A variational formulation of vertical slice models

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A variational framework is defined for vertical slice models with three-dimensional velocity depending only on $x$ and $z$. The models that result from this framework are Hamiltonian, and have a Kelvin–Noether circulation theorem that results in a conserved potential vorticity in the slice geometry. These results are demonstrated for the incompressible Euler–Boussinesq equations with a constant temperature gradient in the $y$-direction (the Eady–Boussinesq model), which is an idealized problem used to study the formation and subsequent evolution of weather fronts. We then introduce a new compressible extension of this model. Unlike the incompressible model, the compressible model does not produce solutions that are also solutions of the three-dimensional equations, but it does reduce to the Eady–Boussinesq model in the low Mach number limit. Hence, the new model could be used in asymptotic limit error testing for compressible weather models running in a vertical slice configuration.

1. Introduction

This paper introduces a variational framework for deriving geophysical fluid dynamics models in a vertical slice geometry (i.e. the $x$–$z$ plane). The work is motivated by the asymptotic limit solutions framework advocated in Cullen [1], where model error in dynamical cores for numerical weather prediction models can be quantified by comparing limits of numerical solutions with solutions from semigeostrophic (SG) models. In particular, the SG solutions of the Eady frontogenesis problem specified in a vertical slice geometry prove very useful because they can be solved in a two-dimensional domain, which means that they can be run quickly on a single workstation. In the incompressible hydrostatic and non-hydrostatic cases, these solutions are
equivalent to exact solutions of the full three-dimensional equations. As described in Cullen [1],
this proves to be a challenging test problem. Using a Lagrangian numerical discretization that
uses the optimal transport formulation, converged numerical integrations of the SG model
indicate an almost periodic cycle in which fronts form, change shape and then relax again
to a smooth solution. However, primitive equation solutions obtained by Garner et al. [2] are
rather dissipative due to the need for eddy viscosity to stabilize the numerics, and the periodic
behaviour is not observed; this leads to a loss of predictability after the formation of the front. In
Cullen [1], it is suggested that greater predictability in this limit might be possible if the numerical
solution exhibits energy and potential vorticity (PV) conservation over long time periods; it is also
suggested that a form of Lagrangian averaging may be required to obtain accurate predictions of
the subsequent front evolution. Because energy conservation can be derived from a variational
framework and PV arises from the particle relabelling symmetry, this has motivated us to develop
such a framework in the case of ‘slice geometries’ in which there are three components of velocity,
but they are functions of $x$ and $z$ only.

Another motivation for our work is that efforts to compare compressible models with the two-
dimensional SG solutions have been thwarted by the fact that it is not possible to construct a
compressible vertical slice model with solutions that are consistent with the full three-dimensional
model, with conserved energy and PV. This is owing to the nonlinear dependence in the equation
of state on the $y$-dependent component of the temperature. Hence, so far, asymptotic limit studies
of compressible models have only been performed over short time intervals corresponding to the
initial stages of front formation [3]. In this paper, we introduce a new slice compressible model
(SCM) that can be used in asymptotic limit studies because it has a conserved energy and PV. The
price to pay is that the solutions are not consistent with the full three-dimensional equations.
However, the model should still be very useful in studying the behaviour of discretization
methods and averaging procedures for numerical weather prediction in the presence of fronts.

Our approach is to derive models in the Euler–Poincaré framework [4]. This framework is a
way of obtaining variational models without resorting to Lagrangian coordinates, by providing
formulae that express how infinitesimal variations in the Lagrangian flow map correspond
to variations in the Eulerian prognostic variables. This paper specializes to the case where
all the Eulerian fields are independent of $y$. This corresponds to a subgroup of the group of
diffeomorphisms in three dimensions, which can be expressed as a semidirect product of two-
dimensional diffeomorphisms in the vertical slice and rigid displacements in the $y$-direction.

Having selected this group, the Euler–Poincaré theory immediately tells us how to perform
Hamilton’s principle. In this framework, the problem of developing slice models reduces to the
problem of choosing which Lagrangian to substitute into the action.

The structure of this paper is as follows. In §2a, we identify the slice subgroup, and set up the
geometrical framework. In §2b, we then obtain the general equations of motion corresponding
to the Euler–Poincaré equation with advected density and tracer variables (temperature). In
§2c, we reformulate the equations in a more geometrical notation, and show that the equations
conserve energy in the case of the Lagrangian without explicit time dependence; this is shown
by recasting the equations in Lie–Poisson form. We also show that the equations have a Kelvin–
Noether circulation theorem. This circulation theorem differs from the usual circulation theorem
for baroclinic fluids, which have a baroclinic circulation production term on the right-hand side
that vanishes only if the circulation loop lies on an isentropic surface. In the slice geometry, this
baroclinic term can be rewritten as the time derivative of another circulation term, and we obtain
conservation of the total circulation on arbitrary curves within the slice. This circulation theorem
leads to a conserved PV that turns out to correspond to the usual three-dimensional Ertel PV. We
then use this framework to present a number of models in the slice geometry. In §3, we show
how to obtain the Euler–Boussinesq–Eady model. We present the corresponding Lagrangian-
averaged Eady model in §4, and introduce our new SCM in §5, comparing it with the model used
in Cullen [3]. Finally, we provide a summary and outlook in §6. The appendices provide proofs
and show how this framework relates to known Lie–Poisson formulations of superfluid models.
This relationship is significant because it shows how to build conservative numerical schemes in the slice geometry. This last point is also discussed in §6.

2. Vertical slice models

(a) Definition

Physically, slice models are used to describe the formation of fronts in the atmosphere and ocean. These fronts arise when there is a strong north–south temperature gradient (maintained by heating at the Equator and cooling at the Pole), which maintains a vertical shear flow in the east–west direction through geostrophic balance. In an idealized situation, neglecting the Earth’s curvature and assuming a constant Coriolis parameter \( f \), this basic steady state can be modelled with a three-dimensional flow in which there is a constant temperature gradient in the \( y \)-direction, and the velocity points in the \( x \)-direction with a linear shear in the \( z \)-direction. This basic flow is unstable to \( y \)-independent perturbations in all three components of velocity and temperature, which rapidly lead to the formation of fronts that vary sharply in the \(-\)-direction. The presence of the constant gradient of the temperature in the \( y \)-direction means that the \( y \)-component of velocity is coupled to other variables because it can lead to a source or sink of temperature in each vertical slice. Because all of the perturbations are \( y \)-independent, we can consider the dynamics in a single vertical slice without loss of generality.

To build a variational vertical slice model of this type, it is assumed that the forward Lagrangian map takes the form

\[
\phi(X, Y, Z, t) = (x(X, Z, t), y(X, Z, t) + Y, z(X, Z, t)),
\]

where \((X, Y, Z)\) are Lagrangian labels, \((x, y, z)\) are particle locations and \( t \) is time, i.e.

\[
\frac{\partial \phi}{\partial Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Such maps form a subgroup of the diffeomorphisms\(^1\) \( \text{Diff}(\Omega \times \mathbb{R}) \) (where \( \Omega \in \mathbb{R}^2 \) is the domain in the \( x-z \) plane, and \( \mathbb{R} \) represents an infinite line in the \( y \)-direction). This subgroup is isomorphic to \( \text{Diff}(\Omega \ltimes F(\Omega)) \), where \( \ltimes \) denotes the semidirect product, and \( F(\Omega) \) denotes an appropriate space of smooth functions on \( \Omega \) that specify the displacement of Lagrangian particles in the \( y \)-direction at each point in \( \Omega \). Multiplication in the semidirect-product group is given by a standard formula [4],

\[
(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, \phi_1 \circ f_2 + f_1).
\]

The corresponding Lie algebra is isomorphic to \( \mathfrak{X}(\Omega) \ltimes F(\Omega) \), where \( \mathfrak{X}(\Omega) \) denotes the vector fields on \( \Omega \), representing the two components of the velocity \( u_S \in \mathfrak{X}(\Omega) \) in the \( x-z \) plane, and the smooth function \( u_T \in F(\Omega) \) represents the \( y \)-component of the velocity. We write elements of \( \mathfrak{X}(\Omega) \ltimes F(\Omega) \) as \((u_S, u_T)\), where \( u_S \) is the ‘slice’ component in the \( x-z \) plane, and \( u_T \) is the ‘transverse’ component in the \( y \)-direction. In component notation, the Lie bracket for the Lie algebra \( \mathfrak{X}(\Omega) \ltimes F(\Omega) \) of the semidirect-product group \( \text{Diff}(\Omega) \ltimes F(\Omega) \) takes the form

\[
[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S], u_S \cdot \nabla w_T - w_S \cdot \nabla u_T),
\]

where \([u_S, w_S] = u_S \cdot \nabla w_S - w_S \cdot \nabla u_S\) is the Lie bracket for the time-dependent vector fields \((u_S, w_S) \in \mathfrak{X}(\Omega))\), and \( \nabla \) denotes the gradient in the \( x-z \) plane.

We introduce two types of advected quantities in this framework.

\(^1\) Diffeomorphisms are smooth invertible maps with smooth inverses.
First, mass is conserved locally, so the mass element $Dd^3x$ is advected in three-dimensional space. That is, the mass density $D(x, y, z, t)$ satisfies

$$(\partial_t + L_{(u_S, u_T)})(Dd^3x) = (\partial_t D + \nabla \cdot (u_S D) + \partial_y (u_T D))d^3x = 0,$$

with partial time derivative $\partial_t = \partial/\partial t$ and partial space derivative $\partial_y = \partial/\partial y$ in the $y$-direction normal to the $x$-$z$ plane. If $u_T$ and $D$ are specified to be $y$-independent consistent with the slice motion assumption, then the last term vanishes, and the equation for conservation of mass reduces to advection of an areal density $DdS \in A^2(\Omega)$, in which $D(x, z, t)$ satisfies the continuity equation

$$\partial_t D + \nabla \cdot (u_S D) = 0. \quad (2.4)$$

Second, in order to represent potential temperature that has a constant gradient in the $y$-direction, $s = \partial \tilde{\theta}/\partial y = \text{constant}$, we shall require advected scalars $\theta(x, y, z, t)$ that may be decomposed into dynamic and static parts, as

$$\theta(x, y, z, t) = \theta_S(x, z, t) + (y - y_0)s. \quad (2.5)$$

Consequently, the three-dimensional scalar tracer equation

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T \partial_y \theta = 0$$

becomes a dynamic equation for $\theta_S(x, z, t) \in \mathcal{F}(\Omega)$ that satisfies

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T = 0, \quad (2.6)$$

in which we keep in mind that $s$ is a constant and $u_T(x, z, t)$ has been specified to be $y$-independent. The space of advected scalars of this type is isomorphic to $\mathcal{F}(\Omega) \times \mathbb{R}$, represented as pairs $(\theta_S, s)$, with infinitesimal Lie algebra action

$$L_{(u_S, u_T)}(\theta_S, s) = (u_S \cdot \nabla \theta_S + u_T s, 0).$$

(b) Variational formulation via Hamilton’s principle

In this section, we show how to perform variational calculus in the slice geometry. Vector fields of infinitesimal variations $(\omega_S, \omega_T)$ in the Lie algebra $\mathfrak{X}(\Omega) \circledast \mathcal{F}(\Omega)$ of the semidirect-product group $\text{Diff}(\Omega) \circledast \mathcal{F}(\Omega)$ induce infinitesimal variations in $(u_S, u_T), D$ and $(\theta_S, s)$ as follows$^2$:

$$\begin{align*}
\delta(u_S, u_T) &= (\partial_t \omega_S + [u_S, \omega_S], \partial_t w_T + u_S \cdot \nabla w_T - w_S \cdot \nabla u_T), \\
\delta D &= -\nabla \cdot (w_S D), \\
\delta(\theta_S, s) &= (w_S \cdot \nabla \theta_S - w_T s, 0)
\end{align*} \quad (2.7)$$

and

For a Lagrangian functional $L((u_S, u_T), (\theta_S, s), D) : (\mathfrak{X} \circledast \mathcal{F}(\Omega)) \circledast ((\mathcal{F}(\Omega) \times \mathbb{R}) \times A^2(\Omega)) \to \mathbb{R}$, we apply Hamilton’s principle and obtain

$$0 = \delta S = \delta \int_0^T L((u_S, u_T), (\theta_S, s), D) dt$$

$$= \int_0^T \left( \frac{\delta l}{\delta (u_S, u_T)} \delta (u_S, u_T) + \frac{\delta l}{\delta (\theta_S, s)} \delta (\theta_S, s) + \frac{\delta l}{\delta D} \delta D \right) dt$$

$^2$ These are standard formulae for defining the variations in Hamilton’s principle. See Holm et al. [4] and appendix A for details.
where the angle brackets indicate $L_2$ inner products with integration over $\mathbb{R}^2$. The last term makes no contribution for velocity variations $(w_s, w_T)$ that vanish at the endpoints in time.

Hence, we obtain the Euler–Poincaré equations on the slice semidirect product with advected density $D$ and scalar $\theta$:

$$
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \nabla \cdot \left( u_s \otimes \frac{\delta l}{\delta u} \right) + (\nabla u_s)^T \cdot \frac{\delta l}{\delta u} + \frac{\delta l}{\delta u_T} \nabla u_T = D \nabla \cdot \left( \frac{\delta l}{\delta \theta} \right)
$$

and

$$
\frac{\partial}{\partial t} \frac{\delta l}{\delta u_T} + \nabla \cdot \left( u_s \frac{\delta l}{\delta u_T} \right) = -\frac{\delta l}{\delta \theta} \cdot s.
$$

System (2.9) is completed by including the advection equations (2.4) and (2.6) for $D$ and $\theta_s$, respectively.

(c) Geometrical reformulation and Kelvin–Noether circulation theorem

Theorem 2.1 (Energy conservation). If the Lagrangian $l$ has no explicit time dependence, then the energy functional defined by the Legendre transformation

$$
h[(m_s, m_T), (\theta_s, s), D] = \langle (m_s, m_T), (u_s, w_T) \rangle - l[(u_s, w_T), (\theta_s, s), D] \tag{2.10}
$$

is conserved for solutions of equations (2.4), (2.6) and (2.9).

Proof. In appendix B, we show that equations (2.4), (2.6) and (2.9) are Hamiltonian, with the Hamiltonian given by $h$ in equation (2.10). If $l$ has no explicit time dependence, then $h$ has no explicit time dependence and is therefore an invariant of the Hamiltonian system. $\blacksquare$

Theorem 2.2 (Kelvin–Noether circulation theorem). Equations (2.4), (2.6) and (2.9) imply a conservation law for circulation,

$$
\frac{d}{dt} \int_{c(u_s)} \left( s \left( \frac{1}{D} \frac{\delta l}{\delta u_s} \right) - \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) \nabla \theta_s \right) \cdot dx = 0, \tag{2.11}
$$

where $c(u_s)$ is a circuit in the vertical slice moving with velocity $u_s$ and $s = s$ is a constant parameter.

Proof. The proof of the theorem is facilitated by rewriting the system of equations (2.4), (2.6) and (2.9) equivalently in the following geometrical form:

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_s} \right) \cdot dx + \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) \cdot dx_T = d \left( \frac{\delta l}{\delta D} \right) - \left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) d \theta_s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{\delta l}{\delta u_s} \right) \cdot dx + \left( \frac{\delta l}{\delta u_T} \right) \cdot dx_T = d \left( \frac{\delta l}{\delta D} \right) - \left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) d \theta_s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) = -\left( \frac{1}{D} \frac{\delta l}{\delta \theta_s} \right) s,
$$

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) \theta_s + w_T s = 0
$$

and

$$
\left( \frac{\partial}{\partial t} + L_{u_s} \right) (DdS) = 0,
where $\mathcal{L}_{u_S}$ denotes Lie derivative along the vector field $u_S$. One may then verify the circulation theorem (2.11) for slice models by applying the relation

$$\frac{d}{dt} \oint_{c(u_S)} v(x, t) \cdot dx = \oint_{c(u_S)} \left( \frac{\partial}{\partial t} + \mathcal{L}_{u_S} \right) (v(x, t) \cdot dx),$$

for any vector $v(x, t)$ in the slice. $\blacksquare$

**Corollary 2.3.** The system of equations (2.12) implies that the following PV (denoted as $q$) is conserved along flow lines of the fluid velocity $u_S$,

$$\partial_t q + u_S \cdot \nabla q = 0 \quad \text{for PV} \quad q := \frac{1}{D} \left( \text{curl} \left( \frac{1}{D} \frac{\delta l}{\delta u_S} + \nabla \theta_S \times \nabla \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) \right) \right) \cdot \hat{y}. \quad (2.13)$$

**Proof.** Applying the differential operation $d$ to the first equation in system (2.12) yields

$$\left( \partial_t + \mathcal{L}_{u_S} \right) \left( \left( \text{curl} \left( \frac{1}{D} \frac{\delta l}{\delta u_S} \right) + s^{-1} \nabla \theta_S \times \nabla \left( \frac{1}{D} \frac{\delta l}{\delta u_T} \right) \right) \cdot \hat{y} dS \right) = 0, \quad (2.14)$$

where $dS$ is the surface element in the vertical slice, whose normal vector is $\hat{y}$. Applying the Lie derivative and using the continuity equation for $D$ then yields the local conservation law (2.13). $\blacksquare$

Upon introducing the new notation

$$v_S := \frac{\delta l}{\delta u_S}, \quad v_T := \frac{\delta l}{\delta u_T}, \quad \pi := \frac{\delta l}{\delta D} \quad \text{and} \quad \gamma_S := \frac{\delta l}{\delta \theta_S}, \quad (2.15)$$

system (2.12) takes a slightly more transparent form,

$$\begin{align*}
\left( \partial_t + \mathcal{L}_{u_S} \right) (v_S \cdot dx) &= d\pi - v_T d\theta_T - \gamma_S d\theta_S, \\
\left( \partial_t + \mathcal{L}_{u_S} \right) v_T &= -s \gamma_S, \\
\left( \partial_t + \mathcal{L}_{u_S} \right) d\theta_S &= -s d\theta_T,
\end{align*} \quad (2.16)$$

and

$$\left( \partial_t + \mathcal{L}_{u_S} \right) (DdS) = 0, \quad$$

in which the differential of the third equation has also been taken. Hence, combining the middle two equations in (2.16) results in

$$\left( \partial_t + \mathcal{L}_{u_S} \right) (v_T d\theta_S) = -s (v_T d\theta_T + \gamma_S d\theta_S), \quad (2.17)$$

Inserting this formula into the first equation in (2.16) implies that

$$\left( \partial_t + \mathcal{L}_{u_S} \right) (sv_S \cdot dx - v_T d\theta_S) = d\pi. \quad (2.18)$$

This relation then yields the Kelvin–Noether circulation theorem as stated earlier in (2.11),

$$\frac{d}{dt} \oint_{c(u_S)} (sv_S - v_T \nabla \theta_S) \cdot dx = \oint_{c(u_S)} \left( \partial_t + \mathcal{L}_{u_S} \right) (sv_S \cdot dx - v_T d\theta_S) = \oint_{c(u_S)} d\pi = 0, \quad (2.19)$$

and PV conservation as in (2.13),

$$\partial_t q + u_S \cdot \nabla q = 0 \quad \text{for PV} \quad q := \frac{1}{D} \left( s \text{curl} v_S + \nabla \theta_S \times \nabla v_T \right) \cdot \hat{y}. \quad (2.20)$$

**Remark 2.4.** Note that this circulation theorem is different from the case of general three-dimensional motions, in which the circulation is preserved only if the loop integral is restricted to lie on a temperature isosurface. In the special case of slice motions, the baroclinic generation term can itself be written as the total derivative of a loop integral. The physical interpretation is that $q$ is in fact the usual three-dimensional PV. Owing to the existence of the linear $y$-variation in $\theta$, it is always possible to find an equivalent three-dimensional loop on a temperature isosurface that projects onto any given two-dimensional loop in the vertical slice plane.
Remark 2.5. In appendix B, we will discuss the geometrical meaning of the Kelvin–Noether circulation theorem (2.11) and the PV conservation law (2.13) from the viewpoint of the Lie–Poisson brackets in the Hamiltonian formulation of these equations.

3. The Euler–Boussinesq–Eady model

(a) Specializing the Euler–Poincaré equations to deal with the Eady model

The Euler–Boussinesq–Eady model, in a periodic channel of width $L$ and height $H$, has Lagrangian

$$\mathcal{L}(u, u_T, D, \theta, p) = \int_\Omega \frac{D}{2}(|u|^2 + |u_T|^2) + Du_T x + \frac{g}{\theta_0} D \left( z - \frac{H}{2} \right) \delta_S + p(1 - D) dV,$$

where $g$ is the acceleration due to gravity, $\theta_0$ is the reference temperature, $f$ is the Coriolis parameter, and we have introduced the Lagrange multiplier $p$ to enforce constant density. We obtain the following variational derivatives of this Lagrangian:

$$u_S = \frac{1}{D} \delta \mathcal{L} = u_S, \quad u_T = \frac{1}{D} \delta \mathcal{L} = u_T + f x,$$

$$\pi = \frac{1}{D} \delta \mathcal{L} = \frac{1}{2} (|u|^2 + |u_T|^2) + Du_T x - p + \frac{g}{\theta_0} \delta_S \left( z - \frac{H}{2} \right),$$

$$\gamma_S = \frac{1}{D} \delta \mathcal{L} = \frac{g}{\theta_0} \left( z - \frac{H}{2} \right) \text{ and } \delta \mathcal{L} = 1 - D.$$

Substitution of these variational derivatives into the Euler–Poincaré equations in (2.9) gives

$$\partial_t u_S + u_S \cdot \nabla u_S + (\nabla u_S)^T \cdot u_S + (u_T + f x) \nabla u_T = \nabla \left( \frac{1}{2} (|u|^2 + |u_T|^2) + u_T f x - p \right) + \frac{g}{\theta_0} \delta_S \left( z - \frac{H}{2} \right) \nabla \delta_S,$$

and

$$\partial_t u_T + u_S \cdot \nabla (u_T + f x) = - \frac{g}{\theta_0} \left( z - \frac{H}{2} \right) s.$$

Upon substituting $D = 1$, $\nabla \cdot u_S = 0$ and combining with equations (2.4) and (2.6) for the advected quantities $D$ and $\theta$, the system of equations (3.3) becomes

$$\partial_t u_S + u_S \cdot \nabla u_S - fu_T \hat{x} = - \nabla p + \frac{g}{\theta_0} \delta_S \hat{z},$$

$$\partial_t u_T + u_S \cdot \nabla u_T + fu_S \cdot \hat{x} = - \frac{g}{\theta_0} \left( z - \frac{H}{2} \right) s,$$

$$\nabla \cdot u_S = 0$$

and

$$\partial_t \delta_S + u_S \cdot \nabla \delta_S + u_T s = 0,$$

where $\hat{x}$ is the unit normal in the $x$-direction.

Remark 3.1. System (3.4) is the standard Euler–Boussinesq–Eady slice model.

(b) Geometrical reformulation and circulation theorem for the Eady model

Substitution of the variational derivatives in (3.2) into the geometrical form of the system of Euler–Poincaré equations in (2.16) gives the following equivalent form of this system:

$$\partial_t (u_S \cdot d) = - dp - (u_T + f x) du_T - \delta_S dy_S,$$

$$\partial_t (u_T + f x) = - s \gamma_S,$$

$$D = 1 \implies \nabla \cdot u_S = 0$$

and

$$\partial_t \delta_S + u_S \cdot \nabla \delta_S = - s du_T.$$
Consequently, we recover the Kelvin circulation conservation law (2.19) for the Eady model in the form
\[ \frac{d}{dt} \oint_{c(u_s)} (u_s - (u_T + fx) \nabla \theta_s) \cdot dx = \oint_{c(u_s)} \left( \frac{1}{2} |u_s|^2 - p + \gamma \theta_s \right) = 0. \] (3.6)

**Corollary 3.2.** Equation (3.6) and incompressibility imply that PV (denoted as \( q \)) is conserved along flow lines of the fluid velocity \( u_s \) in the Eady model,
\[ \partial_t q + u_s \cdot \nabla q = 0 \quad \text{for PV} \quad q := (\text{curl } u_s + \nabla \theta_s \times \nabla (u_T + fx)) \cdot \hat{y}. \] (3.7)

On denoting \( u_s = (u, w) \), \( u_T = v \), this PV may be written as
\[ q = \frac{\partial \hat{\theta}}{\partial y} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \frac{\partial (v + fx, \theta_s)}{\partial (x, z)}. \]

Applying the Legendre transform to the Lagrangian (3.1) yields the energy
\[ h[u_s, u_T, D, \theta, p] = \int_{\Omega} \frac{D}{2} \left( |u_s|^2 + u_T^2 \right) - \frac{g}{\theta_0} D \left( z - \frac{H}{2} \right) \theta_s dV. \] (3.8)

**Corollary 3.3.** The energy (3.8) is conserved for the Eady–Boussinesq slice model.

### 4. Lagrangian-averaged Boussinesq model

Numerical forecast models are restricted in grid resolution owing to the stringent time requirements of operational forecasting, and hence it is necessary to perform some form of averaging on the equations in order to prevent energy and enstrophy accumulating at the gridscale, either explicitly by introducing extra terms (i.e. eddy viscosities, or large Eddy simulation), or implicitly by numerical stabilization in advection schemes. All of these examples amount to some form of Eulerian averaging that leads to dissipation, which is thought to be detrimental to evolution of fronts. To avoid this, Cullen [1] suggested that some form of Lagrangian averaging may be required, also suggesting that it is important for averaging to retain energy and PV conservation if agreement with the SG-limiting solution is to be obtained.

In this section, we obtain a Lagrangian-averaged Boussinesq model from a variational principle, and so energy and PV conservation will follow immediately. Here, we shall interpret Lagrangian averaging as a regularization of the equations that is consistent with the Lagrangian flow map for slice models in equation (2.1). This regularization is obtained by replacing equation (3.1) with
\[ l[u_s, u_T, D, \theta, p] = \int_{\Omega} \left[ \frac{D}{2} \left( |u_s|^2 + \alpha^2 |u_s| + u_T^2 + \alpha^2 |u_T|^2 \right) + Dfu_Tx \right. \]
\[ + \left. \frac{g}{\theta_0} D \left( z - \frac{H}{2} \right) \theta_s + p(1 - D) \right] dV, \] (4.1)

where \( \alpha \) is a regularization length scale. We obtain the following variational derivatives of this Lagrangian:
\[ \tilde{u}_s = \frac{1}{D} \frac{\delta l}{\delta u_s} = (1 - \alpha^2 \nabla^2) u_s, \quad \tilde{u}_T = \frac{1}{D} \frac{\delta l}{\delta u_T} = (1 - \alpha^2 \nabla^2) u_T + fx, \]
\[ \pi = \frac{\delta l}{\delta D} = \frac{1}{2} (|u_s|^2 + u_T^2) + fu_Tx - p + \frac{g}{\theta_0} \theta_s \left( z - \frac{H}{2} \right), \]
\[ \gamma_s = \frac{1}{D} \frac{\delta l}{\delta \theta_s} = \frac{g}{\theta_0} \left( z - \frac{H}{2} \right) \quad \text{and} \quad \frac{\delta l}{\delta p} = 1 - D, \] (4.2)

where
\[ \nabla^2 = \frac{1}{D} \nabla \cdot D \nabla. \]
Substitution into the Euler–Poincaré equations and applying \( D = 1 \) gives

\[
\begin{align*}
\frac{\partial \tilde{u}_S}{\partial t} + u_S \cdot \nabla \tilde{u}_S + \nabla u_S^T \tilde{u}_S - f u_T \hat{x} &= -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \\
\frac{\partial \tilde{u}_T}{\partial t} + u_S \cdot \nabla \tilde{u}_T + f u_S \cdot \hat{x} &= -\frac{g}{\theta_0} \left( z - \frac{H}{2} \right) \hat{s}, \\
\nabla \cdot \tilde{u}_S &= 0, \\
\frac{\partial \theta_S}{\partial t} + u_S \cdot \nabla \theta_S + u_T s &= 0, \\
\tilde{u}_S &= (1 - \alpha^2 \nabla^2) u_S \\
\tilde{u}_T &= (1 - \alpha^2 \nabla^2) u_T.
\end{align*}
\]

(4.3)

This is the Lagrangian-averaged Boussinesq–Eady slice model.

**Corollary 4.1.** Equations (4.3) have conserved energy

\[
h = \int_{\Omega} D \left( |u_S|^2 + \alpha^2 |\nabla u_S|^2 + |u_T|^2 + \alpha^2 |\nabla u_T|^2 \right) - \frac{g}{\theta_0} D \left( z - \frac{H}{2} \right) \theta_S dV.
\]

**Corollary 4.2.** Equations (4.3) have Lagrangian PV conservation

\[
\frac{\partial q}{\partial t} + u_S \cdot \nabla q = 0 \quad \text{for PV} \quad q := \frac{1}{D} (s \text{curl} \tilde{u}_S + \nabla \theta_S \times \nabla \tilde{u}_T) \cdot \hat{y}.
\]

(4.4)

5. **Slice compressible model**

In this section, we present a model that is a compressible extension of the Boussinesq–Eady model described in §4. The aim of the model is to provide a framework where non-hydrostatic compressible dynamical cores can be benchmarked in a slice geometry. Owing to the nonlinear equation of state, it is not possible to write down an SCM with solutions that correspond to solutions of the full three-dimensional equations, and we need to proceed by replacing the full potential temperature \(\theta\) in the internal energy by the slice component \(\theta_S\). This approximation would be valid if the potential temperature were slowly varying in the \(y\)-direction. We derive a model that has conserved energy, PV, and supports baroclinic instability leading to front formation, so that dynamical cores in this configuration can be compared with the corresponding model in the SG limit.

In the present notation, the Lagrangian for the SCM in Eulerian \((x, y, z)\) coordinates is

\[
l[u_S, u_T, D, \theta_S] = \int_{\Omega} \frac{D}{2} (|u_S|^2 + u_T^2) + fD u_T x + gDz - Dc_v \theta_S \Pi dV,
\]

(5.1)

where \(\Pi\) is the Exner function given by

\[
\Pi = \left( \frac{p}{p_0} \right)^{R/c_p},
\]

where \(p_0\) is a reference pressure level and \(c_p\) and \(R\) are gas constants. The equation for an ideal gas becomes

\[
p_0 \Pi^{c_p/R} = DR \theta_S \Pi,
\]

and differentiating with respect to \(\theta_S\) and \(D\) gives

\[
\frac{\partial \Pi}{\partial \theta_S} = \frac{\partial}{\partial \theta_S} \left( \frac{DR \theta_S}{p_0} \right)^{\gamma - 1}
\]

\[
= (\gamma - 1) \frac{DR \theta_S}{p_0} \left( \frac{DR \theta_S}{p_0} \right)^{\gamma - 2}
\]

\[
= \frac{\gamma - 1}{\theta_S} \Pi = \frac{c_p - c_v}{c_v \theta_S} \Pi = \frac{R}{c_v \theta_S} \Pi.
\]
Similarly, we obtain
\[ \frac{\partial \Pi}{\partial D} = \frac{R}{c_i D^2} \Pi. \]

Note that we use \( \theta_S \) in both the internal energy term in the Lagrangian, and in the equation of state. This removes all \( y \)-dependence from the Lagrangian, making a slice model possible.

We obtain the following variational derivatives of this Lagrangian:
\[
\begin{align*}
\delta l = \frac{1}{D} (\delta u_S) = u_S, \\
\delta l = \frac{1}{D} (\delta u_T) = u_T + fx, \\
\delta l = \frac{1}{2} (\delta u_S^2 + u_T^2) + f u_T x + g z - c_p \Pi \theta_S \\
\delta l = \frac{1}{D} (\delta \theta_S) = -c_p \Pi,
\end{align*}
\]
(5.2)

and
\[
\gamma_S = \frac{1}{D} (\delta D) = \frac{1}{D} (\delta \theta_S) = -c_p \Pi,
\]

where we have used the decomposition (2.5) in the last line.

Substitution of the variational derivatives (5.2) of the SCM Lagrangian (5.1) into the Euler–Poincaré equations in (2.9) gives the system
\[
\begin{align*}
(\partial_t + L_{us})(u_S \cdot dx) &= -c_p \theta_S d\Pi + \frac{D}{2} (u_S^2 - g z) + f u_T dx, \\
(\partial_t + L_{us})(u_T + fx) &= s c_p \Pi, \\
\theta_S &= -s u_T \\
(\partial_t + L_{us})(DdS) &= 0.
\end{align*}
\]
(5.3)

Consequently, we recover the expected Kelvin circulation conservation law (2.19) for the SCM in the form
\[
\frac{d}{dt} \int_{c(u_S)} (u_S - s^{-1}(u_T + fx) \nabla \theta_S) \cdot dx = 0.
\]
(5.4)

**Corollary 5.1.** The system of SCM equations in (5.3) implies that PV \( q \) is conserved along flow lines of the fluid velocity \( u_S \),
\[
\partial_t q + u_S \cdot \nabla q = 0 \quad \text{with PV} \quad q := \frac{1}{D} (s \text{curl } u_S + \nabla \theta \times \nabla (u_T + fx)) \cdot \hat{y}.
\]
(5.5)

**Corollary 5.2.** These equations are Hamiltonian, with conserved energy
\[
E = \int_{\Omega} \frac{D}{2} (u_S^2 + u_T^2) - g Dz + c_i D \Pi \theta_S dV.
\]
Next, we check that the basic state of these equations supports a shear profile (and hence allows baroclinic instability and frontogenesis). Reverting to more standard notation \( u_S = (u, w), u_T = v, \)
the balance equations are

\[-fv = -cp\theta_S \frac{\partial \pi}{\partial x}, \tag{5.7}\]

\[fu = \frac{\partial \tilde{\theta}}{\partial y} c_p \Pi \tag{5.8}\]

and

\[0 = -cp\theta \frac{\partial \pi}{\partial z} - g. \tag{5.9}\]

Assuming an \(x\)-independent temperature field, then equation (5.7) implies that \(v = 0\). For positive \(\theta\), equation (5.9) implies that \(\Pi\) will increase with height, and equation (5.8) then implies that \(u\) decreases with height, leading to a shear profile in the basic state.

We now compare our SCM with the SCM in Cullen [3] and identify the differences. On defining velocity \(u = (u_S, u_T)\) with \(u_S\) in the vertical slice, and \(u_T\) transverse to it, the model in Cullen [3] in Eulerian \((x, y, z)\) coordinates becomes, in the present notation,

\[
\begin{align*}
\partial_t u_S + u_S \cdot \nabla u_S - fu_T \hat{x} &= -cp\theta \nabla \Pi - g\hat{z}, \\
\frac{1}{2} d(u_S^2 - g)z + fu_T dx, \\
\partial_t u_T + u_S \cdot \nabla u_T + fu_S \cdot \hat{x} &= -cp\theta \Pi', \\
\partial_t \theta_S + u_S \cdot \nabla \theta_S &= -s u_T, \\
\frac{\partial D}{\partial t} + \nabla \cdot (Du_S) &= 0.
\end{align*}
\tag{5.10}
\]

Writing Cullen’s [3] equations in Lie-derivative form yields, cf. equation (5.6),

\[
\begin{align*}
(\partial_t + \mathcal{L}_{u_S})(u_S \cdot dx) &= -cp\theta d\Pi + d\left(\frac{1}{2} |u_S|^2 - g\hat{z}\right) + fu_T dx, \\
(\partial_t + \mathcal{L}_{u_S})(u_T + fx) &= -cp\theta \Pi', \\
(\partial_t + \mathcal{L}_{u_S})\theta &= -s u_T, \\
(\partial_t + \mathcal{L}_{u_S})(DdS) &= 0.
\end{align*}
\tag{5.11}
\]

These equations differ from the SCM equations in (5.6) by only one term. Namely, the right-hand sides of the second equation in each set differ, with \((-cp\theta \Pi'')\) in these equations and \((sc_p\Pi')\) in (5.6). It turns out that this single difference has important consequences for their respective circulation laws.

The circulation law for the SCMs in Cullen [3] is similar to that for the SCM in §4, but with one important difference. Namely, theorem 5.4.

**Theorem 5.4.** Circulation for the SCMs in Cullen [3] is not conserved. Instead, we find

\[
\frac{d}{dt} \oint_{\ell(u_S)} (u_S - s^{-1}(u_T + fx))\nabla \theta \cdot dx = -\oint_{\ell(u_S)} cp\theta \nabla \Pi \cdot dx. \tag{5.12}
\]

**Proof.** The proof uses the first three equations in system (5.11). The middle two equations yield

\[(\partial_t + \mathcal{L}_{u_S})(-s^{-1}(u_T + fx)d\theta) = \frac{1}{2} d(s^{-1} cp\Pi''\theta^2 + u_T^2) + fxdu_T.\]

Combining this formula with the first equation in system (5.11) then yields the circulation law (5.12).

**Corollary 5.5.** Equation (5.12) implies that PV (still denoted as \(q\)) is created along flow lines of the fluid velocity \(u_S\) as

\[
\partial_t q + u_S \cdot \nabla q = cpD^{-1}\nabla \times \nabla \theta \cdot \hat{y} \quad \text{with PV given by} \quad q := D^{-1}(s \text{ curl } u_S + \nabla \theta \times \nabla (u_T + fx)) \cdot \hat{y}. \tag{5.13}
\]
Proof. Applying Stokes’ theorem to the circulation equation in (5.12) yields
\[
\frac{d}{dt} \int \int_{\partial S(\mathbf{u}_S)} (\text{curl} \mathbf{u}_S + s^{-1} \nabla \theta \times \nabla (\mathbf{u}_T + f) \cdot \hat{y} \, dS = \int \int_{\partial S(\mathbf{u}_S)} c_p \nabla \Pi \times \nabla \theta \cdot \hat{y} \, dS,
\]
(5.14)
where \(\hat{y} \, dS\) is the surface element in the vertical slice, whose normal vector is \(\hat{y}\). Expanding the time derivative in (5.14) and applying the Lie-derivative relation for \(D\) in the last equation of system (5.11), which is the continuity equation for \(D\), then yields the local PV evolution equation in (5.13).

Remark 5.6. This is the main difference between the SCM here and in Cullen [3]. According to corollary 5.1, the PV in the SCM is conserved, and this conservation is a general property of this class of Euler–Poincaré equations, as given by corollary 2.3. By contrast, according to corollary 5.5, the PV in the model of Cullen [3] (when viewed as a slice model) is created whenever the gradients of \(\theta\) and \(\Pi\) are not aligned. This, combined with the lack of a conserved energy, meant that it was not possible to obtain long-time asymptotic convergence results in a compressible model because these quantities are conserved in the equivalent SG model; Cullen [3] restricted to looking at the asymptotic magnitude of the geostrophic imbalance in the solution. Our new model addresses this problem, allowing asymptotic limit tests to be performed with compressible models in a slice configuration.

6. Summary and outlook

In this paper, we have shown how to construct variational models for geophysical fluid dynamics problems in a vertical slice configuration in which there is motion transverse to the slice, but the velocity field is independent of the transverse coordinate. (The vertical slice configuration may be taken as the \(x-z\) plane. Then, the transverse coordinate is \(y\).) Any model developed in this framework has a conserved energy, and corresponding conserved PV. The formulation has a number of interesting geometrical features, arising from the semidirect-product structure of the slice subgroup of the group of three-dimensional diffeomorphisms. First, the formulation leads to a Kelvin–Noether circulation theorem in which circulation is preserved on arbitrary loops in the slice, unlike the usual circulation theorem in which circulation is only preserved on isentropic surfaces. Second, as shown in appendix B, the equations can always be rewritten in terms of a pair of two-dimensional momenta, one comprising the \(x\)- and \(z\)-components of linear momentum, and one formed from the temperature and the \(y\)-component of linear momentum, plus the density. This formulation involving only two-dimensional momenta and density meaning that PV-conserving numerical schemes for the shallow-water equations can be adapted for vertical slice problems. In the shallow-water case, the equations can be written in the form
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) v \cdot dx + d\pi = 0,
\]
where \(u\) is the velocity, \(v\) is the total momentum divided by the layer thickness and \(\pi\) is a pressure. It is possible using mimetic/discrete exterior calculus methods [5] to use \(u\) as a prognostic variable, but to also apply \(d\) to the above equation, use some chosen stable conservative advection scheme for PV, and then to obtain a discrete form of \(\mathcal{L}_u \cdot v \cdot dx\) which is consistent with that scheme (so that PV advection is stabilized even though it is a diagnostic variable). This programme cannot be easily extended to three dimensions when advected temperature is present because we gain an extra term of the form \(G d\theta\) (for some scalar function \(G\)), and it is not currently clear how to obtain a discrete form of \(\mathcal{L}_u \cdot v_\theta\) that is consistent with a stable advection scheme for the Ertel PV. However, for the slice model equation set (2.12), it should be possible to use conservative advection schemes for the last three equations, then apply \(d\) to equation (2.18), apply a stable conservative advection scheme for PV, and obtain a discrete form of \(\mathcal{L}_u, v_\delta \cdot dx\) that is consistent with that scheme. This becomes possible for slice models, the extra term can be moved inside the Lie derivative to obtain equation (2.18).
This work has led to the development of new model equations: a Lagrangian-averaged form of the Eady model of frontogenesis and a new compressible model. We plan to use both of these models to investigate how to improve prediction of front evolution, following the programme set out by Cullen [1]. While solutions of the SCM do not recover solutions of the full three-dimensional equations, this model approximates the slice Boussinesq model in the Boussinesq limit, and is easily obtained by very minor modifications to standard dynamical core slice configurations, so will allow asymptotic limit analysis to be performed with compressible codes, addressing a problem highlighted in Cullen [3].

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Appendix A. Euler–Poincaré semidirect-product formulation

The advection equations (2.4)–(2.6) for \((\theta_s, s)\) and \(D\) may be rewritten in Lie-derivative notation as:

\[
\begin{align*}
\frac{\partial}{\partial t}(\theta_s, s) &= -L_{(u_s, u_T)}(\theta_s, s) = (-u_s \cdot \nabla \theta_s - u_T s, 0) \\
\frac{\partial}{\partial t}(D dS) &= -L_u(D dS) = -\text{div}(u_s D) dS.
\end{align*}
\]

The corresponding infinitesimal variations in \((\theta_s, s)\) and \(D\), in (2.7) induced by the Lie-derivative actions of the Lie algebra of vector fields \(\mathfrak{X}(\Omega) \otimes \mathcal{F}(\Omega)\), are given by:

\[
\begin{align*}
\delta(\theta_s, s) &= -L_{(w_s, w_T)}(\theta_s, s) = (-w_s \cdot \nabla \theta_s - w_T s, 0) \\
\delta D dS &= -L_{w_s}(D dS) = -\text{div}(w_s D) dS.
\end{align*}
\]

The infinitesimal variations in \((u_s, u_T)\) in (2.7) may be expressed in terms of the adjoint action in the Lie algebra \(\mathfrak{X}(\Omega) \otimes \mathcal{F}(\Omega)\) of the semidirect-product group \(\text{Diff}(\Omega) \otimes \mathcal{F}(\Omega)\). Namely,

\[
\delta(u_s, u_T) = (\frac{\partial}{\partial t} w_s, \frac{\partial}{\partial t} w_T) - \text{ad}_{(u_s, u_T)}(w_s, w_T) = (\frac{\partial}{\partial t} w_s + [u_s, w_s], \frac{\partial}{\partial t} w_T + u_s \cdot \nabla w_T - w_s \cdot \nabla u_T).
\]

For a Lagrangian functional \(l[(u_s, u_T), (\theta_s, s), D]: (\mathfrak{X}(\Omega) \otimes \mathcal{F}(\Omega))/((\Lambda^0(\Omega) \times \mathbb{R}) \times A^2(\Omega)) \to \mathbb{R}\), one defines Hamilton’s principle using the \(L^2\) pairing, which is denoted as \(\langle \cdot, \cdot \rangle\). Hence, inserting the infinitesimal variational formulae in (2.7) for \((u_s, u_T), (\theta_s, s)\) and \(D\) yields, in semidirect-product notation,

\[
0 = \delta S, \quad \delta S = \int_0^T \delta l[(u_s, u_T), (\theta_s, s), D] dt
\]

\[
= \int_0^T \left( \frac{\partial}{\partial (u_s, u_T)} \delta l(\theta_s, s), \frac{\partial}{\partial (w_s, w_T)} \delta l(\theta_s, s), \frac{\partial}{\partial D} \delta l, \delta D \right) dt
\]

\[
= \int_0^T \left( \frac{\partial}{\partial (u_s, u_T)} \left( \frac{\partial l}{\partial (u_s, u_T)} \right) \frac{\partial}{\partial t} w_s, \frac{\partial}{\partial t} w_T \right) - \text{ad}_{(u_s, u_T)}(w_s, w_T) \right)
+ \left( \frac{\partial}{\partial (\theta_s, s)} - L_{(w_s, w_T)}(\theta_s, s) \right) + \left( \frac{\partial}{\partial D} \right) - \text{div}(w_s D) \right) dt
\]

\[
= \int_0^T \left( -\frac{\partial}{\partial t} \frac{\partial l}{\partial (u_s, u_T)} - \text{ad}^g_{(u_s, u_T)} \frac{\partial l}{\partial (\theta_s, s)} + \frac{\partial l}{\partial (\theta_s, s)} \circ \frac{\partial l}{\partial D} \circ D, 0, (w_s, w_T) \right) dt
+ \left[ \frac{\partial l}{\partial (u_s, u_T)}, (w_s, w_T) \right]_0^T.
\]
notation in (A 4) denotes the dual of the ad operation with respect to the $L^2$ pairing $\langle \cdot, \cdot \rangle$ [4]. Explicitly, the $L^2$ dual of the ad operation is defined by

$$\left( \text{ad}_{(u_S, u_T)}^* \frac{\delta l}{\delta (u_S, u_T)} \right)(w_S, w_T) = \left( \frac{\delta l}{\delta (u_S, u_T)} \right) \text{ad}_{(u_S, u_T)}(w_S, w_T).$$

(A 5)

Likewise, the diamond ($\diamond$) operation is defined in the present notation by the $L^2$ pairings,

$$\left\{ \frac{\delta l}{\delta (\theta_S, s)} \diamond (\theta_S, s), (w_S, w_T) \right\} := \left\{ \frac{\delta l}{\delta (\theta_S, s)} - L_{(w_S, w_T)}(\theta_S, s) \right\}
\quad \text{and}
\left\{ \left( \frac{\delta l}{\delta D} \diamond D, 0 \right), (w_S, w_T) \right\} := \left\{ \frac{\delta l}{\delta D} - L_{w_S D} \right\} = \left\{ \frac{\delta l}{\delta D} - \text{div}(w_S D) \right\}.$$  

(A 6)

Hence, the last equality of (A 4) yields the Euler–Poincaré equations on the dual Lie algebra $(\mathcal{X}(\Omega) \otimes \mathcal{F}(\Omega))^*$ with the advected areal density $D \in \Lambda^2$ and advected scalars $(\theta_S, s) \in \Lambda^0 \times \mathbb{R}$ in semidirect-product form, as

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta (u_S, u_T)} + \text{ad}_{(u_S, u_T)}^* \frac{\delta l}{\delta (u_S, u_T)} = \frac{\delta l}{\delta (\theta_S, s)} \diamond (\theta_S, s) + \left( \frac{\delta l}{\delta D} \diamond D, 0 \right).$$  

(A 7)

The system (A 7) is completed by including the advection equations (A 1) for $D$ and $(\theta_S, s)$.

### Appendix B. Lie–Poisson Hamiltonian formulation

**B.1. Equations on the dual of $(\mathcal{X} \otimes \mathcal{F}(\Omega)) \otimes ((\Lambda^0(\Omega) \times \mathbb{R}) \times \Lambda^2(\Omega))$**

The Legendre transformation to the Hamiltonian is defined by

$$h[(m_S, m_T), (\theta_S, s), D] = \langle (m_S, m_T), (u_S, u_T) \rangle - l[(u_S, u_T), (\theta_S, s), D].$$  

(B 1)

Therefore, we find the variational relations

$$(m_S, m_T) = \frac{\delta l}{\delta (u_S, u_T)}, \quad (u_S, u_T) = \frac{\delta h}{\delta (m_S, m_T)}, \quad \frac{\delta h}{\delta (\theta_S, s)} = - \frac{\delta l}{\delta (\theta_S, s)} \quad \text{and} \quad \frac{\delta h}{\delta D} = - \frac{\delta l}{\delta D}.$$  

(B 2)

Consequently, system (A 7) may be written in terms of the Hamiltonian as

$$\frac{\partial}{\partial t} \left( m_S, m_T \right) = - \text{ad}_{h/\delta(m_S, m_T)}^* \left( m_S, m_T \right) - \frac{\delta h}{\delta (\theta_S, s)} \diamond (\theta_S, s) - \left( \frac{\delta h}{\delta D} \diamond D, 0 \right).$$  

(B 3)

The advection equations (A 1) for $(\theta_S, s)$ and $D$ are then written as

$$\frac{\partial}{\partial t} (\theta_S, s) = - L_{\delta h/\delta(m_S, m_T)}(\theta_S, s)$$  

and

$$\frac{\partial}{\partial t} (D, 0) = - L_{\delta h/\delta(m_S, m_T)}(D, 0).$$  

(B 4)

Hence, the entire system (B 3)–(B 4) may be written in Hamiltonian form as

$$\frac{\partial}{\partial t} \left[\begin{array}{c} (m_S, m_T) \\ (\theta_S, s) \\ (D, 0) \end{array}\right] = - \left[\begin{array}{ccc} \text{ad}_{h/\delta(m_S, m_T)}^* & \square \diamond (\theta_S, s) & \square \diamond (D, 0) \\ L_{\square}(\theta_S, s) & 0 & 0 \\ L_{\square}(D, 0) & 0 & 0 \end{array}\right] \left[\begin{array}{c} \delta h \\ \delta l \\ \delta (\theta_S, s) \end{array}\right],$$

(B 5)

where the box $\square$ indicates the appropriate substitutions. The matrix operator in (B 5) defines a Lie–Poisson bracket dual to the semidirect-product action $(\mathcal{X} \otimes \mathcal{F}(\Omega)) \otimes ((\Lambda^0(\Omega) \times \mathbb{R}) \times \Lambda^2(\Omega))$ with coordinates $(u_S, u_T) \in \mathcal{X} \otimes \mathcal{F}(\Omega)$, $(\theta_S, s) \in \Lambda^0(\Omega) \times \mathbb{R}$ and $D \in \Lambda^2(\Omega)$. This identification of the Lie–Poisson bracket with the dual of a Lie algebra action guarantees that it satisfies the Jacobi identity.
Explicitly, the Lie–Poisson bracket is the following:

\[
\{f, h\} = - \left[ \frac{\delta f}{\delta (m_S, m_T)}, \frac{\delta h}{\delta (m_S, m_T)} \right]^T \left[ \begin{array}{ccc}
\text{ad}^*_{\nabla} (m_S, m_T) & \delta (m_S, m_T) \\
\delta (\theta_S, s) & \delta (D, 0)
\end{array} \right]^{-1} \left[ \begin{array}{ccc}
\text{ad}^*_{\nabla} (m_S, m_T) & \delta (m_S, m_T) \\
\delta (\theta_S, s) & \delta (D, 0)
\end{array} \right]
\]

(B6)

where \(\langle \cdot, \cdot \rangle\) denotes the \(L^2\) pairing.

Expanding out the operations in (B6) makes it clear that this Lie–Poisson bracket has the required property of being antisymmetric under exchange of \(f\) and \(h\). That is, \(\{h, f\} = -\{f, h\}\), which is evident upon expanding out the expressions to express the bracket in (B6) equivalently as

\[
\{f, h\} = - \left[ (m_S, m_T), \frac{\delta f}{\delta (m_S, m_T)} \right]^{\dagger} \left[ \begin{array}{c}
\frac{\delta}{\delta (m_S, m_T)} \frac{\delta h}{\delta (m_S, m_T)} \\
\frac{\delta}{\delta (\theta_S, s)} \frac{\delta h}{\delta (\theta_S, s)} \\
\frac{\delta}{\delta (D, 0)} \frac{\delta h}{\delta (D, 0)}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\frac{\delta f}{\delta (m_S, m_T)} \\
\frac{\delta f}{\delta (\theta_S, s)} \\
\frac{\delta f}{\delta (D, 0)}
\end{array} \right]
\]

(B7)

Here, \(\mathcal{L}^+\) denotes the \(L^2\) adjoint of the Lie derivative \(\mathcal{L}\). In particular, upon denoting \(\delta f/\delta (m_S, m_T) = \langle w_S, w_T \rangle\), we find the following relations among the operations \(\mathcal{L}^+, \mathcal{L}\) and \(\circ\):

\[
\left( \theta_S, \mathcal{L}^+_{(w_S, w_T)} \frac{\delta h}{\delta (\theta_S, s)} \right) := \left( \frac{\delta h}{\delta (\theta_S, s)} \mathcal{L}_{(w_S, w_T)} (\theta_S, s) \right) = \left( - \frac{\delta h}{\delta (\theta_S, s)} \circ (\theta_S, s), \langle w_S, w_T \rangle \right)
\]

(B8)

**Condition B.1.** If desired, one may now substitute the expressions for Lie derivative (A 1), \(\text{ad}^*\) (A 5) and diamond \(\circ\) (A 6) into the \(L^2\) pairings (B6) or (B7) to find the Lie–Poisson bracket \(\{f, h\}\) as an integral over the slice domain, \(\Omega\), involving ordinary vector calculus operations. However, the present forms (B6) and (B7) readily reveal the semidirect-product nature and suggest further rearrangements, which we pursue next.

### B.2. Equations on the dual of \(\mathcal{X}(\Lambda \oplus \Lambda^\circ \oplus \Lambda^\circ)\)

To explore the particular case at hand further, one may rewrite the system of equations (2.4), (2.6) and (2.9) equivalently as

\[
\frac{\partial}{\partial t} \frac{\delta l}{\delta u_S} = - \text{ad}^*_{u_S} \frac{\delta l}{\delta u_S} - \frac{\delta l}{\delta u_T} \nabla \theta_S - \frac{\delta l}{\delta u_T} \nabla u_T + D V \frac{\delta l}{\delta D}
\]

\[
\frac{\partial}{\partial t} \frac{\delta l}{\delta u_T} = - \mathcal{L}_{u_S} \frac{\delta l}{\delta u_T} - \frac{\delta l}{\delta \theta_S} s,
\]

\[
\frac{\partial}{\partial t} \theta_S = - \mathcal{L}_{u_S} \theta_S - u_T s
\]

\[
\frac{\partial}{\partial t} D = - \mathcal{L}_{u_S} D,
\]

(B9)

and \(\mathcal{L}_{u_S}\) denotes the Lie derivative along the vector field \(u_S\) and we have identified \(\mathcal{L}_{u_S}\) and \(\text{ad}^*_{u_S}\) when acting on the one-form density \(\delta l/\delta u_S\) in the first equation. For more detail on this matter, see Holm et al. [4].

We define the Legendre transformation to the Hamiltonian in this case by

\[
h(m_S, m_T, \theta_S, D; s) = \langle m_S, u_S \rangle + \langle m_T, u_T \rangle - l[u_S, u_T, \theta_S, D; s],
\]

(B10)

where the semicolon \(\ldots; s\) denotes parametric dependence on the constant \(s \in \mathbb{R}\). The Legendre transformation (B10) yields the variational relations

\[
m_S = \frac{\delta l}{\delta u_S}, \quad u_S = \frac{\delta h}{\delta m_S}, \quad m_T = \frac{\delta l}{\delta u_T}, \quad u_T = \frac{\delta h}{\delta m_T}, \quad \frac{\delta h}{\delta \theta_S} = - \frac{\delta l}{\delta \theta_S} \quad \text{and} \quad \frac{\delta h}{\delta D} = - \frac{\delta l}{\delta D}.
\]

(B11)
Consequently, system (A7) may be written in terms of the Hamiltonian as

\[
\begin{aligned}
\frac{\partial}{\partial t} m_S &= -a d^*_{\delta h/\delta m_S} m_S - m_T \nabla \frac{\delta h}{\delta m_T} + \frac{\delta h}{\delta \theta_S} \nabla \theta_S - D \nabla \frac{\delta h}{\delta D}, \\
\frac{\partial}{\partial t} m_T &= -L_{\delta h/\delta m_S} m_T + \frac{\delta h}{\delta \theta_S} s, \\
\frac{\partial}{\partial t} \theta_S &= -L_{\delta h/\delta m_S} \theta_S - \frac{\delta h}{\delta m_T} s, \\
\frac{\partial}{\partial t} D &= -L_{\delta h/\delta m_S} D.
\end{aligned}
\]  

(B 12)

The corresponding Hamiltonian matrix is

\[
\begin{bmatrix}
\frac{\partial}{\partial t} m_S \\
\frac{\partial}{\partial t} m_T \\
\frac{\partial}{\partial t} \theta_S \\
\frac{\partial}{\partial t} D
\end{bmatrix} =
\begin{bmatrix}
ad^*_{\delta h/\delta m_S} m_S & \square \circ m_T & \square \circ \theta_S & \square \circ D \\
L_{\square m_T} & 0 & -s & 0 \\
L_{\square \theta_S} & s & 0 & 0 \\
L_{\square D} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\delta h}{\delta m_S} \\
\frac{\delta h}{\delta m_T} \\
\frac{\delta h}{\delta \theta_S} \\
\frac{\delta h}{\delta D}
\end{bmatrix},
\]  

(B 13)

where the box \(\square\) indicates the appropriate substitutions.

After this rearrangement, one recognizes (B 13) as the Hamiltonian matrix for the Lie–Poisson bracket on the dual of the semidirect-product Lie algebra \(X \otimes (\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^0)\) with a symplectic two-cocycle between \(m_T\) and \(\theta_S\). The Lie bracket for this semidirect-product algebra is

\[
[(X,f,\omega,g), (\tilde{X},\tilde{f},\tilde{\omega},\tilde{g})] = [(X,\tilde{X}), X(\tilde{f}) - \tilde{X}(f), X(\tilde{\omega}) - \tilde{X}(\omega), X(\tilde{g}) - \tilde{X}(g)),
\]  

(B 14)

where, for example, \(X(\tilde{f}) = L_X \tilde{f}\) denotes the Lie derivative of \(\tilde{f}\) by vector field \(X\). The dual coordinates are: \(m_S\) dual to \(X \in \mathcal{X}\); \(m_T\) to \(f \in \Lambda^0\); \(\theta_S\) to \(\omega \in \Lambda^2\); and \(D\) to \(g \in \Lambda^0\). The spaces in which the coordinates themselves are defined are \((m_S, m_T, \theta_S, D) \in (\Lambda^1 \otimes \Lambda^2, \Lambda^2, \Lambda^0, \Lambda^0)\) and \(s \in \mathbb{R}\) is a parameter. The second part of the bracket (B 13) is the standard two-cocycle (symplectic form) on \(\Lambda^0 \oplus \Lambda^2\) arising from the natural projection \(X \otimes (\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^0) \to \Lambda^0 \oplus \Lambda^2\).

**Condition B.2.** The Hamiltonian matrix with the two-cocycle in (B 13) has been seen before. Namely, it is the same as that for \(^4\)He superfluids \([6,7]\) in the spatially two-dimensional case. For \(^4\)He superfluids, the function \(\theta_S\) here plays the role of the phase of the Bose-condensate wave function, whose gradient \(\nabla \theta_S\) is the superfluid velocity. The other variables \(m_S, m_T\) and \(D\) correspond, respectively, to total momentum density, mass density and entropy density of the superfluid.

**B.3. Equations on the dual of \(X_1 \otimes (X_2 \oplus \Lambda^0)\)**

Holm & Kuperschmidt [7] showed that the two-cycle in (B 13) may be removed by transforming to new variables

\[
(m_S, m_T, \theta_S, D) \to (m_S, m_R, D), \quad \text{where } m_R := (s)^{-1} m_T \nabla \theta_S.
\]  

(B 15)

The quantity \(m_R\) is the momentum map for right action of the diffeomorphisms on the buoyancy \(\theta_S\) in two spatial dimensions, see Holm & Marsden [8] for more details. The resulting Lie–Poisson bracket has the standard form dual to the Lie algebra \(X_1 \otimes (X_2 \oplus \Lambda^0)\), whose Lie bracket is

\[
[(X_1, X_2, f), (\tilde{X}_1, \tilde{X}_2, \tilde{f})] = [(X_1, \tilde{X}_1), [X_1, X_2] + [X_1, \tilde{X}_2] - [X_2, X_1], X_1(\tilde{f}) - \tilde{X}_1(f)),
\]  

(B 16)

Dual coordinates in this case are: \(m_S\) dual to \(X_1 \in \mathcal{X}_1\); \(m_R\) to \(X_2 \in \mathcal{X}_2\); and \(D\) to \(f \in \Lambda^0\).
Transformation of the Hamiltonian matrix (B.13) into these variables yields the following Lie–Poisson Hamiltonian system:

\[
\frac{\partial}{\partial t} \begin{bmatrix} m_S \\ m_R \\ D \end{bmatrix} = -\begin{bmatrix} \text{ad}^*_S m_S & \text{ad}^*_S m_R & \Box \circ D \\ \text{ad}^*_R m_R & \text{ad}^*_R m_R & 0 \\ \mathcal{L}_D D & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta h}{\delta m_S} =: u_S \\ \frac{\delta h}{\delta m_R} =: u_R \\ \frac{\delta h}{\delta D} =: p \end{bmatrix}.
\]

(B.17)

This system produces a system of equations for relative momentum \((m_S - m_R)\), momentum map \(m_R = (s)^{-1} m_T \nabla \theta_S\) and mass density \(D\), given by

\[
\begin{align*}
\partial_t(m_S - m_R) &= -\text{ad}^T_S (m_S - m_R) - p \circ D, \\
\partial_t m_R &= -\text{ad}^*_R (u_S + i u_R) m_R \\
\partial_t D &= -\mathcal{L}_{u_S} D.
\end{align*}
\]

(B.18)

Upon evaluating \(p \circ D = D \nabla p\), the first of these equations explains the geometrical origin of the Kelvin–Noether circulation theorem (2.11) that was found by direct manipulation in §2c. Together, the three equations in (B.18) show that the slice dynamics may be expressed in terms of \((m_S, m_R, D)\) as a Lie–Poisson Hamiltonian system on the semidirect product

\[
\text{Diff}_1(\Omega) \circledS (\text{Diff}_2(\Omega) \times \Lambda^2(\Omega)),
\]

in the slice domain \(\Omega\). When \(D = 1\) is imposed, we have \(\nabla \cdot u_S = 0\), and this simplifies to

\[
\text{SDiff}_1(\Omega) \circledS \text{Diff}_2(\Omega).
\]

References