechanical buckling of the plate, which is also an interesting topic that needs further discussion. Figure 7 plots the variation of the normalized critical electric voltage $V_{cr}/V_{cr}^0$ for buckling ($V_{cr}^0$ is the calculated critical electric voltage without surface effects) with the plate thickness $h$. Similar to the observations in the previous figures, the surface effects

Figure 6. Variation of normalized resonant frequency with mode number for PNP with clamped in-plane constraints. (Online version in colour.)

Figure 7. Separate surface effects on the buckling of the PNP ($a$ and $b = 20h$). Dashed line, surface elasticity; dotted line, surface piezoelectricity; dashed-dotted line, residual surface stress; solid line, combined surface effects.
Figure 8. Variation of the normalized critical electric voltage for buckling with the aspect ratio \(a/h\) of the PNP \((a = b)\). Solid line, \(h = 20\,\text{nm}\); dashed line, \(h = 60\,\text{nm}\); dotted line, \(h = 100\,\text{nm}\); dashed-dotted line, \(h = 500\,\text{nm}\).

are more prominent with a decrease in the plate thickness \(h\). It is also found that both the surface elasticity and the residual surface stress increase this critical electric voltage, while the surface piezoelectricity decreases it. In comparison, the residual surface stress and surface piezoelectricity have more effects on this critical electrical load for buckling than the surface elasticity, which again indicates the necessity of considering the surface piezoelectricity model for the piezoelectric nanostructures. Owing to the opposite effect of the surface piezoelectricity on the critical electrical buckling load to the other two separate surface effects, it is natural to believe that the combined surface effects on the critical electric voltage may vanish under some conditions. Figure 8 plots the variation of the normalized critical electric voltage \(V_{cr}/V_{cr}^0\) against the aspect ratio \(a/h\) for a square PNP \((a = b)\) with different thickness. This figure clearly demonstrates how the combined surface effects influence the critical electric voltage for the buckling of the PNP with the change of plate size. It is interesting to note that the combined surface effects decrease the critical electric voltage when \(a/h\) is small, while increase it when \(a/h\) becomes larger. Therefore, a transition point exists within the considered range of the surface material constants, i.e. regardless of the value of the plate thickness \(h\), the influence of the surface effects on the critical electrical load for buckling vanishes at this transition point \((a/h)_t\). Obviously \((a/h)_t\) depends on the in-plane aspect ratio \(b/a\) of the PNP. In particular, for a square PNP, \((a/h)_t\) is derived as

\[
(a/h)_t = \sqrt{\frac{\varepsilon_{31}^2(c_{11} + c_{12} + 2c_{66} + 2(e_{31}^2/k_{33})) - 3e_{31}(c_{11}^s + c_{12}^s + 2c_{66}^s + 2e_{31}^s c_{31}/k_{33})}{12\sigma_0^s c_{31}}}.
\]  

(3.1)
For a positive residual surface stress ($\sigma_0^s > 0$), the surface effects decrease the critical electric voltage when $a/h < (a/h)_t$, while they increase it when $a/h > (a/h)_t$. However, if the residual surface stress $\sigma_0^s \leq 0$, the surface effects always increase the critical electric voltage for buckling.

From the simulation results in this work, it can be concluded that the transverse vibration behaviour of PNPs is substantially influenced by the in-plane boundary conditions. It should also be noted that although the applied electric potential and the residual surface stress have no effect on the transverse vibration of the PNPs with case 1 in-plane constraints, they will significantly influence the in-plane mechanical behaviour of the PNPs, such as the in-plane relaxation as indicated in equation (2.15). Investigation of the in-plane vibration of the PNPs with consideration of relaxation effects is for our future attention. Our current work focuses on quantitative prediction of the vibrational behaviour of the PNPs with modified continuum plate theory, while relevant experimental measurements, atomistic simulations and finite-element analysis with consideration of surface energy to validate the current results will be further pursued.

4. Conclusions

A modified Kirchhoff plate model is developed to investigate the surface effects on the vibration and buckling behaviour of a simply supported PNP under different in-plane constraints. The surface effects are accounted for by employing a surface piezoelectricity model and generalized Young–Laplace equations. Surface effects are found to be more prominent for a plate with smaller thickness, while they decrease with increasing plate thickness. Simulation results show that the surface effects on the vibrational behaviour of the PNP depend on the in-plane constraints. For the PNP with traction-free in-plane conditions, the residual surface stress and the applied electric potential have no effect on the transverse vibration of the PNP, while they induce an in-plane relaxation of the PNP. In addition, the influence of the surface effects on the resonant frequency of the PNP does not change with mode number and plate aspect ratio. However, the applied electric potential, mode number and plate aspect ratio significantly influence surface effects on the vibrational behaviour of the PNP with clamped in-plane conditions. It is also concluded that for the PNP with clamped in-plane constraints, its resonant frequencies for lower vibration modes can be tuned by applying an electrical load. The possible mechanical buckling when the PNP is subject to an electrical load and the surface effects on the buckling behaviour have also been studied. The influence of the surface effects is sensitive to the plate thickness and aspect ratio. It is found that a transition point at which surface effects vanish for all plate thicknesses may exist under certain conditions owing to the combined effects of the surface elasticity, residual surface stress and surface piezoelectricity. This work is expected to be helpful for understanding the size-dependent properties of piezoelectric nanostructured materials and provide guidelines for the design and applications of piezoelectric nanoplate-based devices in nanotechnology.

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Academic Development Fund of UWO.

References


Vibration of piezoelectric nanoplates


piezoelectricity and stretchability in energy harvesting devices fabricated from buckled PZT


Song, F., Huang, G. L., Park, H. S. & Liu, X. N. 2011 A continuum model for the mechanical
behavior of nanowires including surface and surface-induced initial stresses. Int. J. Solids Struct.

Stan, G., Ciobanu, C. V., Parthangal, P. M. & Cook, R. F. 2007 Diameter-dependent radial and

Su, W. S., Chen, Y. F., Hsiao, C. L. & Tu, L. W. 2007 Generation of electricity in GaN nanorods

Tanner, S. M., Gray, J. M., Rogers, C. T., Bertness, K. A. & Sanford, N. A. 2007 High-Q GaN

Wang, G. F. & Feng, X. Q. 2007 Effects of surface elasticity and residual surface tension on the

Wang, G. F. & Feng, X. Q. 2009 Timoshenko beam model for buckling and vibration of nanowires

Wang, G. F. & Feng, X. Q. 2010 Effect of surface stresses on the vibration and buckling of
piezoelectric nanowires. EPL 91, 56007. (doi:10.1209/0295-5075/91/56007)


Yan, Z. & Jiang, L. Y. 2011a Surface effects on the electromechanical coupling and bending

Yan, Z. & Jiang, L. Y. 2011b The vibrational and buckling behaviors of piezoelectric nanobeams
with surface effects. Nanotechnology 22, 245703. (doi:10.1088/0957-4484/22/24/245703)

Yan, Z. & Jiang, L. Y. 2011c Electromechanical response of a curved piezoelectric nanobeam
with the consideration of surface effects. J. Phys. D 44, 365301. (doi:10.1088/0022-
3727/44/36/365301)

Yao, W., Zhong, W. & Lim, C. W. 2009 Symplectic elasticity, 1st edn, pp. 225–226. Singapore:
World Scientific Publishing.

Zhang, L. X. & Huang, H. C. 2006 Young’s moduli of ZnO nanoplates: ab initio

Zhao, M. H., Wang, Z. L. & Mao, S. X. 2004 Piezoelectric characterization of individual zinc
nl035198a)

Zhao, M. H., Qian, C. F., Lee, S. W. R., Tong, P., Suemasu, H. & Zhang, T. Y. 2007 Electro-
15685510779755273)

Zhang, Y. H., Hong, J. W., Liu, B. & Fang, D. N. 2010a Strain effect on ferroelectric behaviors of
BaTiO3 nanowires: a molecular dynamics study. Nanotechnology 21, 015701. (doi:10.1088/0957-
4484/21/1/015701)