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Einstein's special relativity beyond the speed of light

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We propose here two new transformations between inertial frames that apply for relative velocities greater than the speed of light, and that are complementary to the Lorentz transformation, giving rise to the Einstein special theory of relativity that applies to relative velocities less than the speed of light. The new transformations arise from the same mathematical framework as the Lorentz transformation, displaying singular behaviour when the relative velocity approaches the speed of light and generating the same addition law for velocities, but, most importantly, do not involve the need to introduce imaginary masses or complicated physics to provide well-defined expressions. Making use of the dependence on relative velocity of the Lorentz transformation, the paper provides an elementary derivation of the new transformations between inertial frames for relative velocities v in excess of the speed of light c, and further we suggest two possible criteria from which one might infer one set of transformations as physically more likely than the other. If the energy–momentum equations are to be invariant under the new transformations, then the mass and energy are given, respectively, by the formulae $m=(p_\infty/c)[(v/c)^2-1]^{-1/2}$ and $\mathscr{E}=mc^2$, where p_∞ denotes the limiting momentum for infinite relative velocity. If, however, the requirement of invariance is removed, then we may propose new mass and energy equations, and an example having finite non-zero mass in the limit of infinite relative velocity is given. In this highly controversial topic, our particular purpose is not to enter into the merits of existing theories, but rather to present a succinct and carefully reasoned account of a new aspect of Einstein's theory of special relativity, which properly allows for faster than light motion.

Keywords: special relativity; superluminal; Lorentz transformation; relativistic mass

1. Introduction

Einstein's special theory of relativity is well known to provide an accurate description of physical phenomena, providing that the relative velocity $v>0$ does not exceed the speed of light c. The two well-known formulae for the mass m and energy $\mathscr{E}$ are

$$m=\frac{m_0}{[1-(v/c)^2]^{1/2}},\quad \mathscr{E}=mc^2, \tag{1.1}$$

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where m_0 denotes the rest mass, and they apply only for $v < c$ and exhibit singular behaviour when $v = c$. There is a large literature on special relativity and the causality implications of superluminal motions (Fox *et al.* 1969; Camenzind 1970; Bers *et al.* 1971; Recami *et al.* 2000; Liberati *et al.* 2002). Other scientific contributions such as the works of Feinberg (1967), Ehrlich (2003) and Asaro (1996) tend to assume imaginary masses or complex speeds and coupled with complicated physics to provide meaningful formulae. There is also an extensive popular literature on faster than light theories. Here, we describe corresponding special theories of relativity that apply for relative velocities v in excess of c, namely $c < v < \infty$. If the energy–momentum relations are to be invariant under the new transformations, then for the new theories, in place of (1.1), we have simply

$$m = \frac{p_\infty/c}{[(v/c)^2 - 1]^{1/2}}, \quad \mathscr{E} = mc^2, \tag{1.2}$$

where p_∞ denotes the value of the momentum $p = mv$ in the limit of the relative velocity v approaching infinity. We observe that for the Einstein theory (1.1), at $v = 0$ the rest mass is finite while the momentum is zero, while for the new theory, at $v = \infty$ the momentum is finite while the mass is predicted to be zero. However, if the requirement of invariance of the energy–momentum relations is relaxed, then we may propose other relations, such as

$$m = \frac{m_\infty v/c}{[(v/c)^2 - 1]^{1/2}}, \quad \mathscr{E} = \frac{1}{2}m(c^2 + v^2) - \frac{1}{2}m_\infty c^2 \cosh^{-1}\left(\frac{v}{c}\right) + \mathscr{E}_0, \tag{1.3}$$

which predicts a finite mass m_∞ in the limit of infinite relative velocity, and $\mathscr{E}_0$ denotes an arbitrary constant. For the new theories, unlike the Einstein theory, there are no obvious points of identification such as Galilean transformations, finite rest mass m_0 and the familiar kinetic energy expression $m_0 v^2/2$ to help determine the new transformations and the constants m_∞ and $\mathscr{E}_0$. The purpose of this paper is to present a brief but careful derivation of new special theories of relativity valid for $v > c$.

For the new theories, there are three important points that we wish to emphasize. First, as will be seen from the derivation, the new theories are merely the logical extension of Einstein's theory but apply for relative velocities in excess of the speed of light. The new theories are consistent and complementary with the Einstein theory and all arise from the same mathematical framework. The second important point is that all theories exhibit singular behaviour when the relative velocity is exactly equal to the speed of light. The third important point is that the new theories provide a framework in which we may discuss relative velocities in excess of the speed of light without resorting to the introduction of imaginary masses or complicated physics to ensure real outcomes.

As already mentioned, for the regime $0 \le v < c$, we can at least associate Galilean transformations, the rest mass m_0 and the kinetic energy $m_0 v^2/2$ for $v \ll c$ to help determine the theory. However, for the regime $c < v < \infty$, there are no corresponding points of identification, and it is not even known whether it is physically sensible to speak of a finite mass m_∞ for infinite relative velocity v. We have only a formal mathematical structure suggesting firstly that there are corresponding theories of relativity for the regime $c < v < \infty$, and secondly that the velocities in this regime are somehow connected to velocities c^2/v in

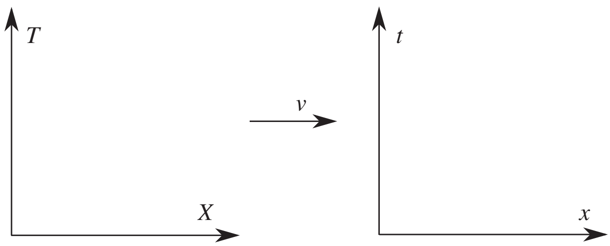

Figure 1. Two inertial frames moving along the x-axis with relative velocity v.

the regime $0 \le v < c$ and vice versa. This is the motivation underlying the mass formula $(1.3)_1$, namely it is formally obtained from $(1.1)_1$ by changing v to c^2/v and re-labelling m_0 to m_∞, and it is an expression with a meaningful and well-defined limit for v tending to infinity. For the uncommon physics applying for motion with relative velocities in excess of the speed of light, we can only be motivated by the suggestive mathematical structure.

As with the Einstein formulae (1.1) that are Lorentz-invariant, the new formulae (1.2) also do not depend on the frame velocity and are therefore invariant under the proposed new transfomations of inertial frames. In fact, (1.2) is the only possibility if the corresponding energy–momentum equations are be invariant under the new transformations, and formulae such as (1.3) have energy–momentum relations that are not invariant under the new transformations. For extensive accounts of modern tests of Lorentz invariance, we refer the reader to Baccetti *et al.* (2012*a*), Coleman & Glashow (1999), Liberati *et al.* (2001), Mattingly (2005) and Mattingly *et al.* (2003).

In §2, we summarize the basic equations that form the Einstein special theory of relativity. In the subsequent section, we show how these equations can be exploited to extend the Einstein theory for relative velocities $v > c$. Following this, we discuss some of the physical implications of the universal equation (2.6) or (4.1) for velocity addition. In the section thereafter we develop the new formulae for mass, momentum and energy, applying to $v > c$. In §6 of the paper, we make some brief conclusions. In appendix A, we give an approximate analysis for $v \approx c$, following two possible criteria that are invoked that may eliminate one of the possibilities for $c < v < \infty$. These criteria are either that the new transformation for this regime should locally admit the same spatial and temporal dependence as that for the Einstein theory, or that we adopt the new transformation that changes the sign of the characteristic $x - ct$, which is singular about $v = c$.

2. Special relativity for $0 \le v < c$

As usual, we consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X- and x-directions as indicated in figure 1. We note that the coordinate notation adopted here is slightly different to that normally used in special relativity involving primed and unprimed variables. We do this purposely because subsequently we wish to view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose, the notation used in nonlinear continuum mechanics is preferable. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, throughout we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest. For a modern approach to inertial frames, we refer the reader to Baccetti *et al.* (2012*a*,*b*) and Padmanabhan & Padmanabhan (2011).

For $0 \le v < c$, the standard Lorentz transformations are

$$X = \frac{x + vt}{[1 - (v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1 - (v/c)^2]^{1/2}},$$

with the inverse transformation characterized by $-v$, thus

$$x = \frac{X - vT}{[1 - (v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.1}$$

and various derivations of these equations can be found in the standard textbooks such as Feynman *et al.* (1966) and Landau & Lifshitz (1980), and other novel derivations are given by Lee & Kalotas (1975) and Levy-Leblond (1976). The above equations reflect, of course, that the two coordinate frames coincide when the relative velocity v is zero, namely

$$x = X, \quad t = T, \quad v = 0, \tag{2.2}$$

so that trivially

$$x + ct = X + cT, \quad x - ct = X - cT, \quad v = 0, \tag{2.3}$$

and subsequently we view (2.2) as a prescribed constraint when the parameter v is zero. Equations (2.2) an (2.3) connect the same physical event as viewed from two distinct moving frames with constant relative velocity v and assuming that $0 \le v < c$.

From equation (2.1), we observe that

$$x + ct = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} (X + cT), \quad x - ct = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} (X - cT), \tag{2.4}$$

which of course coincides with (2.3) when v is zero and in particular we have

$$x^2 - (ct)^2 = X^2 - (cT)^2,$$

which is well known. Further, with $U = \mathrm{d}X/\mathrm{d}T$ and $u = \mathrm{d}x/\mathrm{d}t$, we have from (2.4)

$$\frac{u + c}{u - c} = \left(\frac{1 - v/c}{1 + v/c}\right)\left(\frac{U + c}{U - c}\right), \tag{2.5}$$

or equivalently, we may deduce the well-known law for the addition of velocities, namely

$$u = \frac{U - v}{1 - Uv/c^2}. \tag{2.6}$$

This is an important equation that is perfectly well defined through the singularity $v = c$ and therefore we might argue that it applies in both regimes $0 \leq v < c$ and $c < v < \infty$, and gives $u = U$ for $v = 0$ as is also evident from (2.2), but on the other hand gives the relationship $uU = c^2$ for v tending to infinity. Strictly speaking, for $c < v < \infty$, we assume only that $uU = c^2$ for v tending to infinity and then establish that (2.6) holds throughout.

We comment that for $0 \leq v < c$, there is formally a second theory of Einstein's special relativity arising from the constraint $x = -X$ and $t = -T$ being consistent with $u = U$ and merely giving the Lorentz transformations (2.1) with X and T replaced by $-X$ and $-T$, respectively. Physically, this additional theory corresponds to a reversal of both space and time, and in Einstein's theory for $0 \leq v < c$, the Galilean transformations are invoked to exclude this as a possibility. However, for $c < v < \infty$, such issues are not quite so transparent, and we can be guided only by the mathematical structure.

The question now arises as to what equations might apply if the relative velocity v exceeds the speed of light. Namely, what are the new transformations between inertial frames that are valid for $c < v < \infty$, and we answer this question in §3.

3. Special relativity for $c < v < \infty$

In order to deduce the corresponding new transformations that are valid for $c < v < \infty$, we speculate that the new transformations and the Lorentz transformation are derivable from the same 'pseudo-velocity' equations that give rise to (2.1), with (2.1) arising from the constraint (2.2) and the new transformations arising from different constraints. We have in mind that the Lorentz transformation (2.1) is a one-parameter group of transformations, with the relative velocity v serving as the parameter, and the identity arising from the value $v = 0$. By the 'pseudo-velocity' equations, we refer to the derivatives $\mathrm{d}x/\mathrm{d}v$ and $\mathrm{d}t/\mathrm{d}v$, subject to the constraint (2.2). The basic underlying idea is that any one-parameter group of transformations is completely determined from its infinitesimal vector; so in the regime $c < v < \infty$, we use the same infinitesimal vector as that determined from the Lorentz transformation, but we apply a different constraint. We comment that the general structure of other relativity groups that are distinct from the new transformations given here are examined by Bacry & Levy-Leblond (1968), Gorini (1971) and Lugiato & Gorini (1972).

We now speculate that in the regime $c < v < \infty$ that the relationship arising from (2.6) applies, namely $uU = c^2$, for v tending to infinity, so that for the new transformations valid for $c < v < \infty$, the required constraint might be supplied by either

$$x = -cT, \quad t = -\frac{X}{c}, \quad v = \infty, \tag{3.1}$$

or alternatively

$$x = cT, \quad t = \frac{X}{c}, \quad v = \infty. \tag{3.2}$$

In terms of the characteristic variables, these constraints become respectively

$$x + ct = -(X + cT), \quad x - ct = X - cT, \quad v = \infty, \tag{3.3}$$

and

$$x + ct = X + cT, \quad x - ct = cT - X, \quad v = \infty. \tag{3.4}$$

Again we comment that for the Einstein special relativity theory, there are also two theories arising from $u = U$ for $v = 0$, and one is excluded on the basis of the Galilean transformations. However, for $c < v < \infty$ and in the absence of conflicting information, there are no corresponding Galilean transformations that might be invoked to exclude one possibility, and both sets of new transformations arising from either constraints (3.1) or (3.2) might have a physical meaning.

We now view x and t defined by (2.1) as functions of v, and by the following 'pseudo-velocity' equations

$$\frac{\mathrm{d}x}{\mathrm{d}v} = \frac{-t}{1-(v/c)^2}, \quad \frac{\mathrm{d}t}{\mathrm{d}v} = \frac{-x/c^2}{1-(v/c)^2}, \tag{3.5}$$

where $\mathrm{d}/\mathrm{d}v$ denotes total differentiation with respect to v, keeping the initial variables (X, T) fixed. We further speculate that the new transformations applying for $c < v < \infty$ are captured by the above 'pseudo-velocity' equations subject to either constraints (3.1) or (3.2). We now proceed formally to show how by solving (3.5), Einstein's theory emerges from the constraint (2.2), while two new transformations between inertial frames, valid for $v < c < \infty$ emerge from the constraints (3.1) or (3.2).

From a rearrangement of (3.5), we have

$$\left[1 - \left(\frac{v}{c}\right)^2\right]\frac{\mathrm{d}x}{\mathrm{d}v} = -t, \quad \left[1 - \left(\frac{v}{c}\right)^2\right]\frac{\mathrm{d}t}{\mathrm{d}v} = -\frac{x}{c^2}, \tag{3.6}$$

which becomes an autonomous system if we introduce a new parameter ϵ such that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} = \left[1 - \left(\frac{v}{c}\right)^2\right]\frac{\mathrm{d}}{\mathrm{d}v},$$

so that

$$\frac{\mathrm{d}v}{\mathrm{d}\epsilon} = 1 - \left(\frac{v}{c}\right)^2.$$

On making the substitution $v = c\sin\phi$, this equation becomes

$$\frac{\mathrm{d}\phi}{\cos\phi} = \frac{\mathrm{d}\epsilon}{c},$$

which integrates to give

$$\frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right) = \frac{\epsilon}{c} + \text{constant}.$$

Suppose we assign the value $v = v_0$ at $\epsilon = 0$, then from this equation we may deduce

$$\frac{1+v/c}{1-v/c} = \left(\frac{1+v_0/c}{1-v_0/c}\right)\mathrm{e}^{2\epsilon/c}, \tag{3.7}$$

and the special relativity theories arise from the two values $v_0 = 0$ (corresponding to the constraint (2.2)) and $v_0 = \infty$ (corresponding to either constraints (3.1) or (3.2)). From (3.7), we may readily deduce

$$v = c \tanh\left(\frac{\epsilon}{c}\right), \quad 0 \le v < c \tag{3.8}$$

and

$$v = c \coth\left(\frac{\epsilon}{c}\right), \quad c < v < \infty. \tag{3.9}$$

If we think about the parameter ϵ in these equations as being the same, then they suggest a connection that the new theory reflects the Einstein theory but with velocity replaced by c^2/v, noting that both (3.8) and (3.9), and their intervals of application, are compatible with this correspondence. In either case, the two ordinary differential equations (3.6) become

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = -t, \quad \frac{\mathrm{d}t}{\mathrm{d}\epsilon} = -\frac{x}{c^2},$$

so that on differentiating either with respect to ϵ, we may eventually deduce

$$x(\epsilon) = A \sinh\left(\frac{\epsilon}{c}\right) + B \cosh\left(\frac{\epsilon}{c}\right) \tag{3.10}$$

and

$$t(\epsilon) = -\frac{[A \cosh(\epsilon/c) + B \sinh(\epsilon/c)]}{c}, \tag{3.11}$$

where A and B denote arbitrary constants of integration. In the case of Einstein's special relativity, we have from (2.2), (3.10) and (3.11), that $A = -cT$ and $B = X$, giving the well-known pseudo-Euclidean rotation (Landau & Lifshitz (1980))

$$x(\epsilon) = X \cosh\left(\frac{\epsilon}{c}\right) - cT \sinh\left(\frac{\epsilon}{c}\right) \tag{3.12}$$

and

$$t(\epsilon) = T \cosh\left(\frac{\epsilon}{c}\right) - \frac{X}{c} \sinh\left(\frac{\epsilon}{c}\right), \tag{3.13}$$

noting that in this case v as a function of ϵ is defined by (3.8), which gives

$$\cosh\left(\frac{\epsilon}{c}\right) = \frac{1}{[1-(v/c)^2]^{1/2}}, \quad \sinh\left(\frac{\epsilon}{c}\right) = \frac{v/c}{[1-(v/c)^2]^{1/2}}, \tag{3.14}$$

and together (3.12)–(3.14) yield (2.1), as might be expected.

However, in the event $c < v < \infty$ and assuming the constraint (3.1), we find $A = X$ and $B = -cT$ and hence (3.10) and (3.11) become

$$x(\epsilon) = X \sinh\left(\frac{\epsilon}{c}\right) - cT \cosh\left(\frac{\epsilon}{c}\right)$$

and

$$t(\epsilon) = T \sinh\left(\frac{\epsilon}{c}\right) - \frac{X}{c} \cosh\left(\frac{\epsilon}{c}\right),$$

and from (3.9) we have

$$\sinh\left(\frac{\epsilon}{c}\right) = \frac{1}{[(v/c)^2-1]^{1/2}}, \quad \cosh\left(\frac{\epsilon}{c}\right) = \frac{v/c}{[(v/c)^2-1]^{1/2}}, \tag{3.15}$$

noting in particular that for $\epsilon > 0$ we have adopted the positive square root for $\sinh(\epsilon/c)$ and essentially the above two theories reverse the roles of $\sinh(\epsilon/c)$ and $\cosh(\epsilon/c)$. Altogether, we find that one possible new transformation valid for $c < v < \infty$ is

$$x = \frac{X - vT}{[(v/c)^2 - 1]^{1/2}}, \quad t = \frac{T - vX/c^2}{[(v/c)^2 - 1]^{1/2}}, \tag{3.16}$$

and we observe that these are quite consistent with the constraint (3.1). We also comment that, of course, the transformation (3.16) is simply the standard Lorentz transformation (2.1) but with a well-defined denominator that applies for $c < v < \infty$.

From (3.16), we have the relations

$$x + ct = -\left(\frac{v/c - 1}{v/c + 1}\right)^{1/2}(X + cT), \quad x - ct = \left(\frac{v/c + 1}{v/c - 1}\right)^{1/2}(X - cT), \tag{3.17}$$

which are in agreement with (3.3) and in particular we have

$$x^2 - (ct)^2 = (cT)^2 - X^2.$$

Again with $U = \mathrm{d}X/\mathrm{d}T$, and $u = \mathrm{d}x/\mathrm{d}t$, (3.17) yields a result identical to that of (2.5), which is equivalent to the velocity addition law for the Einstein theory, namely (2.6). Notice also that (2.5) and (2.6) have the rather peculiar feature that they remain invariant under any two changes of u, v and U to c^2/u, c^2/v and c^2/U, respectively. This reflects the fact that under these transformations, terms such as $(u + c)/(u - c)$ in (2.5) change sign, and the equation itself remains unaltered, changing two such terms at a time.

It is clear from (3.17) and (3.3) that there is no particular reason why the positive characteristic should change sign, and equally well, in place of (3.1), we could impose the alternative constraint (3.2) as being entirely consistent with $uU = c^2$. In this case, we have $A = -X$ and $B = cT$ and in place of (3.17), we find that the new transformations valid for $c < v < \infty$ become

$$x = \frac{-X + vT}{[(v/c)^2 - 1]^{1/2}}, \quad t = \frac{-T + vX/c^2}{[(v/c)^2 - 1]^{1/2}}, \tag{3.18}$$

from which we obtain

$$x + ct = \left(\frac{v/c - 1}{v/c + 1}\right)^{1/2}(X + cT), \quad x - ct = -\left(\frac{v/c + 1}{v/c - 1}\right)^{1/2}(X - cT). \tag{3.19}$$

We see from (2.4), (3.17) and (3.19) that the positive characteristic $x + ct$ remains well defined through the singularity $v = c$, while the negative characteristic $x - ct$ is singular, and therefore for this reason we might speculate that the immediately above new transformation (3.18) is more likely to represent the continuation of Einstein's special theory of relativity than (3.16). However, from an approximate analysis presented in appendix A, perhaps the new transformation (3.16) with the constraint (3.1) is more likely to be the physical continuation of the Einstein theory as it gives the same spatial and temporal dependence as the Einstein theory. Accordingly, if we expect the same local spatial and temporal dependence on both sides of the singularity $v = c$, then we propose

(3.16), but if physically we expect the singular characteristic to change sign, then we propose (3.18). In the following section, we discuss some of the physical implications of the universality of equation (2.6) for velocity addition.

4. Implications of universality of (2.6)

We argue in the previous sections that Einstein's equation for the addition of velocities (2.6) applies for all real values of v, u and U. To recapitulate, we assume one inertial frame (X, T) with an observer at rest and who measures a second frame (x, t) moving at velocity v in the positive x-direction (the x-axes of both frames are assumed to be parallel to one another), and a second observer, at rest in this second frame, which measures some particle moving at a relative velocity u, also in the positive direction of x. Then from (2.6) relative to the first observer, the particle will have a velocity U given by

$$U = \frac{u + v}{1 + uv/c^2}. \tag{4.1}$$

When $|u|, |v| < c$, this leads to the result $|U| < c$, or in other words, superluminal velocities cannot be achieved by the accumulation of any number of subluminal velocities. In §3, for $c < v < \infty$, we assume $uU = c^2$ at $v = \infty$, and show that (2.6) holds throughout, so that the addition formula (4.1) is applicable to all velocities (less than, equal to and greater than c) and this has counterintuitive physical implications for the relationship between frames moving relative to one another at these velocities.

The first thing to notice is that if $|u|, |v| > c$, then $|U| < c$, which is also the case when $|u|, |v| < c$. In other words, if the second frame is moving with a velocity greater than c, and the second observer measures a particle travelling faster than light, then relative to the first observer this particle is seen to be travelling less than the speed of light. This leads to a classification of all inertial frames into two sets, *relative to some rest frame*: (i) those with a relative velocity less than c, denoted here as S_{sub}; and (ii) those with a relative velocity greater than c, denoted by S_{sup}. All the frames in S_{sub} have a relative velocity less than c when compared with all other members of that set and a relative velocity greater than c when compared with all members of S_{sup}, and likewise all the frames in S_{sup} have a relative velocity less than c with all other members of S_{sup} and a relative velocity greater than c with all members of S_{sub}. Thus, in the example where $|u|, |v| > c$, if the first frame is taken to be in S_{sub}, then the second frame is in S_{sup} and the rest frame for the particle being measured is superluminal compared with the second frame and therefore it is a member of S_{sub} and hence the relative velocity with the original frame is $|U| < c$.

Other possibilities are when $|u| < c$ and $|v| > c$, or alternatively when $|u| > c$ and $|v| < c$. In the first case $|v| > c$ and hence if we assume the first frame is in S_{sub}, then the second frame is in S_{sup} and since $|u| < c$ then the particle's rest frame is also in S_{sup} and therefore $|U| > c$. Similarly, in the second case, we have $|v| < c$ and therefore the first two frames are both in S_{sub} and because $|u| > c$ then the particle's rest frame is in S_{sup} and again $|U| > c$. We comment that the two sets of frames S_{sub} and S_{sup} are completely symmetric and there is no objective way to identify whether a particular frame is in the subluminal set or in the

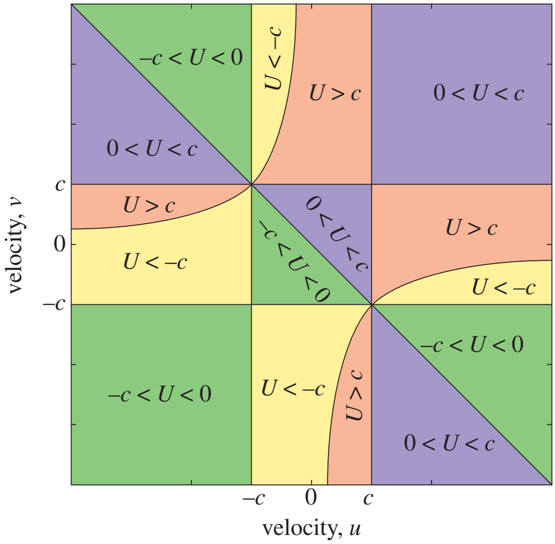

Figure 2. Addition of velocities according to (4.1).

superluminal set of frames other than by reference to some arbitrary rest frame. However, once a rest frame is established, any of the frames in the set $S_{\rm sub}$ provide an equally good rest frame for the purpose of defining these sets, and choosing any member of $S_{\rm sub}$ as the rest frame would lead to exactly the same members in the two sets $S_{\rm sub}$ and $S_{\rm sup}$.

Further insights can be gained by considering the result U arising from equation (4.1) for various ranges of u and v, as indicated in figure 2. In this figure, the velocities u and v appear as the axes and the resulting value of U is in one of four ranges: either less than $-c$, which is shown as yellow; between $-c$ and zero, which is shown as green; between 0 and c, which is shown as blue; and, greater than c, which is shown as red. In figure 2, there are a number lines that correspond to the lines $U = \pm\infty$, $U = \pm c$ and $U = 0$, which divide the plane into 16 distinct regions, each being one of the four types described previously. We note that the $U = 0$ isoline occurs only for $v = -u$ which is a statement on the uniqueness of rest frames.

We observe that across the isoline $U = 0$ $(v = -u)$ the sign of U changes, and in particular if $|u|, |v| < c$ then $v > -u$ implies that $U > 0$. However, in the regions $|u|, |v| > c$ where $v > -u$ then $U < 0$. The counterintuitive physical implication being that if for the second frame $v > c$, and the observer in this frame measures a particle with velocity $u < -c$ and $|u| < |v|$, then relative to an observer in the rest frame, the particle has a negative rather than positive velocity. The $U = c$ isolines occur for $u = c$ or $v = c$ and conversely the $U = -c$ isolines occur for $u = -c$ or $v = -c$, indicating that the addition of velocities, either subluminal or superluminal, will not result in $U = \pm c$ unless one of the input velocities itself is $\pm c$. The rectangular hyperbola $v = -c^2/u$ prescribes an isoline over which U is discontinuous and approaches either $\pm\infty$, depending on the direction that one approaches the line. Of particular interest are the points $(u, v) = (c, -c)$ and $(u, v) = (-c, c)$, and we note that all of the isolines mentioned so far pass though

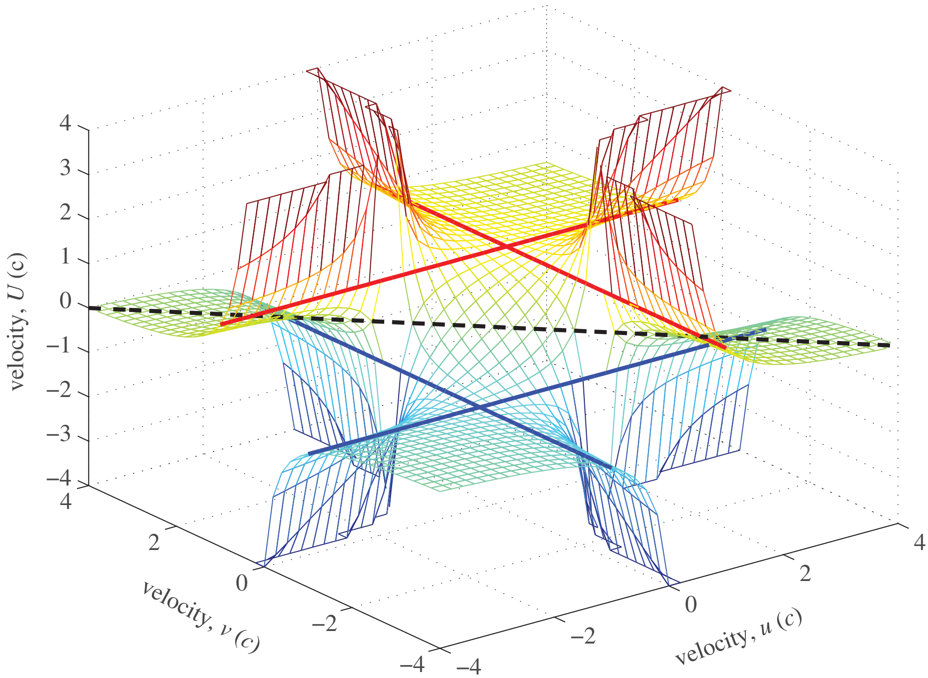

Figure 3. Three-dimensional view of U as function of u and v with all units multiples of c showing isolines of $U = \pm c$ and $U = 0$.

these two points. Indeed, it is possible to construct isolines for any value of U, that likewise will pass through these two points, and hence U might be referred to as being highly singular at these points. In figure 3, we show a three-dimensional view of U as a function of u and v. In this figure, the hyperbolic isolines of $U = \pm\infty$ are identified as well as the central region where $|u|, |v| < c$. Now we consider fixing one of the velocities v to examine the behaviour of the resultant final velocity U as a function of u. With reference to figure 4, we see from (4.1) that when $v = 0$ then $U = u$, which appears as a straight line on the (u, U) plane, but when $0 < v < c$, then the relationship is a curve in the (u, U) until we reach the value $v = c$, at which point we again reach the straight line $U = c$. Subsequently, as $v \to \infty$ the relationship $uU = c^2$ leads to the hyperbola $U = c^2/u$ in this limit. From this graph, we also see the relationship described earlier in this section, that $|U| < c$ either when $|u|$ and $|v|$ are both less than c or when they are both greater than c, so that $|U| > c$ implies one of $|u|$ or $|v|$ is less than c and the other is greater than c. In §5, we summarize formulae for mass, momentum and energy.

5. Mass, momentum and energy

The now widely accepted formula $(1.1)_1$ for the special relativistic variation of mass with velocity is derived from the following formal calculations. Using $\mathscr{E} = mc^2$, Einstein derives the mass formula $(1.1)_1$ from the energy and momentum

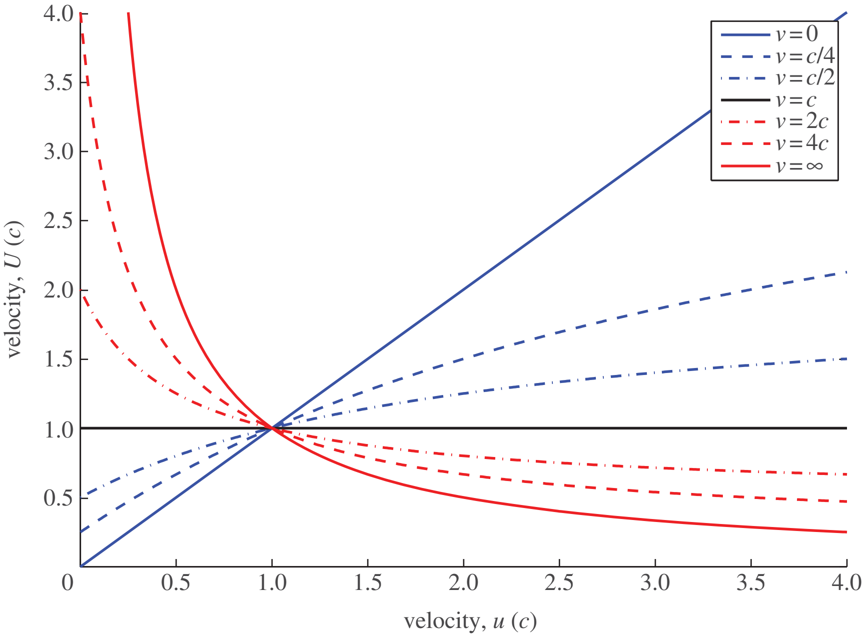

Figure 4. U as function of u for various values of v.

equations (Feynman *et al.* 1966)

$$\frac{d\mathscr{E}}{dt} = Fv, \quad F = \frac{d}{dt}(mv), \tag{5.1}$$

so that altogether we have

$$\frac{d\mathscr{E}}{dt} - v\frac{d}{dt}(mv) = \frac{c^2}{2m}\frac{d}{dt}\left\{m^2\left[1 - \left(\frac{v}{c}\right)^2\right]\right\},$$

and from which it is clear that the energy and momentum equations are properly satisfied, providing that m as a function of v is given by $(1.1)_1$. For the Einstein theory for $0 \le v < c$, we may deduce from $p = mu$ and $P = MU$ and the velocity addition law (2.6) the Lorentz invariant energy–momentum relations

$$p = \frac{P - vM}{[1 - (v/c)^2]^{1/2}}, \quad m = \frac{M - vP/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{5.2}$$

which give

$$p^2 - (cm)^2 = P^2 - (cM)^2.$$

Now again using $\mathscr{E} = mc^2$, for $c < v < \infty$, we may deduce the mass formula $m = (p_\infty/c)[(v/c)^2 - 1]^{-1/2}$, because the above formal device again carries through, yielding

$$\frac{d\mathscr{E}}{dt} - v\frac{d}{dt}(mv) = -\frac{c^2}{2m}\frac{d}{dt}\left\{m^2\left[\left(\frac{v}{c}\right)^2 - 1\right]\right\},$$

which, of course, is identically zero for m defined by $m=(p_\infty/c)[(v/c)^2-1]^{-1/2}$. If now for $c<v<\infty$, we assume the energy–momentum relations

$$p=\frac{P-vM}{[(v/c)^2-1]^{1/2}},\quad m=\frac{M-vP/c^2}{[(v/c)^2-1]^{1/2}}, \tag{5.3}$$

so that

$$p^2-(cm)^2=(cM)^2-P^2,$$

then from this equation, $p=mu$ and $P=MU$ we may readily deduce the frame invariant relation

$$m\left[\left(\frac{u}{c}\right)^2-1\right]^{1/2}=M\left[1-\left(\frac{U}{c}\right)^2\right]^{1/2}=\text{constant},$$

indicating $(1.1)_1$ and $(1.2)_1$ as the only possibilities for the mass variation giving rise to the invariant energy–momentum relations (5.3), noting again that at least one of $|u|$, $|v|$ and $|U|$ must be less than c.

If we do not insist on the energy–momentum relations (5.3) and we wish to propose a mass variation expression that has a finite value for v tending to infinity, then given the widespread acceptance of (1.1) and the suggestive nature of the two sets of expressions (3.14) and (3.15), it is reasonable to propose $(1.3)_1$ as a possible mass variation for $v>c$, because this is the only likely expression that tends to a finite value m_∞ as v tends to infinity. Thus, if we do not impose the invariance requirement, the corresponding energy calculation is as follows. We assume that the energy $\mathscr{E}$ has the structure $\mathscr{E}=mc^2\phi(\xi)$ for some function $\phi(\xi)$, where $\xi=(v/c)^2$. In terms of ξ, the above energy balance (5.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{\xi}{\xi-1}\right)^{1/2}\phi(\xi)\right]=\xi^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(\xi-1)^{1/2}}\right),$$

which we regard as an ordinary differential equation for the determination of $\phi(\xi)$. The above equation can be rearranged to yield

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{\xi}{\xi-1}\right)^{1/2}(\phi(\xi)-1)\right]=\frac{1}{2}\left(\frac{\xi-1}{\xi}\right)^{1/2},$$

which upon integration gives

$$\phi(\xi)=\frac{1}{2}(1+\xi)-\frac{1}{2}\left(\frac{\xi-1}{\xi}\right)^{1/2}\cosh^{-1}\left(\xi^{1/2}\right)+\phi_0\left(\frac{\xi-1}{\xi}\right)^{1/2},$$

where ϕ_0 denotes an arbitrary constant of integration. In terms of the original variables, we may eventually deduce

$$\mathscr{E}=\frac{m}{2}(c^2+v^2)-\frac{m_\infty c^2}{2}\cosh^{-1}\left(\frac{v}{c}\right)+\phi_0 m_\infty c^2, \tag{5.4}$$

so that the constant ϕ_0 reflects an arbitrary datum point for zero energy.

As two simple illustrations of the mass formula $(1.3)_1$ for the regime $c < v < \infty$, at time $t = 0$ consider a particle travelling at n times the speed of light ($n > 1$), and subject to either (i) a constant force F or (ii) a force linearly dependent on velocity, namely fv, where f is a constant. In both cases with $x(0) = 0$ and $v(0) = nc$ and assuming the mass variation $(1.3)_1$, we may solve the momentum equation to determine the subsequent particle trajectory. Thus, for problem (i), we have

$$v(t) = c[1 + y_0(t)^2]^{1/2}$$

and

$$x(t) = ct_0\left[\frac{(1+y^2)^{3/2}}{y} - \frac{y(1+y^2)^{1/2}}{2} - \frac{1}{2}\sinh^{-1}(y)\right]_{\alpha}^{y_0(t)},$$

where t_0 is a characteristic time scale defined by $t_0 = m_\infty c/F$, $\alpha = (n^2-1)^{1/2}$ and $y_0(t)$ is given by

$$y_0(t) = \left[\frac{1}{2}\left(\frac{t}{t_0} + \beta\right)^{1/2} + \frac{1}{2}\left(\frac{t}{t_0} + \gamma\right)^{1/2}\right]^2,$$

where $\beta = (\alpha-1)^2/\alpha$, and $\gamma = (\alpha+1)^2/\alpha$. For problem (ii), we find

$$v(x) = c[1 + y_0(x)^2]^{1/2}$$

and

$$t(x) = \frac{x_0}{c}\left[\frac{(1+y^2)^{1/2}}{y} + \sinh^{-1}(y)\right]_{\alpha}^{y_0(x)},$$

where, in this case, the characteristic length scale x_0 is defined by $x_0 = m_\infty c/f$, and $y_0(x)$ is given by

$$y_0(x) = \left[\frac{1}{2}\left(\frac{x}{x_0} + \beta\right)^{1/2} + \frac{1}{2}\left(\frac{x}{x_0} + \gamma\right)^{1/2}\right]^2,$$

and where α, β and γ are as previously defined.

Finally, we comment that for any mass variation $m(v)$, the energy equations (5.1) admit the general integral, say for $c < v < \infty$,

$$\mathscr{E} + \int_c^v u m(u)\,\mathrm{d}u = mv^2 + \text{constant}, \tag{5.5}$$

since on writing (5.1) as

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}v} = v\frac{\mathrm{d}}{\mathrm{d}v}(mv),$$

and so by addition of mv on both sides, we obtain the integral (5.5). We may exploit (5.5) to deduce the energy expression for any assumed mass variation $m(v)$. For example, for $c < v < \infty$ and $m(v)$ defined by

$$m(v) = \frac{m_0 + m_\infty(v/c)}{[(v/c)^2 - 1]^{1/2}},$$

we find

$$\mathscr{E} = mc^2 + \frac{m_\infty c^2}{2}\left\{\left(\frac{v}{c}\right)\left[\left(\frac{v}{c}\right)^2 - 1\right]^{1/2} - \cosh^{-1}\left(\frac{v}{c}\right)\right\} + \mathscr{E}_0,$$

including both the classical formula when m_∞ is zero, and equation (5.4) when m_0 is zero.

6. Conclusions

We have proposed two possible new transformations between inertial frames as extensions of the Einstein special theory of relativity that apply for relative velocities v such that $c < v < \infty$, and do not necessitate the introduction of imaginary masses and complicated physics to make the theory sensible. The proposed new transformations connecting inertial frames, namely (3.16) and (3.18), are similar to the standard formulae (2.1) but have well-defined denominators. These new transformations are derived from certain 'pseudo-velocity' equations and constraints. In the regime $0 \leq v < c$, the constraint arises from $v = 0$ and the coincidence of velocities $u = \mathrm{d}x/\mathrm{d}t$ and $U = \mathrm{d}X/\mathrm{d}T$, namely $u = U$. However, in the regime $c < v < \infty$, the constraints arise from $v = \infty$ and the equation $uU = c^2$. Formally, in each regime, there are two possible theories. For $0 \leq v < c$, the second theory is excluded on the basis that the Galilean transformation applies for $v \ll c$. For $c < v < \infty$, the situation is not quite so transparent, and there are two possibilities. We may suggest the new transformation (3.16) in preference to the transformation (3.18) on the basis that the approximate analysis for $v \approx c$ given in appendix A predicts the same spatial and temporal dependence for (3.16) but not for (3.18). However, from (3.17) and (3.19), we see that (3.16) preserves the sign of $x - ct$ but changes the sign of $x + ct$, whereas (3.18) preserves the sign of $x + ct$ and changes the sign of $x - ct$. Because $x + ct$ is well defined through $v = c$, whereas $x - ct$ is singular at $v = c$, there may be an argument to suggest that the changing of sign should occur on the singular characteristic, which recommends (3.18) in preference to (3.16). Neither argument is completely convincing, and conceivably Einstein's special relativity could bifurcate through the singularity $v = c$ to become two equally physically plausible theories.

The well-established Einstein formulae (1.1) involve Lorentz invariant energy–momentum relations (5.2), and if for $c < v < \infty$ we also require the corresponding relations to be invariant under the new transformations, then inevitably the only mass variation with this feature is $(1.2)_1$ along with the well-known formula $\mathscr{E} = mc^2$. Together, the two equations (1.1) and (1.2) show that at $v = 0$ the rest mass is finite while the momentum is zero, while for the new theory, at $v = \infty$ the momentum is finite while the mass is predicted to be zero.

The mathematical formalism suggests a connection with v for $0 \le v < c$ and c^2/v for $c < v < \infty$. On the basis of this connection and on Einstein's well-known formula $(1.1)_1$ for the mass variation, we have also proposed the new formula $(1.3)_1$ for the variation of the mass with relative velocity in the regime $c < v < \infty$. In this case, Einstein's energy formula $\mathscr{E} = mc^2$ no longer holds, and instead on expansion, the respective energy expressions $(1.1)_2$, $(1.2)_2$ and $(1.3)_2$ give

$$\mathscr{E} = \begin{cases} m_0 c^2 + \dfrac{m_0}{2} v^2 + \cdots, & 0 \le v < c, \\ \dfrac{p_\infty c^2}{v}\left[1 + \dfrac{1}{2}\left(\dfrac{c}{v}\right)^2\right] + \cdots, & c < v < \infty, \\ \dfrac{m_\infty c^2}{2}\left[\left(\dfrac{v}{c}\right)^2 - \log\left(\dfrac{2v}{c}\right) + \dfrac{3}{2} + \dfrac{9}{8}\left(\dfrac{c}{v}\right)^2\right] + \mathscr{E}_0 + \cdots, & c < v < \infty, \end{cases}$$

where m_0 denotes the usual rest mass, p_∞ denotes the limiting momentum for infinite relative velocity and m_∞ is the value of the mass for infinite relative velocity. As far as the authors are aware, the relationship of m_0 with either p_∞/c or m_∞ is unknown through the theory of special relativity, and no doubt requires a higher theory to establish any connection.

Our primary purpose here has been to provide a deliberate account of two possible extensions of the Einstein special theory of relativity, and lay down the basic equations of the extended theories. At this stage, we particularly do not want to enter into the debate regarding the merits of existing theories for the uncommon physics applying for motion with velocities in excess of the speed of light. However, the formulae elucidated here will have many physical implications for issues such as light-cones, causality and space- and time-like intervals. It will have implications for the stipulations on the structure of space–time, such as those proposed by Ignatowsky (1911), and it will be of interest in many other areas of theoretical physics.

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Appendix A. Approximate analysis of (3.6) for $v \approx c$

We observe from (3.6) that in the limit of v tending to c, the coefficients of the highest order derivatives tend to zero, and therefore an approximate asymptotic analysis of (3.6) is necessary. In the regime $0 \le v < c$, we look for the solution of (3.6) about $v = c$, and we make the substitution $v = c(1 - \xi)$ where $\xi \ll 1$, and we have $1 - (v/c)^2 \approx 2\xi$ and $\mathrm{d}v = -cd\xi$. From these relations, we see that (3.6) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{ct}{2\xi}, \quad \frac{\mathrm{d}t}{\mathrm{d}\xi} = \frac{x}{2c\xi}, \tag{A 1}$$

which by differentiation of either equation with respect to ξ, and solving the resulting second-order ordinary differential equation, we may readily deduce that

$$x = A\xi^{1/2} + B\xi^{-1/2}, \quad ct = A\xi^{1/2} - B\xi^{-1/2}, \tag{A 2}$$

where A and B denote arbitrary constants. From (A 2), it is clear that

$$x + ct = 2A\xi^{1/2}, \quad x - ct = 2B\xi^{-1/2},$$

but from (2.4) we may deduce

$$x + ct = \left(\frac{\xi}{2}\right)^{1/2}(X + cT), \quad x - ct = \left(\frac{2}{\xi}\right)^{1/2}(X - cT), \tag{A 3}$$

and therefore we obtain

$$A = \frac{(X + cT)}{2^{3/2}}, \quad B = \frac{(X - cT)}{2^{1/2}}, \tag{A 4}$$

so that either from (A 2) and (A 4) or directly from (A 3), we may deduce the asymptotic expressions

$$x = \frac{1}{2}\left\{\left(\frac{\xi}{2}\right)^{1/2}(X + cT) + \left(\frac{2}{\xi}\right)^{1/2}(X - cT)\right\}$$

and

$$ct = \frac{1}{2}\left\{\left(\frac{\xi}{2}\right)^{1/2}(X + cT) - \left(\frac{2}{\xi}\right)^{1/2}(X - cT)\right\},$$

which are valid in the limit $\xi \to 0$. Thus to leading order, we have

$$x \approx \frac{(X - cT)}{(2\xi)^{1/2}}, \quad t \approx \frac{(T - X/c)}{(2\xi)^{1/2}}, \tag{A 5}$$

valid in the limit $v \to c$ from below and where $\xi = 1 - (v/c)$.

We now repeat the same analysis for the regime $c < v < \infty$ and v tending to c from above. Accordingly, we make the substitution $v = c(1 + \eta)$, where $\eta \ll 1$. In this case, we have $1 - (v/c)^2 \approx -2\eta$ and $dv = cd\eta$ and from these relations, we see that (3.6) becomes the same equation as (A 1) but with η in place of ξ, so that we have the same formal solution:

$$x = C\eta^{1/2} + D\eta^{-1/2}, \quad ct = C\eta^{1/2} - D\eta^{-1/2},$$

where C and D denote arbitrary constants. From these relations, it is clear that

$$x + ct = 2C\eta^{1/2}, \quad x - ct = 2D\eta^{-1/2},$$

while the two alternatives (3.17) and (3.19) yield respectively

$$x + ct = -\left(\frac{\eta}{2}\right)^{1/2}(X + cT), \quad x - ct = \left(\frac{2}{\eta}\right)^{1/2}(X - cT)$$

and

$$x + ct = \left(\frac{\eta}{2}\right)^{1/2}(X + cT), \quad x - ct = -\left(\frac{2}{\eta}\right)^{1/2}(X - cT).$$

In the former case, we have the asymptotic relations

$$x = -\frac{1}{2}\left\{\left(\frac{\eta}{2}\right)^{1/2}(X + cT) - \left(\frac{2}{\eta}\right)^{1/2}(X - cT)\right\}$$

and

$$ct = -\frac{1}{2}\left\{\left(\frac{\eta}{2}\right)^{1/2}(X + cT) + \left(\frac{2}{\eta}\right)^{1/2}(X - cT)\right\}$$

while in the latter case, we obtain

$$x = \frac{1}{2}\left\{\left(\frac{\eta}{2}\right)^{1/2}(X + cT) - \left(\frac{2}{\eta}\right)^{1/2}(X - cT)\right\}$$

and

$$ct = \frac{1}{2}\left\{\left(\frac{\eta}{2}\right)^{1/2}(X + cT) + \left(\frac{2}{\eta}\right)^{1/2}(X - cT)\right\},$$

so that in the limit $\eta \to 0$, we obtain the respective limits

$$x \approx \frac{(X - cT)}{(2\eta)^{1/2}}, \quad t \approx \frac{(T - X/c)}{(2\eta)^{1/2}}$$

and

$$x \approx \frac{(cT - X)}{(2\eta)^{1/2}}, \quad t \approx \frac{(X/c - T)}{(2\eta)^{1/2}}.$$

On comparison of these expressions with (A 5), it is clear that the former gives the same local spatial and temporal dependence, while the latter corresponds to the alternative theory for the regime $0 \le v < c$, which has both space and time reversed. On the basis of the above approximate analysis, we might suggest that the new transformation (3.16) provides the continuation of the Einstein special theory of relativity into the regime $c < v < \infty$. Finally, we comment that the asymptotic formulae derived are entirely consistent with those obtained directly from the appropriate Lorentz transformation and the new transformations.

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