Cauchy integral formula for generalized analytic functions in hydrodynamics

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It is shown that for several classes of generalized analytic functions arising in linearized equations of hydrodynamics and magnetohydrodynamics, the Cauchy integral formulae follow from the one for generalized holomorphic vectors in a uniform fashion. If hydrodynamic fields (velocity, pressure and vorticity) admit representations in terms of corresponding generalized analytic functions, those representations and the Cauchy integral formulae form two essential parts of the generalized analytic function approach, which readily yields either closed-form solutions or boundary integral equations. This approach is demonstrated for problems of axisymmetric and asymmetric Stokes flows, two-phase axisymmetric Stokes flows, two-dimensional and axisymmetric Oseen flows.

Keywords: generalized analytic function; Cauchy integral formula; Stokes flow; Oseen flow

1. Introduction

(a) Vector fields in linear hydrodynamics

A vector field \( \mathbf{\Omega} = \mathbf{\Omega}(\mathbf{x}) \) and a scalar field \( \Psi = \Psi(\mathbf{x}) \) related by

\[
\text{curl} \, \mathbf{\Omega} + 2[a \times \mathbf{\Omega}] = -\nabla \Psi, \quad \text{div} \, \mathbf{\Omega} = 0,
\]

where \( a \) is a constant real-valued vector and \( \mathbf{x} \) is the position vector (multiplier 2 is introduced for convenience), arise in linearized equations of hydrodynamics and magnetohydrodynamics (MHD) (Happel & Brenner 1983; Zabarankin & Krokhmal 2007; Zabarankin 2010, 2011a). For \( a = 0 \), (1.1) is known as the Moisil–Theodorescu system (Moisil & Theodorescu 1931; Bitsadze 1969), whereas for \( \Psi = 0 \) and \( a = 0 \), (1.1) simplifies to the classical potential flow equations (Mises 1944; Bitsadze 1969).

Example 1.1 (Ideal fluid). The velocity field \( \mathbf{u} \) of an ideal fluid is irrotational and incompressible (solenoidal), i.e.

\[
\text{curl} \, \mathbf{u} = 0, \quad \text{div} \, \mathbf{u} = 0,
\]

which corresponds to (1.1) with \( \mathbf{\Omega} = \mathbf{u}, \, \Psi = 0 \) and \( a = 0 \).

Example 1.2 (Stokes flows). Under the assumption of negligible inertial and thermal effects, the time-independent velocity field \( \mathbf{u} \) of a viscous incompressible
fluid is governed by the Stokes equations

$$\mu \Delta \mathbf{u} = \nabla p, \quad \text{div } \mathbf{u} = 0,$$

(1.3)

where \( p \) is the pressure in the fluid, \( \mu \) is shear viscosity and \( \Delta \mathbf{u} = \nabla (\text{div } \mathbf{u}) - \text{curl}(\text{curl } \mathbf{u}) \) (Happel & Brenner 1983). The Stokes equations (1.3) imply that the vorticity \( \mathbf{\omega} = \text{curl } \mathbf{u} \) and pressure \( p \) are related by

$$\mu \text{curl } \mathbf{\omega} = -\nabla p, \quad \text{div } \mathbf{\omega} = 0,$$

(1.4)

which corresponds to (1.1) with \( \Psi = p, \ \Omega = \mu \mathbf{\omega} \) and \( a = 0 \).

**Example 1.3 (Oseen flows).** Suppose a solid body translates with constant velocity \( \mathbf{v} \) in a quiescent viscous incompressible fluid. If the Reynolds number is sufficiently small, the time-independent velocity field \( \mathbf{u} \) with partially accounted inertial effects can be described by the Oseen equations

$$\mu \Delta \mathbf{u} + \rho (\mathbf{v} \cdot \nabla) \mathbf{u} = \nabla p, \quad \text{div } \mathbf{u} = 0,$$

(1.5)

where \( p \) is the pressure, and \( \mu \) and \( \rho \) are fluid shear viscosity and density, respectively (Happel & Brenner 1983). Let \( \mathbf{v} \cdot \mathbf{\omega} = 0 \) with \( \mathbf{\omega} = \text{curl } \mathbf{u} \). Then the Oseen equations (1.5) can be recast in two equivalent forms:

$$\text{curl } (\mu \mathbf{\omega} + \rho [\mathbf{v} \times \mathbf{u}]) = -\nabla p, \quad \text{div } (\mu \mathbf{\omega} + \rho [\mathbf{v} \times \mathbf{u}]) = 0$$

(1.6)

and

$$\mu \text{curl } \mathbf{\omega} + \rho [\mathbf{v} \times \mathbf{\omega}] = -\nabla (\rho - \rho (\mathbf{v} \cdot \mathbf{u})), \quad \text{div } \mathbf{\omega} = 0,$$

(1.7)

which are both particular cases of (1.1): \( \Psi = p, \ \Omega = \mu \mathbf{\omega} + \rho [\mathbf{v} \times \mathbf{u}] \) and \( a = 0 \) in (1.6), and \( \Psi = p - \rho (\mathbf{v} \cdot \mathbf{u}), \ \Omega = \mu \mathbf{\omega} \) and \( 2a = \rho \mathbf{v} / \mu \) in (1.7).

Let \((x, y, z)\) be a Cartesian coordinate system with basis \((\mathbf{i}, \mathbf{j}, \mathbf{k})\), and let \((r, \varphi, z)\) be a cylindrical coordinate system with basis \((\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{k})\). The both coordinate systems have the same \( z \)-axis and are related in the ordinary way.

**Example 1.4 (Magnetohydrodynamics).** Let a non-magnetic solid body rotate at constant velocity in an electrically conducting viscous incompressible fluid under the presence of an initially constant and uniform magnetic field. It is assumed that body’s axis of revolution, body’s velocity and the direction of the undisturbed magnetic field are all parallel to the \( z \)-axis. The fluid velocity \( \mathbf{u} \), fluid pressure \( p \) and magnetic field disturbances \( \mathbf{h}^+ \) and \( \mathbf{h}^- \) inside and outside the body, respectively, can be described by linearized dimensionless equations of MHD, provided that \( \mathbf{u} \) and \( \mathbf{h}^- \) are small:

$$R(k \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u} + M^2 [(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}] \times \mathbf{k}, \quad \text{div } \mathbf{u} = 0,$$

(1.8)

and

$$\text{curl } \mathbf{h}^+ = 0, \quad \text{curl } \mathbf{h}^- = R_m [(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}], \quad \text{div } \mathbf{h}^+ = 0,$$

where \( R \) is the Reynolds number, \( M \) is the Hartmann number and \( R_m \) is the magnetic Reynolds number. In this case, the electric field is zero everywhere, and \( \mathbf{u} = u_r (r, z) \mathbf{e}_r + u_z (r, z) \mathbf{k}, \ \mathbf{\omega} = \text{curl } \mathbf{u} = \omega (r, z) \mathbf{e}_\varphi, \ p = p (r, z) \) and \( \mathbf{h}^\pm = h^\pm_r (r, z) \mathbf{e}_r + h^\pm_z (r, z) \mathbf{k} \). Let \( \lambda_{1,2} = (R + R_m \pm \sqrt{(R - R_m)^2 + 4M^2})/4 \). It is shown in Zabarankin (2011a) that (1.8) can be recast in the form of (1.1) with \( \Psi_j = p + 2\lambda_j u_z + (R - 2\lambda_j)h^-_z, \ \Omega_j = \omega + (2\lambda_j - R)(u_r - h^-_r) \mathbf{e}_\varphi \) and \( a = -\lambda_j \mathbf{k}, \ j = 1, 2 \).
(b) Generalized analytic functions

A generalized analytic (pseudoanalytic) function $G = u + iu$ with the real and imaginary parts $u = u(\xi, \eta)$ and $v = v(\xi, \eta)$, respectively, and with $i = \sqrt{-1}$ is defined by the Bers–Vekua system (Bers 1953; Vekua 1962):

\[ u'_\xi - v'_\eta + au + bv = 0 \quad \text{and} \quad u'_\eta + v'_\xi + cu + dv = 0, \]

(1.1) where $a = a(\xi, \eta)$, $b = b(\xi, \eta)$, $c = c(\xi, \eta)$ and $d = d(\xi, \eta)$ are known real-valued functions. For $a \equiv b \equiv c \equiv d \equiv 0$, (1.9) simplifies to the Cauchy–Riemann system.

The relationship (1.1) leads to several classes of generalized analytic functions. It can be readily seen that with the substitution

\[ \Omega = e^{-a x}A \quad \text{and} \quad \Psi = e^{-a x}Y, \]

(1.1) takes the form

\[ \nabla \times [a \times A] = \nabla Y, \quad \nabla \cdot A = a \cdot A, \]

(1.10) so that $\Delta A - \|a\|^2 A = 0$ and $\Delta Y - \|a\|^2 Y = 0$, where $\Delta Y = \nabla \cdot \nabla Y$. The system (1.10) defines a generalized holomorphic vector $(Y, A)$ introduced by Obolashvili (1975) (see also Liede (1990)) and is a particular case of the quaternionic equation (Kravchenko & Shapiro 1996; Kravchenko 2003) related to time-harmonic Maxwell’s equations. Next two corollaries show that under certain assumptions on the symmetry of $A$ and $Y$, the system (1.10) defines $h$-analytic and $H$-analytic functions.

**Corollary 1.5 (h-analytic functions).** If $a = -\lambda i$, where $\lambda$ is a real-valued constant, and

\[ A = u(x, y)k \quad \text{and} \quad Y = v(x, y), \]

(1.11) then (1.10) reduces to the system

\[ u'_{xx} - \lambda u = v'_{yy} \quad \text{and} \quad u'_{yy} = -v'_{xx} - \lambda v, \]

(1.12) which defines an $h$-analytic function $G = u + iv$. In this case, both $u$ and $v$ satisfy the modified Helmholtz equation $u''_{x} + u''_{y} - \lambda^2 u = 0$ and $v''_{x} + v''_{y} - \lambda^2 v = 0$.

The system (1.12) was introduced by Duffin (1971) in context of the Yukawan potential theory and also arises in the two-dimensional Oseen equations.

**Corollary 1.6 (H-analytic functions).** Let $a = -\lambda k$ and

\[ Y = u(r, z) \cos n\varphi \]

and

\[ A = -e_r v(r, z) \sin n\varphi + e_{\varphi} v(r, z) \cos n\varphi - ku(r, z) \sin n\varphi, \]

(1.13) where $\lambda$ is a real-valued constant and $n$ is a nonnegative integer. In this case, the vectorial relationship (1.10) simplifies to two equations

\[ \left( \frac{\partial}{\partial r} - \frac{n}{r} \right) u = \left( \frac{\partial}{\partial z} - \lambda \right) v \quad \text{and} \quad \left( \frac{\partial}{\partial z} + \lambda \right) u = -\left( \frac{\partial}{\partial r} + \frac{n+1}{r} \right) v, \]

(1.14) which define an $n$th-order $H$-analytic function $G = u + iv$ and imply that

\[ \Delta_n u - \lambda^2 u = 0, \quad \Delta_{n+1} v - \lambda^2 v = 0 \quad \text{and} \quad \Delta_n = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{n^2}{r^2}. \]
Obviously, (1.14) is a particular case of (1.9). Also, if \( u \) and \( v \) satisfy (1.14), then \( e^{i\lambda}r^{-n}u + ie^{-i\lambda}r^{n+1}v \) is a so-called \( p \)-analytic function (with \( p = e^{-2i\lambda}r^{2n+1} \)) introduced by Polozhii (1973) for arbitrary real-valued characteristic \( p = p(r, z) \) as yet another generalization of ordinary analytic functions.

The system (1.14) has several important cases:

(a) For \( n = 0 \) and \( \lambda = 0 \), (1.14) defines a zero-order \( r \)-analytic function and arises in axially symmetric problems of Stokes flows (Zabarankin & Krokhmal 2007; Zabarankin 2008\(^a\)) and isotropic elastic medium (Polozhii 1973; Alexandrov & Soloviev 1978).

(b) For \( \lambda = 0 \), (1.14) defines an \( n \)th-order \( r \)-analytic function\(^1\) and appears in asymmetric Stokes flow problems (Zabarankin 2008\(^b\)).

(c) For \( n = 0 \), (1.14) defines a zero-order \( H \)-analytic function and arises in axially symmetric problems of Oseen flows (Zabarankin 2010) and linearized MHD (Zabarankin 2011\(^a\)).

\(^c\) Generalized Cauchy integral formula and its application

The theories of \( p \)-analytic functions and generalized analytic functions defined by (1.9) furnish general forms for the Cauchy integral formula, which often need to be specialized and refined for particular classes of generalized analytic functions (Chemeris 1995; Kravchenko 2008; Zabarankin 2008\(^a\)). For example, the theory of \( p \)-analytic functions facilitated obtaining the Cauchy integral formula for zero-order \( H \)-analytic functions (Zabarankin 2010). However, it does not readily yield an explicit-form Cauchy kernel for \( n \)th-order \( r \)-analytic functions, which was derived via an integral representation involving ordinary analytic functions similar to the representation for zero-order \( r \)-analytic functions (Alexandrov & Soloviev 1978; Zabarankin 2008\(^a\)). Section 2 shows that the Cauchy integral formulae for \( h \)-analytic functions and \( H \)-analytic functions with particular cases of \( n = 0 \) and \( \lambda = 0 \) follow in a uniform fashion from the Cauchy integral formula for a generalized holomorphic vector defined by (1.10).

In applications, the generalized Cauchy integral formula is indispensable only if involved physical fields (e.g. velocity, pressure, vorticity, etc.) admit representations in terms of corresponding generalized analytic functions. In some cases as for an ideal fluid, the velocity \( \mathbf{u} \) is already a generalized analytic function, whereas, for example, for a two-dimensional Stokes flow, the fields \( \mathbf{u} \), \( p \) and \( \omega \) are represented by Kolosov’s formulae with two ordinary analytic functions involving their derivatives. Analogues of Kolosov’s formulae with generalized analytic functions are available for three-dimensional Stokes flows (Zabarankin 2008\(^a,b\)), axisymmetric Oseen flows (Zabarankin 2010) and axisymmetric linearized MHD (Zabarankin 2011\(^a\)). However, in contrast to Kolosov’s formulae, they contain no derivatives of the involved generalized analytic functions, which considerably simplifies obtaining closed-form solutions and deriving boundary-integral equations based on the generalized Cauchy integral formula (Zabarankin 2008\(^a,b\), 2010, 2011\(^a\)). Thus, the Cauchy integral formula and the analogues of Kolosov’s formulae for \( \mathbf{u} \), \( p \), and \( \omega \) form two essential parts of the generalized analytic function approach to linear hydrodynamics.

\(^1\)This function is called \( n \)-harmonically analytic in Zabarankin (2008\(^a,b\)).

Section 3 demonstrates this approach in problems of three-dimensional Stokes flows, two-phase axisymmetric Stokes flows, two-dimensional Oseen flows and axisymmetric Oseen flows.

2. Generalized Cauchy integral formula

This section shows that for several classes of generalized analytic functions, the Cauchy integral formulae follow from the one for generalized holomorphic vectors defined by (1.10). The next theorem is the vector-form restatement of the matrix-form Cauchy integral formula for generalized holomorphic vectors given by (8) and (16) in Obolashvili (1975).

**Theorem 2.1 (Cauchy integral formula for generalized holomorphic vectors).** Let $D$ be a bounded open region in $\mathbb{R}^3$ with a piecewise smooth boundary $\partial D$, and let $\Gamma$ and $A$ satisfy (1.10) in $D$, be continuously differentiable in $D$ and continuous in $D \cup \partial D$. Then $\Gamma$ and $A$ can be represented in $D$ via their boundary values by

\[
\Gamma(x) = -\int_{\partial D} \left((\Gamma(y) n_y + [n_y \times A(y)]) \cdot (\nabla_y - a)\Phi(y-x)\right) dS(y) \tag{2.1}
\]

and

\[
A(x) = -\int_{\partial D} \left((\Gamma(y) n_y + [n_y \times A(y)]) \times (\nabla_y + a)\Phi(y-x)\right)
+ (n_y \cdot A(y)) (\nabla_y - a)\Phi(y-x)) dS(y), \tag{2.2}
\]

where $dS(y)$ is the surface area element, $n_y$ is the outward normal of $\partial D$ at point $y$, $\nabla_y$ is the gradient with respect to $y$, $(\nabla_y \pm a)\Phi(y-x) \equiv \nabla_y \Phi(y-x) \pm \Phi(y-x)a$, and

\[
\Phi(y-x) = \begin{cases} 
\frac{e^{-\|a\| \|y-x\|}}{4\pi \|y-x\|} & \text{in the three-dimensional case,} \\
\frac{1}{2\pi} K_0(\|a\| \|y-x\|) & \text{in the two-dimensional case}
\end{cases}
\]

is the fundamental solution of the modified Helmholtz equation\(^2\) with $K_m(\cdot)$ being the $m$th-order modified Bessel function of the second kind.\(^3\) If $a = 0$, then in the two-dimensional case, $\Phi(y-x) = -1/(2\pi) \ln \|y-x\|$.

For brevity, let $I_1(\Gamma, A)$ and $I_2(\Gamma, A)$ denote the right-hand sides of (2.1) and (2.2), respectively. Two remarks are in order.

**Remark 2.2.** Let $D^-$ be the complement of $D \cup \partial D$ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) and let $\{\Gamma, A\}$ be a generalized holomorphic vector continuously differentiable in $D^-$ and continuous in $D^- \cup \partial D$. Then the Cauchy integral formula (2.1) and (2.2) takes the form: $\Gamma(x) = -I_1(\Gamma, A)$, $x \in D^-$ and $A(x) = -I_2(\Gamma, A)$, $x \in D^-$, provided that $\Gamma$ and $A$ vanish at infinity (Liede 1990).

\(^2\)The fundamental solution $\Phi(x)$ solves $(\Delta - \|a\|^2)\Phi(x) = -\delta(0)$, where $\delta(\cdot)$ is the Dirac delta function.

\(^3\)In the two-dimensional case, $D$ is a region in the $xy$-plane, and $\Gamma$ and $A$ are represented by (1.11) with $a$ being a constant vector parallel to the $z$-axis.
Remark 2.3. Let $f$ and $g$ be scalar and vector functions, respectively, that are continuous on $\partial D$. Then $I_1(f, g)$ and $I_2(f, g)$ determine a generalized holomorphic vector for $x \notin \partial D$. If $f$ and $g$ are Hölder continuous\(^1\) on $\partial D$ and $x_0 \in \partial D$, then the limits of $I_1(f, g)$ and $I_2(f, g)$ as $x = x^+$ and $x = x^-$ approach $x_0$ from inside and outside $D$, respectively, are determined by $\lim_{x \to x_0^+} I_1(f, g) = \pm \frac{1}{2}f(x_0) + I_1(f, g)|_{x=x_0}$ and $\lim_{x \to x_0^-} I_2(f, g) = \pm \frac{1}{2}g(x_0) + I_2(f, g)|_{x=x_0}$, which are analogues of the Sokhotski–Plemelj formulae (Obolashvili 1975, eqn 19).

Corollary 2.4 (Ideal fluid). The equations (1.2) governing the velocity field $u$ of an ideal fluid correspond to (1.10) with $\gamma = 0$, $A = u$, and $a = 0$, so that for (1.2) in the three-dimensional case, (2.2) simplifies to the formula (Mises 1944; Bitsadze 1969; Morgunov 1974)

$$u(x) = \frac{1}{4\pi} \int_{\partial D} \left( \frac{n_y \cdot (y-x)}{||y-x||^3} u(y) + \frac{n_y \times (y-x)}{||y-x||^3} \times u(y) \right) dS(y),$$

whereas in the two-dimensional case, the components $u_x$ and $-u_y$ form an ordinary analytic function $u_x - iu_y$, and the linear combination (2.2) $i - i(2.2)j$ with $\Phi(y - x) = -1/(2\pi) \ln ||y - x||$ reduces to the ordinary Cauchy integral formula.

Corollary 2.5 (Stokes flow). The equations (1.4) that relate the pressure $p$ and vorticity $\omega$ in a Stokes (creeping) flow correspond to (1.10) with $\gamma = 0$, $A = \mu \omega$ and $a = 0$. Thus, for (1.4), the Cauchy integral formula (2.1)–(2.2) yields

$$p(x) = \frac{1}{4\pi} \int_{\partial D} (p(y) n_y + \mu [n_y \times \omega(y)]) \cdot \frac{y-x}{||y-x||^3} dS(y)$$

and

$$\omega(x) = \frac{1}{4\pi\mu} \int_{\partial D} \frac{p(y) [n_y \times (y-x)] + \mu [\omega(y), y-x, n_y]}{||y-x||^3} dS(y),$$

where $[a, b, c] = (b \cdot c)a + (a \cdot b)b - (a \cdot b)c$ is the triple product of vectors $a$, $b$ and $c$. These formulae are a vector-form restatement of the matrix-form Cauchy integral formula for the generalized analytic functions defined by the Moisil–Theodorescu system (Bitsadze 1969).

The Cauchy integral formula for $h$-analytic functions defined by (1.12) is stated in theorem 18 in Duffin (1971) for $\lambda > 0$. The next corollary shows that it follows from (2.1) and (2.2) and holds for positive and negative $\lambda$.

Corollary 2.6 (Cauchy integral formula for $h$-analytic functions). Let $D$ be a bounded open region in the $xy$-plane with a piecewise smooth positively oriented boundary $\ell$, and let $G$ be an $h$-analytic function in $D$ and Hölder continuous on $\ell$, then

$$G(\zeta) = \mathcal{C}_h(G; \ell) \equiv \frac{1}{2\pi i} \oint_{\ell} G(\tau) \frac{\sigma K_1(\sigma)}{\tau - \zeta} d\tau - \lambda G(\tau) K_0(\sigma) d\tau, \quad \zeta \in D,$$

where $\mathcal{C}_h$ denotes the Cauchy operator for $h$-analytic functions, $\zeta = x + iy$, $\tau = x_1 + iy_1$, and $\sigma = |\lambda| |\tau - \zeta|$. For $\lambda \to 0$, (2.4) reduces to the Cauchy integral formula for ordinary analytic functions.

\(^{1}\)A function $f(x)$ is Hölder continuous on $\partial D$, if $|f(x_2) - f(x_1)| \leq c \|x_2 - x_1\|^\alpha$ for any $x_1, x_2 \in \partial D$, some $\alpha \in (0, 1]$ and non-negative constant $c$. 

Generalized Cauchy integral formula

**Detail.** Let $A$ and $Y$ be represented by (1.11) with $a = -\lambda i$, where $\lambda$ is a real-valued constant, and let $x = (x, y)$ and $y = (x_1, y_1)$ be vectors in the $xy$-plane. In this case, the formula (2.2), projected onto $k$, and the formula (2.1) simplify to

\[
u(x) = A \cdot k = -\int_{Y} (v(y)[k \times n_Y] + u(y) n_y) \cdot (\nabla y - i\lambda \Phi(y - x)ds(y) \tag{2.5}
\]

and

\[
u(x) = Y = -\int_{Y} (v(y)n_y - u(y)[k \times n_y]) \cdot (\nabla_y + i\lambda \Phi(y - x)ds(y). \tag{2.6}
\]

respectively, where $\Phi(y - x) = K_0(|\lambda| \tau - \zeta|)/(2\pi)$ and $ds(y)$ is the curve length element. Let $G(\zeta) = u(x) + iv(x)$. Then with $u(x) = (G(\zeta) + G(\zeta))/2$ and $v(x) = (G(\zeta) - G(\zeta))/2i$, the expression (2.5) + i(2.6) reduces to

\[
G(\zeta) = -\int_{Y} (G(\tau)(n_y - i[k \times n_y] \cdot \nabla y \Phi(y - x)
- \lambda G(\tau)(n_y + i[k \times n_y]) \cdot i\Phi(y - x)ds(y). \tag{2.7}
\]

For positively oriented $\ell$ with the outward normal $n_y$, we have $n_y ds(y) = i dy_1 - j dx_1$ and

\[
(n_y \mp i[k \times n_y])ds(y) = (\mp i + j)(dr_1 \pm i dz_1). \tag{2.8}
\]

Now, with the identity

\[
\nabla \Phi(y - x) = -\frac{|\lambda|}{2\pi} \frac{(|x_1 - x| i + (y_1 - y)j)}{|\tau - \zeta|} K_1(|\lambda| \tau - \zeta|)
\]

and the relationship (2.8), the representation (2.7) takes the form of (2.4).

Finally, the limits $\lim_{\ell \to 0} \epsilon K_0(\epsilon) = 0$ and $\lim_{\ell \to 0} \epsilon K_1(\epsilon) = 1$ imply that (2.4) reduces to the Cauchy integral formula for ordinary analytic functions as $\lambda \to 0$.

**Theorem 2.7 (Cauchy integral formula for $H$-analytic functions).** Let $D$ be a bounded open region in the rz-half plane (in the cylindrical coordinates $(r, \varphi, z)$, $r \geq 0$) with a piecewise smooth positively oriented boundary $\ell$, which is either closed or an open curve with the endpoints lying on the z-axis (if $D$ contains a segment of the z-axis). Also, let $G$ be an nth-order $H$-analytic function in $D$ and Hölder continuous on $\ell$. The Cauchy integral formula for $G$ is given by

\[
G(\zeta) = \mathcal{C}(G; \ell) \equiv \frac{1}{2\pi i} \int_{Y} \left( G(\tau) \frac{\Omega_+(\zeta, \tau)}{\tau - \zeta} d\tau - G(\tau) \frac{\Omega_-(\zeta, \tau)}{\tau + \zeta} d\tau \right), \quad \zeta \in D, \tag{2.9}
\]

where $\mathcal{C}$ is the generalized Cauchy operator, $\zeta = r + iz$, $\tau = r_1 + iz_1$ and $\Omega_+(\zeta, \tau)$ and $\Omega_-(\zeta, \tau)$ are determined by

\[
\Omega_\pm(\zeta, \tau) = 2(-1)^n r_1 \int_0^{\pi/2} L_\pm(\zeta, \tau, t) e^{-|\lambda|2\zeta} dt, \tag{2.10}
\]

with

\[
L_\pm(\zeta, \tau, t) = \left( \frac{2n + 1 + i\lambda(\tau - \zeta)}{g(\zeta, \tau, t)} \right) \cos t \cos(2n + 1)t + |\lambda| \sin t \sin(2n + 1)t \bigg| \\
L_{-}(\zeta, \tau, t) = \left( \frac{2n + 1 - i\lambda(\tau + \zeta)}{g(\zeta, \tau, t)} \right) \sin t \sin(2n + 1)t + |\lambda| \cos t \cos(2n + 1)t \bigg| \\
and \quad g(\zeta, \tau, t) = \sqrt{(r_1 + r)^2 + (z_1 - z)^2 - 4r_1r \sin^2 t}.
\]

\[ (2.11) \]

\textbf{Proof.} Let \( D \) be the axially symmetric region obtained by revolving \( \mathcal{D} \) around the \( z \)-axis in the cylindrical coordinate system \((r, \varphi, z)\). For \( n \)-th-order \( H \)-analytic functions, \( A \) and \( Y \) are represented by (1.13) with \( a = -\lambda k \), where \( \lambda \) is a real-valued constant. Since \( u(r, z) \) and \( v(r, z) \) in (1.13) do not depend on the angular coordinate \( \varphi \), the formulae (2.1) and (2.2) can be considered in the \( rz \)-half plane corresponding to \( \varphi = 0 \). In this case, \( x = (r e_r + z e_z)|_{\varphi=0} = ri + zk \), and the coordinate \( \varphi \) will be used to describe the vector \( y \) as \( y = r_1 \cos \varphi i + r_1 \sin \varphi j + z_1 k \), so that

\[
y - x = (r_1 \cos \varphi - r)i + r_1 \sin \varphi j + (z_1 - z)k. \quad (2.12)
\]

Then in the identified \( rz \)-half plane, (2.1) and (2.2) projected onto \( j \) take the form

\[
u(r, z) = -\int_{0}^{2\pi} \int_{\ell} (v(r_1, z_1)[(i \sin(n + 1)\varphi - j \cos(n + 1)\varphi) \times n_y]) \\
\quad + u(r_1, z_1)(n_y \cos n\varphi + [k \times n_y] \sin n\varphi)) \cdot (\nabla_y + \lambda k) \Phi r_1 \, ds \, d\varphi \quad (2.13)
\]

and

\[
v(r, z) = A(x) \cdot j \\
\quad = -\int_{0}^{2\pi} \int_{\ell} (v(r_1, z_1)(n_y \cos(n + 1)\varphi - [k \times n_y] \sin(n + 1)\varphi) \\
\quad + u(r_1, z_1)([i \sin n\varphi + j \cos n\varphi] \times n_y)) \cdot (\nabla_y + \lambda k) \Phi r_1 \, ds \, d\varphi, \quad (2.14)
\]

respectively, where \( \Phi \equiv \Phi(y - x) = \Phi(r, z_1, r_1, z_1, \varphi) \), \( dS(y) = r_1 \, ds \, d\varphi \) and \( ds = ds(r_1, z_1) \) is the curve length element.

Let \( G(\zeta) = u(r, z) + iu(r, z) \), so that \( u(r, z) = (G(\zeta) + G(\zeta))/2 \) and \( u(r, z) = (G(\zeta) - G(\zeta))/(2i) \). In this case,

\[
\nabla_y \Phi = -\frac{y - x}{4\pi} \left( \frac{1}{||y - x||^3} + \frac{|\lambda|}{||y - x||^2} \right) e^{-|\lambda||y - x||},
\]

where \( y - x \) is given by (2.12). Then, with the relationship \( n_y \, ds = (i \cos \varphi + j \sin \varphi) d\zeta_1 - k \, dr_1 \), which holds for the outward normal \( n_y \) and positively oriented \( \ell \), and with the identity

\[
(y - x) \cdot ((\cos n\varphi \pm \cos(n + 1)\varphi)(i + k) \pm i(\sin(n + 1)\varphi \mp \sin n\varphi)j) \\
\quad = \pm i(r_1 \mp i\zeta_1 \mp \zeta)(\cos n\varphi \pm \cos(n + 1)\varphi),
\]

the combination (2.13)+i(2.14) simplifies to

$$G(\zeta) = \frac{1}{8\pi i} \int_0^{2\pi} r_1(G(\tau) \Xi_+(\zeta, \tau, \varphi) d\tau - \overline{G(\tau)} \Xi_-(\zeta, \tau, \varphi) d\tau) d\varphi, \tag{2.15}$$

where

$$\Xi_\pm(\zeta, \tau, \varphi) = \left( r_1 \mp i z_1 \mp \bar{\zeta} \right) \left( \cos n\varphi \pm \cos(n+1)\varphi \right) \left( \frac{1}{\|y - x\|^3 + \|y - x\|^2} + \frac{|\lambda|}{\|y - x\|^2} \right)$$

$$\pm i\lambda \frac{\cos n\varphi \mp \cos(n+1)\varphi}{\|y - x\|} e^{-|\lambda| \|y - x\|}.$$ 

Changing the variable $\varphi = \pi - 2t$ and using $\varrho(\zeta, \tau, t)$ in (2.11), we have

$$\int_0^{2\pi} (\cos n\varphi + \cos(n+1)\varphi) \left( \frac{1}{\|y - x\|^3 + \|y - x\|^2} + \frac{|\lambda|}{\|y - x\|^2} \right) e^{-|\lambda| \|y - x\|} d\varphi$$

$$= 8(-1)^n \int_0^{\pi/2} \sin t \sin(2n+1)t \left( \frac{1}{\varrho^3(\zeta, \tau, t)} + \frac{|\lambda|}{\varrho^2(\zeta, \tau, t)} \right) e^{-|\lambda| \varrho(\zeta, \tau, t)} dt$$

$$= 8(-1)^n \int_0^{\pi/2} \left( \frac{2n+1}{\varrho^2(\zeta, \tau, t)} e^{-|\lambda| \sin t \sin(2n+1)t} \right)$$

$$\times e^{-|\lambda| \varrho(\zeta, \tau, t)} dt \tag{2.16}$$

and

$$\int_0^{2\pi} (\cos n\varphi - \cos(n+1)\varphi) \left( \frac{1}{\|y - x\|^3 + \|y - x\|^2} + \frac{|\lambda|}{\|y - x\|^2} \right) e^{-|\lambda| \|y - x\|} d\varphi$$

$$= 8(-1)^n \int_0^{\pi/2} \cos t \cos(2n+1)t \left( \frac{1}{\varrho^3(\zeta, \tau, t)} + \frac{|\lambda|}{\varrho^2(\zeta, \tau, t)} \right) e^{-|\lambda| \varrho(\zeta, \tau, t)} dt$$

$$= 8(-1)^n \int_0^{\pi/2} \left( \frac{2n+1}{\varrho^2(\zeta, \tau, t)} e^{-|\lambda| \cos t \cos(2n+1)t} \right)$$

$$\times e^{-|\lambda| \varrho(\zeta, \tau, t)} dt, \tag{2.17}$$

where the middle integrals in (2.16) and (2.17) are integrated by parts with the relationships

$$\frac{\sin t}{\varrho^3(\zeta, \tau, t)} = -\frac{1}{|\tau - \zeta|^2} \frac{d}{dt} \left( \frac{\cos t}{\varrho(\zeta, \tau, t)} \right)$$

and

$$\frac{\cos t}{\varrho^3(\zeta, \tau, t)} = \frac{1}{|\tau + \zeta|^2} \frac{d}{dt} \left( \frac{\sin t}{\varrho(\zeta, \tau, t)} \right),$$

respectively. Now, (2.15) with (2.16) and (2.17) yields (2.9)–(2.11).

The generalized Cauchy integral formula (2.9)–(2.11) extends the one for zero-order $H$-analytic functions derived in Zabarankin (2010, theorem 1) via the theory of $p$-analytic functions (Polozhii 1973), whereas the following result was obtained in Zabarankin (2008a) based on an integral representation of $n$th-order $r$-analytic functions through ordinary analytic functions.
Corollary 2.8 (Cauchy integral formula for nth-order r-analytic functions). For \( \lambda = 0 \), the kernels (2.10) become real-valued functions determined by

\[
\Omega_\pm(\zeta, \tau) = 2r_1 \frac{\Gamma(n+3/2)^2 \kappa^{2n}(\zeta, \tau)}{(2n+1)!} \left[ n + \frac{1}{2}, n + 1 \mp \frac{1}{2}, 2(n+1), \kappa^2(\zeta, \tau) \right],
\]

where \( \Gamma(\cdot) \) is the gamma function, \( \mathbb{F}(a, b, c) = \Gamma(c)/(\Gamma(b)\Gamma(c-b)) \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-kt)^{-d} \, dt \) is the hypergeometric function, and \( \kappa^2(\zeta, \tau) = 4r_1r|\zeta + \bar{\tau}|^{-2} \).

Detail. With \( g(\zeta, \tau, t) = |\tau + \bar{\zeta}| \sqrt{1 - \kappa^2(\zeta, \tau)^2} \sin^2 t \), the formula (2.18) follows from the relationships

\[
I_1 = \int_0^{\pi/2} \cos t \cos(2n+1)t \, dt = 2^{n+1}(1-n)! \mathbb{F} \left[ n + \frac{1}{2}, n + 1, 2(n+1), \kappa^2 \right],
\]

and

\[
I_2 = \int_0^{\pi/2} \sin t \sin(2n+1)t \, dt = 2^{n+1}(1-n)! \mathbb{F} \left[ n + \frac{1}{2}, n + 1, 2(n+1), \kappa^2 \right],
\]

see the appendix A, and the fact that \( \Gamma(n+3/2) = \sqrt{\pi}(2n+1)!/(2^{n+1}n!) \).

Corollary 2.9 (Cauchy integral formula for zero-order r-analytic functions). The case of \( \lambda = 0 \) and \( n = 0 \) in (1.14) corresponds to zero-order r-analytic functions, for which (2.18) simplifies to

\[
\Omega_+(\zeta, \tau) = \frac{|\tau + \bar{\zeta}|}{2r} \{ \mathbb{E}(\kappa(\zeta, \tau)) - (1 - \kappa^2(\zeta, \tau))\mathbb{K}(\kappa(\zeta, \tau)) \}
\]

and

\[
\Omega_-(\zeta, \tau) = \frac{|\tau + \bar{\zeta}|}{2r} \{ \mathbb{K}(\kappa(\zeta, \tau)) - \mathbb{E}(\kappa(\zeta, \tau)) \},
\]

where \( \mathbb{K}(\kappa) = \int_0^{\pi/2}(1 - \kappa^2 \sin^2 t)^{-1/2} \, dt \) and \( \mathbb{E}(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 t} \, dt \) are complete elliptic integrals of the first and second kinds, respectively. The function \( \mathbb{K}(\kappa) \) has a logarithmic singularity as \( \kappa \to 1^- : \mathbb{K}(\kappa) = -(\ln(1-\kappa^2))/2 + O(1) \).

Corollary 2.10 (Cauchy integral formula for first-order r-analytic functions). The case of \( \lambda = 0 \) and \( n = 1 \) in (1.14) corresponds to first-order r-analytic functions, for which (2.18) simplifies to

\[
\Omega_+(\zeta, \tau) = \frac{|\tau + \bar{\zeta}|}{2r} \{ (8 - 7\kappa^2)\mathbb{E}(\kappa) - (8 - 3\kappa^2)(1 - \kappa^2)\mathbb{K}(\kappa) \}
\]

and

\[
\Omega_-(\zeta, \tau) = \frac{|\tau + \bar{\zeta}|}{2r} \{ (\kappa^2 - 8)\mathbb{E}(\kappa) + (8 - 5\kappa^2)\mathbb{K}(\kappa) \},
\]

where \( \kappa^2 = 4r_1r|\zeta + \bar{\tau}|^{-2} \).
3. Application to linear hydrodynamics

This section demonstrates the approach of generalized analytic functions to problems of linear hydrodynamics. Advantages of this approach are in the convenience of representations of hydrodynamic fields (velocity, vorticity and pressure) and key characteristics (drag, torque and lift) in terms of generalized analytic functions and in the simplicity of obtaining closed-form solutions and boundary-integral equations via the generalized Cauchy integral formulae; compare to the methods, e.g. in Pozrikidis (1992) and Bardzokas et al. (2007).

For clarity, \( C_0, C_1 \) and \( C_H \) will denote the Cauchy operator \( C \) in (2.9) for zero-order \( r \)-analytic, first-order \( r \)-analytic and zero-order \( H \)-analytic functions, respectively, whereas \( C_h \) is the Cauchy operator in (2.4) for \( h \)-analytic functions.

(a) Ideal fluid

In the two-dimensional case, equations (1.2) for the velocity field \( \mathbf{u} = u_x(x, y)\mathbf{i} + u_y(x, y)\mathbf{j} \) of an ideal fluid reduce to the Cauchy–Riemann system for \( u_x \) and \(-u_y\), so that \( g = u_x - iu_y \) is an ordinary analytic function of a complex variable \( \zeta = x + iy \). If a solid infinitely long airfoil is aligned with the \( z \)-axis and is immersed into a uniform flow \( u_\infty^x \mathbf{i} + u_\infty^y \mathbf{j} \), then \( \lim_{\zeta \to \infty} g(\zeta) = u_\infty^x - iu_\infty^y \equiv v_\infty \) and \( \mathbf{u} \cdot \mathbf{n} \equiv \text{Re}[g\partial\zeta/\partial n] = 0 \) on the boundary \( \ell \) of the airfoil cross section in the \( xy \)-plane, where \( \mathbf{n} \) is the outward normal for \( \ell \). In this case, \( F_x + iF_y = -(i\rho/2) \oint_{\ell} \overline{g'(\zeta)}\,d\zeta \) is the airfoil lifting force (Blasius–Chaplygin formula), where \( \rho \) is the fluid density. It simplifies to the Kutta–Joukowski formula \( F_x + iF_y = -i\rho v_\infty \oint_{\ell} g(\zeta)\,d\zeta \).

In the three-dimensional case, \( \mathbf{u} \) satisfying (1.2) is itself a generalized analytic function for which the Cauchy integral formula is given by (2.3); see Mises (1944) and Morgunov (1974) for application of (2.3) to the ideal fluid. In particular, in an axisymmetric flow with the \( z \)-axis of revolution, \( u_z \) and \( u_r \) are independent of the angular coordinate \( \phi \) and form a zero-order \( r \)-analytic function \( G = u_z + iu_r \) with \( \text{Im}[G\partial\zeta/\partial n] = 0 \) on any fixed solid boundary, where \( \zeta = r + iz \).

(b) Stokes flows

Under the zero Reynolds number assumption, the time-independent velocity field \( \mathbf{u} \) and pressure \( p \) of a viscous incompressible fluid are governed by the Stokes equations (1.3). The relationship (1.4) plays a pivotal role in constructing solution forms for (1.3) in terms of generalized analytic functions; see example 1.2.

(i) Two-dimensional case

In the two-dimensional case, \( \mathbf{u} \) and \( p \) depend on the Cartesian coordinates \( x \) and \( y \) only, i.e. \( \mathbf{u} = u_x(x, y)\mathbf{i} + u_y(x, y)\mathbf{j} \), \( \mathbf{\omega} = \omega(x, y)\mathbf{k} \) and \( p = p(x, y) \), and consequently, the system (1.4) implies that \( p/\mu - i\omega \) is an ordinary analytic function. In this case, \( u_x + iu_y = g_1(\zeta) - \zeta \overline{g_1'(\zeta)} - \overline{g_2(\zeta)} \) and \( \frac{p}{\mu} - i\omega = -4\overline{g_1'(\zeta)} \),

\[
(3.1)
\]

where \( \zeta = x + iy \) is a complex variable, and \( g_1(\zeta) \) and \( g_2(\zeta) \) are analytic functions; see (4.1) and (4.2) in Richardson (1995). In fact, the representation (3.1) is Kolosov’s formulae (Kolosov 1909) for an incompressible elastic medium (with Poisson’s ratio 1/2). It is used to reduce two-dimensional Stokes flow problems.
to boundary-integral equations via the Cauchy integral formula (Muskhelishvili 1977, 1992). If a solid infinitely long cylinder of arbitrary cross section is aligned with the z-axis, then \( F_x + i F_y = -4\mu i \int_\ell g_1(\xi)d\xi \) is the total flow reaction force per unit length of cylinder’s span, where \( \ell \) is the cross section’s boundary in the \( xy \)-plane. However, the problem of unbounded two-dimensional Stokes flow with solid obstacles has no solution bounded at infinity (Happel & Brenner 1983). This well-known paradox is resolved by the Oseen approximation of the Navier–Stokes equations.

(ii) Three-dimensional axially symmetric flows

If a flow is axisymmetric with the \( z \)-axis of revolution, then in the cylindrical coordinates \((r, \varphi, z)\), \( u \) and \( p \) are independent of the angular coordinate \( \varphi \): \( u = u_r(r, z) \mathbf{e}_r + u_z(r, z) \mathbf{e}_z \), \( u_\varphi = 0 \), \( p = p(r, z) \) and \( \mathbf{\omega} = \mathbf{\omega}(r, z) \mathbf{e}_\varphi \) \((\mathbf{\omega} = \mathbf{e}_\varphi \cdot \text{curl} \, \mathbf{u})\). In this case, a complex variable is introduced by \( \zeta = r + iz \), and proposition 7 in Zabarankin (2008a) states that

\[
 u_z + i u_r = \left(z - \frac{ir}{2}\right) G_1 + G_2 \quad \text{and} \quad \frac{p}{\mu} + i\omega = G_1, 
\]

where \( G_1 \) and \( G_2 \) are zero-order \( r \)-analytic functions. The representation (3.2) is a three-dimensional analogue of (3.1), but in contrast to the latter, it involves no derivatives of \( G_1 \) and \( G_2 \).

Suppose a solid axisymmetric finite body with the \( z \)-axis of revolution translates in the quiescent fluid at constant velocity \( v \mathbf{k} \). Then \( \mathbf{u} = v_z \mathbf{k} \) on body’s surface \( S \) (no-slip boundary condition), and \( u \) and \( p \) vanish at infinity. Let open regions \( D^+ \) and \( D^- \) be the interior and exterior of the body’s cross section in the \( rz \)-half plane \((r \geq 0)\) with common positively oriented boundary \( \ell \) (cross section of \( S \)). Then (3.2) implies that \((z - ir/2) G_1 + G_2 = v_z \) on \( \ell \) and that \( G_1 \) and \( G_2 \) vanish at infinity.\(^{5}\) If \( G_1 \) on \( \ell \) is known, then \( G_2 = v_z - (z - ir/2) G_1 \) on \( \ell \), and with the Cauchy integral formula (2.9) for zero-order \( r \)-analytic functions (see corollary 2.9), (3.2) yields representations for \( u \), \( p \) and \( \omega \) in \( D^- \):

\[
 u_z + i u_r = \mathcal{A}(G_1; \ell) \quad \text{and} \quad \frac{p}{\mu} + i\omega = -\mathcal{C}_0(G_1; \ell), \quad \zeta \in D^-, \]

where the operator \( \mathcal{A} \) is determined by

\[
 \mathcal{A}(G_1; \ell) = \mathcal{C}_0 \left( \left(z - \frac{ir}{2}\right) G_1; \ell\right) - \left(z - \frac{ir}{2}\right) \mathcal{C}_0(G_1; \ell). \quad (3.3)
\]

Note that \( \mathcal{A} \) has no Cauchy-type singularity on \( \ell \), and thus, \( \mathcal{A}(G_1; \ell) \) is continuous as \( \zeta \) approaches \( \ell \) from within \( D^- \). This fact and the boundary condition \( u_z + i u_r = v_z \) on \( \ell \) imply that \( G_1 \) on \( \ell \) satisfies the boundary-integral equation

\[
 \mathcal{A}(G_1; \ell) = v_z, \quad \zeta \in \ell. \quad (3.4)
\]

\(^{5}\) A zero-order \( r \)-analytic function \( G \) vanishing at infinity behaves as \( G = o(|\zeta|^{-1}) \) as \(|\zeta| \to \infty \) (Zabarankin 2008a). Since the second equation in (3.2) implies that \( G_1 \) vanishes at infinity, i.e. \( G_1 = o(|\zeta|^{-1}) \) as \(|\zeta| \to \infty \), the first one in (3.2) yields \( G_2 = o(1) \) as \(|\zeta| \to \infty \).

Theorem 10 in Zabarankin (2008a) proves that a homogeneous solution of (3.4) is any real constant and that 
\[ G_1 = w - c, \quad \zeta \in \ell, \] where \( w \) is a solution of (3.4) and 
\( c \) is the real constant determined by 
\[ c = w(\zeta)/2 + c_0(w; \ell), \quad \zeta \in \ell. \]

The operator (3.3) has only a logarithmic-type singularity, and (3.4) is solved as follows. Let \( \ell \) be parametrized by \( \zeta = \zeta(t), \quad t \in [t_1, t_2] \), then \( G_1 \) on \( \ell \) is approximated by a finite functional series 
\[ G_1(t) = v_z \sum_{k=1}^{m} a_k \phi_{1k}(t) + i b_k \phi_{2k}(t), \quad t \in [t_1, t_2], \] with basis functions \( \{\phi_{1k}(t)\}_{k=1}^{m}, \quad \{\phi_{2k}(t)\}_{k=1}^{m} \) for null \( \phi_{1k}(t) \) is constant, and coefficients \( a_k, b_k \) are found by minimizing the total quadratic error on \([t_1, t_2]:\)
\[ \min_{a_k, b_k} \left\| \sum_{k=1}^{m} a_k \mathcal{A}(\phi_{1k}; \ell) + b_k \mathcal{A}(i\phi_{2k}; \ell) - 1 \right\|^2. \]

The representation (3.2) and the boundary condition \( u_z + au_r = v_z \) on \( \ell \) yield \( \partial u_z/\partial n + i \partial u_r/\partial n = -(\partial \zeta/\partial n)\text{Im} \ G_1, \quad \zeta \in \ell, \) and proposition 11 in Zabarankin (2008a) shows that the drag exerted on the body is given by
\[ F_z = 2\pi \mu \text{Re} \left[ \int_{\ell} r G_1(\zeta) d\zeta \right]. \]

Note that \( F_z \) is unaffected if \( G_1 \) is added a real constant; so \( F_z \) holds for any solution of (3.4). For example, if in the \( rz \)-half plane, a sphere of radius \( a \) is parametrized by \( \zeta(t) = ae^{\pi i t^2}, \quad t \in [-1, 1] \), then \( G_1(t) = -3v_z i/(2a)e^{-\pi i t^2}, \quad t \in [-1, 1] \) and \( F_z = -6\pi \mu v_z a \), which is the well-known Stokes formula for the sphere drag. Zabarankin (2008a) showed that for prolate and oblate spheroids, biconvex lens and torus of circular cross section, solutions of the boundary-integral equation (3.4) coincide with corresponding analytical solutions in Happel & Brenner (1983), Zabarankin & Krokhmal (2007) and Zabarankin & Ulitko (2006). Also, Zabarankin (2008a) solved (3.4) for solid bi-spheroids (two separate spheroids of equal size) and torus of elliptical cross section, whereas Zabarankin & Molyboha (2010) used (3.4) to find minimum-drag shapes for solid bodies of revolution subject to constraints on body’s volume and body’s shape.

(iii) Two-phase three-dimensional axially symmetric flow

Suppose an initially spherical liquid drop with radius \( a \) is placed into an extensional flow \( u^\infty = \zeta(-re_r + 2z k) \) at the origin \( (r = z = 0) \), where \( \zeta \) is the share rate, and suppose \( D^+ \) and \( D^- \) are the regions occupied by the deformed drop and the ambient fluid, respectively, with common boundary \( S \). Let \( u^+ \) and \( u^- \) be the actual velocity in the drop and the velocity disturbance of the extensional flow, respectively, and let \( p^\pm \) be the pressure in \( D^\pm \). It is assumed that the drop and the ambient fluid are incompressible and viscous with the same viscosity \( \mu \) and that for fixed \( S, u^\pm \) and \( p^\pm \) satisfy the Stokes equations (1.3) in \( D^\pm \). For convenience, let the linear dimensions, \( u^\pm, \) and \( p^\pm \) be rescaled by \( a, a \zeta \) and \( \mu, \zeta \), respectively, and let \( Ca = \mu a \zeta^2/\gamma \) be the capillary number, where \( \gamma \) is the interfacial tension.

This problem is axisymmetric with the \( z \)-axis of revolution, and the boundary conditions on \( S \) are given by
\[ u^+ = u^- + u^\infty, \quad p^+ - p^- = Ca^{-1} \text{div} \, n \quad \text{and} \quad \omega^+ = \omega^-, \tag{3.5} \]
where \( n \) is the outward normal for \( S \) and \( \omega^\pm = e_\phi \cdot \text{curl} \, u^\pm \) (Zabarankin & Nir 2011, proposition 3.1). Also, \( u^- \) and \( p^- \) vanish at infinity.
At the steady state, \( \mathbf{u}^+ \cdot \mathbf{n} = 0 \) on \( S \) (kinematic condition), and in this case, \( \mathbf{u}^\pm, \omega^\pm \) and \( p^\pm \) are time-independent and have a representation similar to that of (3.2):

\[
\begin{align*}
\mathbf{u}^\pm + i\mathbf{u}^\mp &= \left( z - \frac{ir}{2} \right) C^\pm_1 + C^\pm_2 \\
p^+ &= \text{Re } C^+_1 + c, \quad p^- = \text{Re } C^-_1 \quad \text{and} \quad \omega^\pm = \text{Im } C^\pm_1,
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{u}^\pm + i\mathbf{u}^\mp &= f^\pm - \text{Ca}^{-1} \mathcal{A} \left( \nabla \mathbf{n} ; \ell \right), \quad \zeta \in \mathcal{D}^\pm \cup \ell, \\
p^\pm + i\omega^\pm &= \text{Ca}^{-1} \mathcal{E}_0 \left( \nabla \mathbf{n} ; \ell \right), \quad \zeta \in \mathcal{D}^\pm
\end{align*}
\]

where \( C^\pm_i(\zeta) \) and \( C^\pm_i(\zeta) \) are zero-order \( r \)-analytic functions in \( \mathcal{D}^\pm \) with \( \zeta = r + iz \) and with \( C^-_1 \) and \( C^-_2 \) vanishing at infinity, and \( c \) is a real constant.

Let open regions \( \mathcal{D}^\pm \) be the interior of cross sections of \( \mathcal{D}^\pm \) in the \( rz \)-half plane, and let \( \ell \) be common positively oriented smooth boundary of \( \mathcal{D}^\pm \). With the Cauchy integral formula for zero-order \( r \)-analytic functions (corollary 2.9), the representations (3.6) and (3.7) and the boundary conditions (3.5) yield a closed-form solution for \( \mathbf{u}^\pm, p^\pm \) and \( \omega^\pm \) (Zabarankin & Nir 2011, theorem 3.3):

\[
\begin{align*}
\mathbf{u}^\pm + i\mathbf{u}^\mp &= f^\pm - \text{Ca}^{-1} \mathcal{A} \left( \nabla \mathbf{n} ; \ell \right), \quad \zeta \in \mathcal{D}^\pm \cup \ell, \\
p^\pm + i\omega^\pm &= \text{Ca}^{-1} \mathcal{E}_0 \left( \nabla \mathbf{n} ; \ell \right), \quad \zeta \in \mathcal{D}^\pm
\end{align*}
\]

where \( f^+ = 2z - ir, f^- = 0 \) and the operator \( \mathcal{A} \) is defined by (3.3). Zabarankin & Nir (2011) used (3.8) to find the drop’s steady shape from the kinematic condition.

(iv) Three-dimensional asymmetric flows

Suppose a solid axisymmetric finite body with the \( z \)-axis of revolution either translates in the quiescent fluid along the \( x \)-axis (\( x \)-translation) or rotates around the \( y \)-axis (\( y \)-rotation). Proposition 2 in Zabarankin (2008b) shows that in this case, \( \mathbf{u} \) and \( p \) are represented by

\[
\begin{align*}
\mathbf{u}(r, \phi, z) + i\mathbf{u}(r, \phi, z) &= \left((r + \zeta) G_1 + iG_2 + G_3\right) \cos \phi, \\
u_\rho(r, \phi, z) &= \text{Im}[-\zeta G_1 - iG_2 + G_3] \sin \phi \\
p(r, \phi, z) &= 2\mu \text{Im } G_1 \cos \phi,
\end{align*}
\]

where \( \zeta = r + iz \) is a complex variable, \( G_1 = G_1(\zeta) \) and \( G_2 = G_2(\zeta) \) are zero-order \( r \)-analytic functions, \( G_3 = G_3(\zeta) \) is a first-order \( r \)-analytic function, and all three \( G_1, G_2 \) and \( G_3 \) vanish at infinity. On the body’s surface \( S \), the boundary conditions are given by \( \mathbf{u} = v_x \mathbf{i} \) for the \( x \)-translation and by \( \mathbf{u} = [\sigma_y \mathbf{j} \times (\mathbf{x} i + z \mathbf{k})] \) for the \( y \)-rotation, where \( v_x \) and \( \sigma_y \) are constants.

Let \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) be open regions corresponding to the interior and exterior of the body’s cross section in the \( rz \)-half plane, and let \( \ell \) be common positively oriented boundary of \( \mathcal{D}^\pm \) (cross section of body’s surface \( S \)). Then \( (r + \zeta) G_1 + iG_2 + G_3 = f_1 \) and \( \text{Im}[\zeta G_1 + iG_2 - G_3] = f_2 \) on \( \ell \), where \( f_1 = iv_x \) and \( f_2 = v_x \) for the \( x \)-translation and \( f_1 = -\sigma_y \zeta \) and \( f_2 = \sigma_y z \) for the \( y \)-rotation. If \( G_1, G_2 \) and \( G_3 \) on \( \ell \) are determined, then representing \( u_r, u_\phi \) and \( u_z \) in \( \mathcal{D}^- \) with (3.9) and with the Cauchy operators \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) is straightforward (see corollaries 2.9 and 2.10).

Suppose that \( G_1 \) and \( \text{Re } G_3 \) on \( \ell \) are known, then \( G_2 = f_3 + i((r + \zeta) G_1 + G_3) \) and \( \text{Im } G_3 = -\frac{1}{2} r \text{Im } G_1 \) on \( \ell \), where \( f_3 = v_x \) for the \( x \)-translation and \( f_3 = i\sigma_y \zeta \) for

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This text is likely a continuation of a mathematical discussion, possibly related to fluid dynamics, given the context and the use of notation such as \( \mathbf{u}, G, \mathcal{A}, \mathcal{E}, \mathbf{n}, \ell, \zeta, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mu, \sigma \). The document or the reference text seems to delve into the study of fluid shapes and boundary conditions in a specific context, possibly involving axisymmetric bodies in quiescent fluids.
the \( y \)-rotation. Theorem 3 in Zabarankin (2008b) proves that \( G_1 \) and \( \operatorname{Re} G_3 \) on \( \ell \) are found from a system of two boundary-integral equations:

\[
\begin{align*}
\mathcal{C}_0(i(\zeta + r) G_1 + i G_3; \ell) - i(\zeta + r)\mathcal{C}_0(G_1; \ell) - i\mathcal{C}_1(G_1; G_3; \ell) = f_1, & \quad \zeta \in \ell \\
2\operatorname{Im} G_3 = -r\operatorname{Im} G_1 \quad \text{and} \quad G_3 + 2\mathcal{C}_1(G_3; \ell) = 0, & \quad \zeta \in \ell,
\end{align*}
\]

which is equivalent to a single real-valued equation in the sense that if \( \operatorname{Re} G_3 \), \( \mathcal{C}_1(G_3; \ell) \), is known then \( \operatorname{Im} G_3 \) on \( \ell \) is found from this formula, and vice versa. Thus, the system (3.10) is viewed as three real-valued equations with three real-valued unknowns \( \operatorname{Re} G_1, \operatorname{Im} G_1 \) and \( \operatorname{Re} G_3 \). Zabarankin (2008b) solved (3.10) with the quadratic error minimization technique for the \( x \)-translation of bi-spheres and for the \( y \)-rotation of torus of elliptical cross section and showed that the solutions for bi-spheres and torus of circular cross section coincide with corresponding analytical solutions in Goren & O’Neill (1980), Wakiya (1967) and Zabarankin (2007).

Propositions 7 and 8 in Zabarankin (2008b) state that the drag exerted on the body in the \( x \)-translation and the torque in the \( y \)-rotation are determined by

\[
F_x = 4\pi \mu \operatorname{Re} \left[ \int_\ell rG_1(\zeta) d\zeta \right] \quad \text{and} \quad T_y = 4\pi \mu \operatorname{Re} \left[ \int_\ell r((2z - i\ell)G_1(\zeta) + G_2(\zeta)) d\zeta \right],
\]

which compared with (48a) and (54a) in Zabarankin (2008b), respectively, have additional multiplier 2 because multiplier 1/2 was omitted from the right-hand side in (3.9). Observe that \( F_x \) and \( F_y \) in the axisymmetric translation have similar expressions.

(c) Oseen flows

Suppose a solid body translates in a viscous incompressible fluid with constant velocity. Under the low Reynolds number assumption, the time-independent velocity field and pressure are governed by the Oseen equations (1.5). Example 1.3 implies that \( \mathbf{u} \) and \( p \) can be represented by \( h \)-analytic functions in the two-dimensional case and by \( H \)-analytic functions in the three-dimensional case.

(i) Two-dimensional case

If the body is a solid infinitely long cylinder of arbitrary cross section, which is aligned with the \( z \)-axis and translates in the quiescent fluid at constant velocity \( \mathbf{v} = -v_x \mathbf{i} \), then \( \mathbf{u} = u_x(x, y) \mathbf{i} + u_y(x, y) \mathbf{j} \), \( \mathbf{\omega} = \omega(x, y) \mathbf{k} \) and \( p = p(x, y) \), and \( \mathbf{u} \) and \( p \) vanish at infinity in the \( xy \)-plane. In this case, (1.6) and (1.7) determine an ordinary analytic function \( g \) and \( h \)-analytic function \( h \), respectively, with a complex variable \( \zeta = x + iy \), and thus, \( \mathbf{u} \), \( p \) and \( \omega \) have a representation

\[
u_y + iu_x = g + e^{\lambda \zeta} h, \quad p = -2\lambda \mu \operatorname{Im} g \quad \text{and} \quad \omega = 2\lambda e^{\lambda \zeta} \operatorname{Re} h,
\]

where \( \lambda = \rho v_x/(2\mu) \neq 0 \), and \( g \) and \( h \) vanish as \( |\zeta| \to \infty \).

Remark 3.1. The functions \( g \) and \( h \) are uniquely determined. Indeed, let pairs \( g_1, h_1 \) and \( g_2, h_2 \) be two different solutions, then \( g = g_2 - g_1 \) and \( h = h_2 - h_1 \) solve the exterior homogeneous two-dimensional Oseen flow problem, which with \( \mathbf{u} \) and
$p$ vanishing as $|\zeta| \to \infty$ has only zero solution (Finn 1965). Thus, (3.11) with $p \equiv 0$ and $\omega \equiv 0$ yields $\text{Im} \, g \equiv 0$ and $\text{Re} \, h \equiv 0$, and then (3.11) and $u \equiv 0$ imply $\text{Re} \, g \equiv 0$ and $\text{Im} \, h \equiv 0$.

Let open regions $D^+$ and $D^-$ be cross sections of the cylindrical body and fluid in the $xy$-plane, respectively, with common boundary $\ell$ (positively oriented Jordan curve). The boundary condition $u = -v_i$ on $\ell$ and the representation (3.11) imply $u_y + i u_x = -i v_x$ on $\ell$. Suppose the value of $g$ on $\ell$ is known. Then $h = -e^{-2x}(g + iv_x)$ on $\ell$, and with the Cauchy integral formulae for analytic and $h$-analytic functions, (3.11) yields representations for $u$, $p$ and $\omega$ in $D^-$:

$$u_y + i u_x = \mathcal{B}_h(g; \ell), \quad p = 2\lambda \mu \text{Im} \left[ \frac{1}{2\pi i} \oint_{\ell} \frac{g(\tau)}{\tau - \zeta} \, d\tau \right], \quad \zeta \in D^-,$$

and $\omega = 2\lambda e^{2x} \text{Re} \left[ \mathcal{C}_h(e^{-2x}g; \ell) \right], \quad \zeta \in D^-$, where the operator $\mathcal{B}_h$ is given by

$$\mathcal{B}_h(g; \ell) = e^{2x} \mathcal{C}_h(e^{-2x}g; \ell) - \frac{1}{2\pi i} \oint_{\ell} \frac{g(\tau)}{\tau - \zeta} \, d\tau \equiv \frac{1}{2\pi i} \oint_{\ell} \frac{\sigma K_1(\sigma)e^{(x-x_1)}}{\tau - \zeta} - \frac{1}{\tau - \zeta} g(\tau) \, d\tau - \lambda K_0(\sigma)e^{(x-x_1)}g(\tau) \, d\tau$$

with $\sigma = |\lambda| |\tau - \zeta|$ and $K_m(\cdot)$ being the $m$th-order modified Bessel function of the second kind. Observe that the operator $\mathcal{B}_h$ has no Cauchy-type singularity on $\ell$. Consequently, $\mathcal{B}_h(g; \ell)$ is continuous as $\zeta$ approaches $\ell$ from within $D^-$, and $u_y + i u_x = -i v_x$ on $\ell$ implies that $g$ satisfies the boundary-integral equation

$$\mathcal{B}_h(g; \ell) = -i v_x, \quad \zeta \in \ell. \quad (3.12)$$

**Theorem 3.2.** Let $\lambda \neq 0$, then a homogeneous solution of (3.12) is any imaginary constant and $g = \tilde{g} - c$, $\zeta \in \ell$, where $\tilde{g}$ is a solution of (3.12), and $c$ is the imaginary constant determined by $c \equiv \tilde{g}(\zeta)/2 + 1/(2\pi i) \oint_{\ell} (\tilde{g}(\tau)/\tau - \zeta) \, d\tau, \quad \zeta \in \ell$.

**Proof.** The proof follows the approach of Muskhelishvili (1992). The necessary and sufficient condition for a complex-valued function $g$ on $\ell$ to be the boundary value of an analytic function in $D^-$ that vanishes at infinity is given by the Sokhotski–Plemelj formula

$$g(\zeta) = \frac{g(\zeta)}{2} - \frac{1}{2\pi i} \oint_{\ell} \frac{g(\tau)}{\tau - \zeta} \, d\tau, \quad \zeta \in \ell. \quad (3.13)$$

Let $\tilde{g}(\zeta)$, $\zeta \in \ell$, be a solution of (3.12), then the Cauchy-type integrals

$$g_1(\zeta) = \frac{1}{2\pi i} \oint_{\ell} \frac{i v_x - \tilde{g}(\tau)}{\tau - \zeta} \, d\tau, \quad h_1(\zeta) = \mathcal{C}_h(e^{-2x}\tilde{g}; \ell), \quad \zeta \in D^+,$$

define ordinary analytic and $h$-analytic functions, respectively, in $D^+$. Then the boundary-integral equation (3.12) is equivalent to $g_1 + e^{2x}h_1 = 0$ on $\ell$ (through Sokhotski–Plemelj formulae as $\zeta$ approaches $\ell$ from within $D^+$). Observe that similar to (3.11), $u_\ell + i u_\ell = g_1 + e^{2x}h_1$ is a formal solution of the two-dimensional Oseen equations (1.5) in $D^+$. However, (1.5) with $u_\ell = 0$ on $\ell$ yields $u_\ell \equiv 0$ in $D^+$. Thus, $g_1 \equiv -e^{2x}h_1$ in $D^+$, which holds only if $g_1 \equiv c_1$ in $D^+$, where $c_1$ is an imaginary constant. As $\zeta$ approaches $\ell$ from within $D^+$, the Cauchy-type integral that defines $g_1$ implies $\tilde{g}(\zeta)/2 + 1/(2\pi i) \oint_{\ell} (\tilde{g}(\tau)/\tau - \zeta) \, d\tau = iv_x - c_1 = c$ for any
\[ \zeta \in \ell. \] Since \( g \) should satisfy (3.13), the solution \( g = \tilde{g} - c, \zeta \in \ell \), follows. Now if \( \tilde{g}^* \) is another solution of (3.12) with a corresponding imaginary constant \( c^* \), then because \( g \) is uniquely determined (remark 3.1), \( \tilde{g}^* - c^* = \tilde{g} - c, \zeta \in \ell \) or, equivalently, \( \tilde{g}^* - \tilde{g} = c^* - c, \zeta \in \ell \), and thus, a homogeneous solution of (3.12) is only an imaginary constant. 

With (3.11), (3.12) is equivalent to the two-dimensional Oseenlet-based boundary-integral eqn (12) in Williams (1965).

It follows from (3.11) and \( u_y + iu_{z} = -v_xi \) on \( \ell \) that \( p/\mu - i\omega = 2\lambda ig \) and \( \partial u_y/\partial n + i\partial u_x/\partial n = -2\lambda (\partial \zeta/\partial n) \Re g, \zeta \in \ell \), which yields the drag exerted on the cylinder per unit length of cylinder’s span:

\[
F_x = i \oint_{\ell} \left( 2\mu \frac{\partial u}{\partial n} + \mu [n \times \omega] - pn \right) ds = -2\lambda \mu \Re \left\{ \oint_{\ell} g(\zeta) d\zeta \right\},
\]

where \( n \) is the outward normal for \( \ell \) and \( ds \) is the curve length element. Since \( F_x \) is unchanged by adding a constant to \( g \), it holds for any solution of (3.12). Observe that \( 2\lambda \mu = \rho v_x \) and (3.14) bears a close resemblance to the Kutta–Joukowski formula (see §3a).

As a verification, (3.12) is solved for a circular cylinder with radius 1. In this case, \( \zeta(t) = e^{it} \) and \( g(t) = v_x \sum_{k=1}^{10} a_k \sin kt + ib_k \cos kt, t \in [-\pi, \pi] \), with \( a_k \) and \( b_k \) found by the quadratic error minimization technique (see §3b(ii)). The obtained drag coefficient \( C_D = F_x/(2\lambda \mu v_x) \) for Reynolds numbers \( R = 4l = 0.2, 2, 20, 200 \) and 2000 is \( 34.6541, 8.0847, 3.4658, 2.5387, 2.3237 \), respectively, versus 34.655, 8.090, 3.469, 2.545 and 2.324, respectively, computed in Miyagi (1974).

(ii) Translation of solid bodies of revolution

Suppose a solid axisymmetric finite body with the \( z \)-axis of revolution translates in the quiescent fluid at constant velocity \(-v_zk\), so that \( u = -v_zk \) on the body’s surface \( S \), and \( u \) and \( p \) vanish at infinity. In this case, \( u \) and \( p \) are independent of the angular coordinate \( \phi \): \( u = u_r(r, z)e_r + u_z(r, z)e_z, u_x = 0, p = p(r, z) \) and \( \omega = \omega(r, z)e_\phi \), and proposition 1 in Zabarankin (2010) states that

\[
u_z + iu_r = G + e^{i\phi}H, \quad p = -2\lambda \mu \Re G, \quad \omega = 2\lambda e^{i\phi} \Im H,
\]

where \( \lambda = \rho v_z/(2\mu) \neq 0 \), and \( G \) and \( H \) are zero-order \( r \)-analytic and zero-order \( H \)-analytic functions, respectively, that both vanish at infinity.

Let open regions \( D^+ \) and \( D^- \) be the interior and exterior of the body’s cross section in the rz-half plane \( (r \geq 0) \), respectively, with common positively oriented boundary \( \ell \). Then (3.15) implies that \( G + e^{i\phi}H = -v_z \) on \( \ell \). If the boundary value of \( G \) on \( \ell \) is known, then \( H = -e^{-i\phi}(v_z + G) \) on \( \ell \) and with the Cauchy integral formula (2.9), (3.15) furnishes representations for \( u \), \( p \) and \( \omega \) in \( D^- \):

\[
u_z + iu_r = \mathcal{B}_H(G; \ell), \quad p = 2\lambda \mu \Re [C_0(G; \ell)], \quad \zeta \in D^-,
\]

and \( \omega = 2\lambda e^{i\phi} \Im [C_0(e^{-i\phi}G; \ell)], \zeta \in D^- \), where \( \mathcal{B}_H \) is determined by

\[
\mathcal{B}_H(G; \ell) = e^{i\phi}C_H(e^{-i\phi}G; \ell) - C_0(G; \ell).
\]

The operator \( \mathcal{B}_H \) has no Cauchy-type singularity on \( \ell \), so that \( \mathcal{B}_H(G; \ell) \) is continuous as \( \zeta \) approaches \( \ell \) from within \( D^- \), and the boundary condition \( u_z +

iur = −vz on ℓ implies that G on ℓ satisfies the boundary-integral equation

\[ B_H(G; \ell) = -v_z, \quad \zeta \in \ell. \] (3.16)

Theorem 10 in Zabarankin (2010) proves that a homogeneous solution of (3.16) is any real constant and that \( G = w - c, \zeta \in \ell \), where \( w \) is a solution of (3.16) and \( c \) is the real constant determined by \( c = w/2 + C_0(w; \ell), \zeta \in \ell. \)

The representation (3.15) and \( uz + iur = -vz \) on ℓ yield

\[ \frac{p}{m} + iur = -2lG \quad \text{and} \quad \frac{vz}{n} + jvur = 2l(\bar{z}/n) \text{Im} G, \zeta \in \ell, \]

and proposition 10 in Zabarankin (2010) shows that the drag exerted on the body is given by

\[ F_z = -4\pi \lambda \mu \text{Re} \left[ \int_\ell rG(\zeta) d\zeta \right]. \]

Observe that a real constant added to \( G \) has no effect on the drag.

As an illustration, Zabarankin (2010) solved the boundary-integral equation (3.16) with the quadratic error minimization technique for sphere, spheroids and bi-spheroids, and showed that the use of the conjugate kernels in the Cauchy integral formula for zero-order \( H \)-analytic functions improves technique’s running time by several times. Zabarankin & Molyboha (2011) used (3.16) with the conjugate kernels in an iterative procedure for finding minimum-drag shapes for solid bodies under the low Reynolds number assumption subject to constraints on body’s volume and body’s shape.

**Appendix A**

For \(|\beta| < 1\) and \(n \in \mathbb{Z}_0^+\), formula (10) in §2.4 in Bateman & Erdelyi (1953) yields

\[
F\left[ \frac{1}{2}, n + \frac{1}{2}, n + 1, \beta^2 \right] = \frac{n!}{2\sqrt{\pi} \Gamma \left( n + 1/2 \right) \beta^n} \int_0^{2\pi} \frac{\cos n\phi \, d\phi}{\sqrt{1 - 2\beta \cos \phi + \beta^2}} = \frac{2(-1)^nn!}{\sqrt{\pi} \Gamma(n + 1/2)\beta^n(1 + \beta)} \int_0^{\pi/2} \frac{\cos(2nt)dt}{\sqrt{1 - 4\beta/(1 + \beta)^2 \sin^2 t}},
\]

whereas formula (24) in §2.1.5 in Bateman & Erdelyi (1953) implies

\[
F\left[ \frac{1}{2}, n + \frac{1}{2}, n + 1, \beta^2 \right] = (1 + \beta)^{-2n-1}F\left[ n + \frac{1}{2}, n + \frac{1}{2}, 2n + 1, \frac{4\beta}{(1 + \beta)^2} \right].
\]
Consequently,
\[
\int_0^{\pi/2} \frac{\cos(2nt)dt}{\sqrt{1 - k^2 \sin^2 t}} = \frac{\sqrt{\pi} \Gamma(n + 1/2)k^{2n}}{2^{2n+1}(-1)^n n!} \mathcal{F}_1,
\]
where \( \mathcal{F}_1 = \mathbb{F}[n + 1/2, n + 1/2, 2n + 1, \kappa^2] \), so that
\[
I_1 = \int_0^{\pi/2} \frac{\cos(2nt) + \cos 2(n+1)t}{2\sqrt{1-k^2 \sin^2 t}} \, dt = \frac{\sqrt{\pi} \Gamma(n + 1/2)k^{2n}}{2^{2(n+1)}(-1)^n n!} \left( \mathcal{F}_1 - \frac{(n + 1/2)k^2}{4(n + 1)} \mathcal{F}_2 \right),
\]
where \( \mathcal{F}_2 = \mathbb{F}[n + 3/2, n + 3/2, 2n + 3, \kappa^2] \). For any positive \( a, b \) and \( c \), formulae (20) and (22) in §2.8 in Bateman & Erdelyi (1953) yield
\[
\mathbb{F} \left[ a + 1, b + 1, c + 1, \kappa^2 \right] = \frac{c(c-1)}{a b \kappa^2} \left( \mathbb{F} \left[ a, b, c - 1, \kappa^2 \right] - \mathbb{F} \left[ a, b, c, \kappa^2 \right] \right),
\]
so that \( \mathcal{F}_2 = 4(n+1)/((n+1/2)k^2)(\mathcal{F}_1 - \mathcal{F}_3) \), where \( \mathcal{F}_3 = \mathbb{F}[n + 1/2, n + 1/2, 2n + 2, \kappa^2] \), and thus, \( I_1 \) simplifies to (2.19). Similarly,
\[
I_2 = \int_0^{\pi/2} \frac{\cos(2nt) - \cos 2(n+1)t}{2\sqrt{1-k^2 \sin^2 t}} \, dt = \frac{\sqrt{\pi} \Gamma(n + 1/2)k^{2n}}{2^{2(n+1)}(-1)^n n!} (2 \mathcal{F}_1 - \mathcal{F}_3).
\]
Now, (34) in §2.8 in Bateman & Erdelyi (1953) implies that
\[
\mathcal{F}_1 = \mathbb{F} \left[ n + \frac{3}{2}, n + \frac{1}{2}, 2n + 1, \kappa^2 \right] - \frac{k^2}{2(1-k^2)} \mathcal{F}_3,
\]
which, with formula (39) in §2.8 in Bateman & Erdelyi (1953), results in (2.20).

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