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The Laplace equation in the exterior of the Hankel contour and novel identities for hypergeometric functions

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By using conformal mappings, it is possible to express the solution of certain boundary-value problems for the Laplace equation in terms of a single integral involving the given boundary data. We show that such explicit formulae can be used to obtain novel identity for special functions. A convenient tool for deriving this type of identity is the so-called *global relation*, which has appeared recently in a wide range of boundary-value problems. As a concrete application, we analyse the Neumann boundary-value problem for the Laplace equation in the exterior of the Hankel contour, which appears in the definition of both the gamma and the Riemann zeta functions. By using the explicit solution of this problem, we derive a number of novel identities involving the hypergeometric function. Also, we point out an interesting connection between the solution of the above Neumann boundary-value problem for a particular set of Neumann data and the Riemann hypothesis.

1. Introduction

Simple boundary-value problems for the Laplace equation in two dimensions can be solved by the powerful technique of conformal mappings. However, the usual implementation of the conformal mappings presented in most text books is limited because it applies only to the case that the given boundary data is piecewise constant. In §2, we present a technique that can be

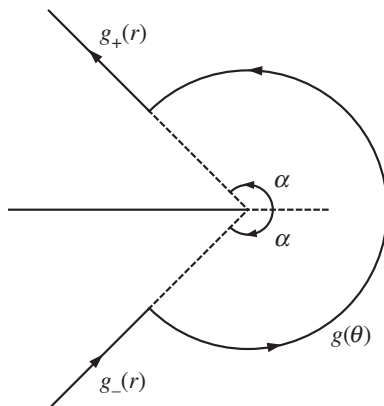


Figure 1. The Hankel contour.

applied to the general case of arbitrary boundary data (see also [1]). By using this technique, we construct in §2 a solution of the Neumann boundary-value problem of the Laplace equation in the domain D , which is the exterior of the so-called Hankel contour H (figure 1) defined by

$$H = \{z = r e^{-i\alpha}, a < r < \infty\} \cup \{z = a e^{-i\theta}, -\alpha < \theta < \alpha\} \cup \{z = r e^{i\alpha}, a < r < \infty\}, \quad (1.1)$$

where

$$0 < a < 2\pi, \quad |\alpha| < \pi.$$

A solution of the above Neumann boundary-value problem is expressed in theorem 2.2 in terms of a single integral involving the three functions $\{g_+(r), g(\theta), g_-(r)\}$ defining the Neumann data. Using this explicit solution, we derive in §3 four integral identities. A particular case of these four identities can be derived by employing the explicit solutions

$$q(r, \theta) = \operatorname{Re}(z^k) \quad \text{and} \quad q(r, \theta) = \operatorname{Im}(z^k), \quad z = r e^{i\theta}, \quad r > 0, \quad \theta \in \mathbb{R}, \quad \operatorname{Re} k > 0. \quad (1.2)$$

In the general case, these identities are derived using the so-called *global relation*. We recall that this relation plays a crucial role for the construction of the solution of a wide class of initial boundary-value problems, for both evolution and elliptic PDEs [2–17]. Here, instead of employing the global relation to solve the given boundary-value problem, having already constructed the solution via conformal mappings, we use the global relation to obtain the integral identities mentioned earlier.

Regarding the global relation, we note that in addition to its basic role for the analytical solution of a large class of PDEs [18–20], it has also been used in the following contexts: (i) it yields a novel non-local formulation of the classical problem of water waves [21–25], (ii) it provides a useful approach to Hele–Shaw type problems [26], and (iii) it gives rise to novel numerical techniques for elliptic PDEs in the interior of a convex polygon [27–32].

We also show that the Riemann hypothesis is valid if and only if the above Neumann boundary-value problem, where the functions $\{g_+(r), g(\theta), g_-(r)\}$ have certain concrete representations (see equations (2.17) and (2.19)), does not have a solution that is bounded as $r \rightarrow \infty$.

2. A Neumann boundary-value problem for the Laplace equation

The Dirichlet and Neumann boundary-value problems for the Laplace equation in the exterior of the Hankel contour can be solved via conformal mappings. In this respect, we note that the usual implementation of the conformal mappings presented in most text books fails here because it only applies to the case of piecewise constant Dirichlet or Neumann data. For the case of arbitrary Neumann data, the following result is useful.

Proposition 2.1. Let $q(r, \theta)$ satisfy a Neumann boundary-value problem for the Laplace equation. Let $\omega(z)$ be the conformal mapping of the relevant domain to the upper half of the complex ω -plane, where

$$\omega(z) = x + iy, \quad z = r e^{i\theta}, \quad x, y, \theta \in \mathbb{R}, \quad r > 0. \quad (2.1)$$

Then,

$$q(r, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln[(\xi - x)^2 + y^2] q_y(\xi, 0) d\xi, \quad (2.2)$$

where

$$x = \operatorname{Re} \omega(r, \theta) \quad \text{and} \quad y = \operatorname{Im} \omega(r, \theta), \quad (2.3)$$

and $q_y(\xi, 0)$ can be computed in terms of the given Neumann data using the identity

$$q_y(x, 0) dx = \left[\frac{1}{r} q_\theta dr - r q_r d\theta \right]_{y=0}. \quad (2.4)$$

Proof. Observe that

$$q_y dx|_{y=0} = i[q_\omega d\omega - q_{\bar{\omega}} d\bar{\omega}]_{y=0}. \quad (2.5)$$

Indeed, the r.h.s. of this equation equals

$$\frac{i}{2} [(q_x - iq_y) dx - (q_x + iq_y) dx]_{y=0},$$

which equals the l.h.s. of (2.5).

The conformal mapping implies

$$q_\omega d\omega = q_z dz.$$

Furthermore, using $z = r e^{i\theta}$, we find

$$\begin{aligned} q_z dz &= \left[\frac{e^{-i\theta}}{2} \left(q_r - \frac{i}{r} q_\theta \right) \right] [e^{i\theta} (dr + ir d\theta)] \\ &= \frac{1}{2} [q_r dr + q_\theta d\theta] + \frac{i}{2} \left[r q_r d\theta - \frac{1}{r} q_\theta dr \right]. \end{aligned}$$

Hence,

$$q_\omega d\omega - q_{\bar{\omega}} d\bar{\omega} = i \left[r q_r d\theta - \frac{1}{r} q_\theta dr \right],$$

and equation (2.5) becomes (2.4).

Equation (2.2) is the well-known Poisson formula for the Neumann problem. ■

Theorem 2.2. Let the domain D be defined by

$$D = \{a < r < \infty, -\alpha < \theta < \alpha, \}, \quad 0 < a < 2\pi, \quad \frac{\pi}{2} < \alpha \leq \pi. \quad (2.6)$$

Let $q(r, \theta)$ solve the following Neumann boundary-value problem for the Laplace equation in the domain D :

$$q_\theta(r, -\alpha) = g_-(r), \quad a < r < \infty, \quad (2.7a)$$

$$q_r(a, \theta) = g(\theta), \quad -\alpha < \theta < \alpha, \quad (2.7b)$$

and

$$q_\theta(r, \alpha) = g_+(r), \quad a < r < \infty, \quad (2.7c)$$

where the functions $g_{\pm}(r)$ and $g(\theta)$ have appropriate smoothness and decay. A solution of this boundary-value problem is given by

$$\begin{aligned}
 q(r, \theta) = & -\frac{1}{2\pi} \int_a^{\infty} \frac{g_+(\rho)}{\rho} \ln \left\{ \frac{1}{4} \left[\left(\frac{r}{a} \right)^{\pi/2\alpha} - \left(\frac{r}{a} \right)^{-\pi/2\alpha} \right]^2 \cos^2 \left(\frac{\pi\theta}{2\alpha} \right) \right. \\
 & \left. + \left[\frac{R(\rho)}{2} - \frac{R(r)}{2} \sin \left(\frac{\pi\theta}{2\alpha} \right) \right]^2 \right\} d\rho \\
 & + \frac{1}{2\pi} \int_a^{\infty} \frac{g_-(\rho)}{\rho} \ln \left\{ \frac{1}{4} \left[\left(\frac{r}{a} \right)^{\pi/2\alpha} - \left(\frac{r}{a} \right)^{-\pi/2\alpha} \right]^2 \cos^2 \left(\frac{\pi\theta}{2\alpha} \right) \right. \\
 & \left. + \left[\frac{R(\rho)}{2} + \frac{R(r)}{2} \sin \left(\frac{\pi\theta}{2\alpha} \right) \right]^2 \right\} d\rho \\
 & + \frac{a}{2\pi} \int_{-\alpha}^{\alpha} g(\varphi) \ln \left\{ \frac{1}{4} \left[\left(\frac{r}{a} \right)^{\pi/2\alpha} - \left(\frac{r}{a} \right)^{-\pi/2\alpha} \right]^2 \cos^2 \left(\frac{\pi\theta}{2\alpha} \right) \right. \\
 & \left. + \left[\sin \left(\frac{\pi\varphi}{2\alpha} \right) - \frac{R(r)}{2} \sin \left(\frac{\pi\theta}{2\alpha} \right) \right]^2 \right\} d\varphi,
 \end{aligned} \tag{2.8a}$$

where $R(\rho)$ is defined by

$$R(\rho) = \left(\frac{\rho}{a} \right)^{\pi/2\alpha} + \left(\frac{\rho}{a} \right)^{-\pi/2\alpha}, \quad a < \rho < \infty, \tag{2.8b}$$

and principal-value integrals are assumed if needed.

Proof. It is straightforward to verify that the function $\omega(z)$, defined by

$$\omega(z) = \frac{i}{2} \left[\left(\frac{z}{a} \right)^{\pi/2\alpha} - \left(\frac{z}{a} \right)^{-\pi/2\alpha} \right], \tag{2.9}$$

maps the domain D defined in (2.6) to the upper half complex ω -plane. The points $\{a e^{i\alpha}, a e^{-i\alpha}, a\}$ are mapped to the points $\{-1, 1, 0\}$, respectively. Also,

$$\begin{aligned}
 x &= -\frac{1}{2} \sin \left(\frac{\pi\theta}{2\alpha} \right) \left[\left(\frac{r}{a} \right)^{\pi/2\alpha} + \left(\frac{r}{a} \right)^{-\pi/2\alpha} \right] \\
 y &= \frac{1}{2} \cos \left(\frac{\pi\theta}{2\alpha} \right) \left[\left(\frac{r}{a} \right)^{\pi/2\alpha} - \left(\frac{r}{a} \right)^{-\pi/2\alpha} \right].
 \end{aligned} \tag{2.10}$$

and

The condition $y = 0$ implies either $\theta = \pm\alpha$ or $r = a$. Hence,

$$q_y(\xi, 0) d\xi = \begin{cases} -aq_r(a, \varphi) d\varphi & \text{if } r = a, \\ \frac{1}{\rho} q_{\theta}(\rho, \pm\alpha) d\rho & \text{if } \theta = \pm\alpha. \end{cases}$$

Thus, equation (2.2) becomes

$$\begin{aligned}
 q(r, \theta) = & \frac{1}{2\pi} \int_{\infty}^a \ln \tilde{F}(\rho, \alpha; r, \theta) q_{\theta}(\rho, \alpha) \frac{d\rho}{\rho} + \frac{1}{2\pi} \int_a^{\infty} \ln \tilde{F}(\rho, -\alpha; r, \theta) q_{\theta}(\rho, -\alpha) \frac{d\rho}{\rho} \\
 & + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \ln \tilde{F}(a, \varphi; r, \theta) [-aq_r(a, \varphi)] d\varphi,
 \end{aligned} \tag{2.11a}$$

where

$$\tilde{F}(\rho, \varphi; r, \theta) = [\xi(\rho, \varphi) - x(r, \theta)]^2 + y^2(r, \theta). \tag{2.11b}$$

Using

$$\xi(\rho, \alpha) = -\frac{1}{2} \sin\left(\frac{\pi}{2}\right) \left[\left(\frac{\rho}{\alpha}\right)^{\pi/2\alpha} + \left(\frac{\rho}{\alpha}\right)^{-\pi/2\alpha} \right] = -\frac{1}{2} R(\rho),$$

$$\xi(\rho, -\alpha) = -\frac{1}{2} \sin\left(-\frac{\pi}{2}\right) \left[\left(\frac{\rho}{\alpha}\right)^{\pi/2\alpha} + \left(\frac{\rho}{\alpha}\right)^{-\pi/2\alpha} \right] = \frac{1}{2} R(\rho)$$

and

$$\xi(a, \varphi) = -\sin\left(\frac{\pi\varphi}{2\alpha}\right),$$

as well as the boundary conditions (2.7), equation (2.11a) becomes the following:

$$\begin{aligned} q(r, \theta) = & -\frac{1}{2\pi} \int_a^\infty \frac{g_+(\rho)}{\rho} \ln \left[\frac{1}{4} R^2(r) - \cos^2\left(\frac{\pi\theta}{2\alpha}\right) + \frac{1}{4} R^2(\rho) - \frac{1}{2} \sin\left(\frac{\pi\theta}{2\alpha}\right) R(\rho) R(r) \right] d\rho \\ & + \frac{1}{2\pi} \int_a^\infty \frac{g_-(\rho)}{\rho} \ln \left[\frac{1}{4} R^2(r) - \cos^2\left(\frac{\pi\theta}{2\alpha}\right) + \frac{1}{4} R^2(\rho) + \frac{1}{2} \sin\left(\frac{\pi\theta}{2\alpha}\right) R(\rho) R(r) \right] d\rho \\ & + \frac{a}{2\pi} \int_{-\alpha}^\alpha g(\rho) \ln \left[\frac{1}{4} R^2(r) - \cos^2\left(\frac{\pi\theta}{2\alpha}\right) + \sin^2\left(\frac{\pi\varphi}{2\alpha}\right) - \sin\left(\frac{\pi\varphi}{2\alpha}\right) \sin\left(\frac{\pi\theta}{2\alpha}\right) R(r) \right] d\varphi. \end{aligned} \quad (2.12)$$

By employing the identity

$$\left(\frac{R(r)}{2}\right)^2 - 1 = \frac{1}{4} \left[\left(\frac{r}{a}\right)^{\pi/2\alpha} - \left(\frac{r}{a}\right)^{-\pi/2\alpha} \right]^2,$$

equation (2.12) becomes equation (2.8a). ■

Remark 2.3. (Asymptotics) The large r asymptotic of the solution $q(r, \theta)$ defined by (2.8a) is given by

$$q(r, \theta) = \frac{1}{2\pi} \left[\ln\left(\frac{1}{4}\right) + \frac{\pi}{\alpha} \ln\left(\frac{r}{a}\right) \right] S + \frac{1}{\pi} \sin\left(\frac{\pi\theta}{2\alpha}\right) \left(\frac{r}{a}\right)^{-\pi/2\alpha} \tilde{S} + O\left[\left(\frac{r}{a}\right)^{-2\pi/\alpha}\right], \quad r \rightarrow \infty, \quad (2.13)$$

where the constants S and \tilde{S} are defined as follows:

$$S = \int_a^\infty [g_-(\rho) - g_+(\rho)] \frac{d\rho}{\rho} + a \int_{-\alpha}^\alpha g(\varphi) d\varphi \quad (2.14)$$

and

$$\tilde{S} = \int_a^\infty [g_+(\rho) + g_-(\rho)] R(\rho) \frac{d\rho}{\rho} - 2a \int_{-\alpha}^\alpha g(\varphi) \sin\left(\frac{\pi\varphi}{2\alpha}\right) d\varphi. \quad (2.15)$$

Indeed, each of the brackets of the r.h.s. of (2.8a) can be written in the form

$$F(R(r), T) = \ln \left[\frac{R^2(r)}{4} - TR(r) + O(1) \right], \quad r \rightarrow \infty,$$

where T is given, respectively, for each bracket by

$$\frac{1}{2} \sin\left(\frac{\pi\theta}{2\alpha}\right) R(\rho), \quad -\frac{1}{2} \sin\left(\frac{\pi\theta}{2\alpha}\right) R(\rho) \quad \text{and} \quad \sin\left(\frac{\pi\theta}{2\alpha}\right) \sin\left(\frac{\pi\varphi}{2\alpha}\right).$$

For large r , we find the following estimate:

$$\begin{aligned} F(R(r), T) &= \ln \left\{ \frac{R^2(r)}{4} \left[1 - 4T \left(\frac{r}{a}\right)^{-\pi/2\alpha} + O\left(\left(\frac{r}{a}\right)^{-\pi/\alpha}\right) \right] \right\} \\ &= \ln\left(\frac{1}{4}\right) + 2 \ln[R(r)] - 4T \left(\frac{r}{a}\right)^{-\pi/2\alpha} + O\left(\left(\frac{r}{a}\right)^{-\pi/\alpha}\right) \\ &= \ln\left(\frac{1}{4}\right) + \frac{\pi}{\alpha} \ln\left(\frac{r}{a}\right) - 4T \left(\frac{r}{a}\right)^{-\pi/2\alpha} + O\left(\left(\frac{r}{a}\right)^{-\pi/\alpha}\right), \quad r \rightarrow \infty. \end{aligned} \quad (2.16)$$

Using the estimate (2.16), equation (2.8a) implies (2.13).

Remark 2.4. (A particular example) Let $u(z)$ be a complex valued function such that $z^{s-1}u(z)$ is bounded for all z on the Hankel contour H , $s \in \mathbb{C}$. Suppose that we choose the functions $\{g_+(\rho), g_-(\rho), g(\varphi)\}$ as follows:

$$g_{\pm}(\rho) = (\rho e^{\pm i\alpha})^s u(\rho e^{\pm i\alpha}) \quad \text{and} \quad g(\varphi) = -\frac{i}{a} (a e^{i\varphi})^s u(a e^{i\varphi}), \quad s \in \mathbb{C}. \quad (2.17)$$

Using these functions in the definition (2.14) of S , we find

$$S(s) = - \int_H z^{s-1} u(z) dz, \quad s \in \mathbb{C}. \quad (2.18)$$

Indeed, recall that H involves three integrals; making in the first, second and third integrals of the r.h.s. of (2.18) the substitutions

$$z = \rho e^{i\alpha}, \quad z = \rho e^{-i\alpha} \quad \text{and} \quad z = a e^{i\varphi},$$

respectively, we find that the r.h.s. of (2.18) equals $-S$.

Remark 2.5. (The Riemann function) Suppose we chose the functions $\{g_+(\rho), g_-(\rho), g(\varphi)\}$ by (2.17), where

$$u(z) = \frac{1}{e^{-z} - 1}, \quad z \in \mathbb{C}. \quad (2.19)$$

Then, the function $S(s)$ is proportional to the Riemann zeta function, namely,

$$S(s) = -\frac{2i\pi \zeta(s)}{\Gamma(1-s)}, \quad s \in \mathbb{C}. \quad (2.20)$$

Hence, the Riemann hypothesis is valid iff there does *not* exist a solution of the Neumann boundary-value problem defined in theorem 2.2 with the functions $\{g_+(\rho), g_-(\rho), g(\varphi)\}$ defined by equations (2.17) and (2.19), which is bounded as $r \rightarrow \infty$.

Unfortunately, we have not been able to use the above interesting connection between the Riemann hypothesis and the solution of the concrete Neumann boundary-value problem specified by the function $u(z)$ defined in (2.19), in order to make any progress on the Riemann hypothesis. However, this connection does suggest a possible way of deriving certain identities involving the Riemann function, which will be presented in a future publication. Here, we illustrate the general approach of deriving such identities for the simpler case of the hypergeometric function.

(a) The Dirichlet boundary-values

By evaluating the r.h.s. of equation (2.8a) at $r = a$ and at $\theta = \pm\alpha$, we find the following expressions for the Dirichlet boundary-values:

$$\begin{aligned} q(a, \theta) = & -\frac{1}{\pi} \int_a^\infty \frac{g_+(\rho)}{\rho} \ln \left| \frac{1}{2} R(\rho) - \sin \left(\frac{\pi\theta}{2\alpha} \right) \right| d\rho + \frac{1}{\pi} \int_a^\infty \frac{g_-(\rho)}{\rho} \ln \left| \frac{1}{2} R(\rho) + \sin \left(\frac{\pi\theta}{2\alpha} \right) \right| d\rho, \\ & + \frac{a}{\pi} \int_{-\alpha}^\alpha g(\varphi) \ln \left| \sin \left(\frac{\pi\varphi}{2\alpha} \right) - \sin \left(\frac{\pi\theta}{2\alpha} \right) \right| d\varphi, \quad -\alpha < \theta < \alpha, \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} q(r, \pm\alpha) = & -\frac{1}{\pi} \int_a^\infty g_+(\rho) \ln \left| \frac{R(r)}{2} \mp \frac{R(\rho)}{2} \right| \frac{d\rho}{\rho} + \frac{1}{\pi} \int_a^\infty g_-(\rho) \ln \left| \frac{R(r)}{2} \pm \frac{R(\rho)}{2} \right| \frac{d\rho}{\rho} \\ & + \frac{a}{\pi} \int_{-\alpha}^\alpha g(\varphi) \ln \left| \frac{R(r)}{2} \mp \sin \left(\frac{\pi\varphi}{2\alpha} \right) \right| d\varphi, \quad a < r < \infty. \end{aligned} \quad (2.22)$$

Adding and subtracting equations (2.22)⁺ and (2.22)[−], we find the following equations:

$$q(r, -\alpha) + q(r, \alpha) = \frac{1}{\pi} \int_a^\infty [g_-(\rho) - g_+(\rho)] \ln \left| \frac{R^2(\rho)}{4} - \frac{R^2(r)}{4} \right| \frac{d\rho}{\rho} \\ + \frac{a}{\pi} \int_{-\alpha}^\alpha g(\varphi) \ln \left| \frac{R^2(r)}{4} - \left(\sin \frac{\pi\varphi}{2\alpha} \right)^2 \right| d\varphi \quad (2.23)$$

and

$$q(r, -\alpha) - q(r, \alpha) = \frac{1}{\pi} \int_a^\infty [g_-(\rho) + g_+(\rho)] \ln \left| \frac{R(r) - R(\rho)}{R(r) + R(\rho)} \right| \frac{d\rho}{\rho} \\ + \frac{a}{\pi} \int_{-\alpha}^\alpha g(\varphi) \ln \left| \frac{R(r) + 2 \sin(\pi\varphi/2\alpha)}{R(r) - 2 \sin(\pi\varphi/2\alpha)} \right| d\varphi. \quad (2.24)$$

In the following, we assume that the given functions $g_\pm(r)$ and $g(\theta)$ satisfy the constraint $S = 0$. Let c be a constant, then

$$\ln \left| \frac{R(r)^2}{4} + c \right| = G(r) + O(r^{-\pi/\alpha}), \quad r \rightarrow \infty, \quad (2.25)$$

where $G(r)$ is defined by

$$G(r) = \frac{\pi}{\alpha} \ln \left(\frac{r}{a} \right) - \ln 4, \quad r > 0. \quad (2.26)$$

Each of the two logarithmic terms on the r.h.s. of equation (2.23) grows logarithmically as $r \rightarrow \infty$; however, the condition $S = 0$ implies that these two terms cancel. Indeed, using the fact that $G(r)S = 0$, equation (2.23) can be rewritten in the form

$$q(r, -\alpha) + q(r, \alpha) = \frac{1}{\pi} \int_a^\infty [g_-(\rho) - g_+(\rho)] \left[\ln \left| \frac{R^2(\rho)}{4} - \frac{R^2(r)}{4} \right| - G(r) \right] \frac{d\rho}{\rho} \\ + \frac{a}{\pi} \int_{-\alpha}^\alpha g(\varphi) \left[\ln \left| \frac{R^2(r)}{4} - \left(\sin \frac{\pi\varphi}{2\alpha} \right)^2 \right| - G(r) \right] d\varphi. \quad (2.27)$$

Equations (2.24) and (2.27) are the basic equations needed for the derivation of certain integral identities.

3. The global relation and certain integral identities

The global relations for the Laplace equation in the domain D are the following two equations:

$$\int_{\partial D} e^{\pm ik\theta} r^k \left[(-q_\theta \pm ikq) \frac{dr}{r} + (rq_r - kq) d\theta \right] = 0, \quad \operatorname{Re} k < \frac{\pi}{2\alpha}. \quad (3.1)$$

The above restriction in k is the consequence of the large r behaviour of q , namely of the estimate

$$q = O(r^{-\pi/2\alpha}), \quad r \rightarrow \infty.$$

If the domain D is defined by (2.6), then equations (3.1)[±] become

$$k \left\{ a^k \int_{-\alpha}^\alpha e^{\pm ik\theta} q(a, \theta) d\theta \pm i \int_a^\infty r^k \left[e^{\mp i\alpha k} q(r, -\alpha) - e^{\pm i\alpha k} q(r, \alpha) \right] \frac{dr}{r} \right\} \\ = a^{k+1} \int_{-\alpha}^\alpha e^{\pm ik\theta} g(\theta) d\theta + \int_a^\infty r^k \left[e^{\mp i\alpha k} g_-(r) - e^{\pm i\alpha k} g_+(r) \right] \frac{dr}{r}. \quad (3.2)$$

Adding and subtracting (3.2)[±], we obtain the following basic equations:

$$k \left\{ a^k \int_{-\alpha}^\alpha \cos(k\theta) q(a, \theta) d\theta + \sin(\alpha k) \int_a^\infty r^k [q(r, -\alpha) + q(r, \alpha)] \frac{dr}{r} \right\} \\ = a^{k+1} \int_{-\alpha}^\alpha \cos(k\theta) g(\theta) d\theta + \cos(\alpha k) \int_a^\infty r^k [g_-(r) - g_+(r)] \frac{dr}{r} \quad (3.3)$$

and

$$\begin{aligned} & k \left\{ a^k \int_{-\alpha}^{\alpha} \sin(k\theta) q(a, \theta) d\theta + \cos(\alpha k) \int_a^{\infty} r^k [q(r, -\alpha) - q(r, \alpha)] \frac{dr}{r} \right\} \\ &= a^{k+1} \int_{-\alpha}^{\alpha} \sin(k\theta) g(\theta) d\theta - \sin(\alpha k) \int_a^{\infty} r^k [g_-(r) + g_+(r)] \frac{dr}{r}. \end{aligned} \quad (3.4)$$

Replacing, on the r.h.s. of the relations (3.3) and (3.4), $q(r, -\alpha) \pm q(r, \alpha)$ by the r.h.s. of (2.24) and (2.27), we find the following equations, which are valid for $\text{Re } k < \alpha/2\pi$:

$$a \int_{-\alpha}^{\alpha} g(\varphi) F_1(\varphi, k) d\varphi + \int_a^{\infty} [g_-(\rho) - g_+(\rho)] F_2(\rho, k) \frac{d\rho}{\rho} = 0 \quad (3.5)$$

and

$$a \int_{-\alpha}^{\alpha} g(\varphi) F_3(\varphi, k) d\varphi + \int_a^{\infty} [g_-(\rho) + g_+(\rho)] F_4(\rho, k) \frac{d\rho}{\rho} = 0, \quad (3.6)$$

where the functions $\{F_j\}_1^4$ are defined as follows:

$$\begin{aligned} F_1(\varphi, k) &= k \int_{-\alpha}^{\alpha} \cos(k\theta) \ln \left| \sin\left(\frac{\pi\varphi}{2\alpha}\right) + \sin\left(\frac{\pi\theta}{2\alpha}\right) \right| d\theta \\ &+ k \sin(\alpha k) \int_a^{\infty} \left(\frac{r}{a}\right)^k \left[\ln \left| \frac{R^2(r)}{4} - \left(\sin\frac{\pi\varphi}{2\alpha}\right)^2 \right| - G(r) \right] \frac{dr}{r} - \pi \cos(k\varphi), \\ &- \alpha < \varphi < \alpha, \quad \text{Re } k < \frac{\pi}{2\alpha}, \end{aligned} \quad (3.7)$$

$$\begin{aligned} F_2(\rho, k) &= k \int_{-\alpha}^{\alpha} \cos(k\theta) \ln \left| \frac{R(\rho)}{2} + \sin\left(\frac{\pi\theta}{2\alpha}\right) \right| d\theta \\ &+ k \sin(\alpha k) \int_a^{\infty} \left(\frac{r}{a}\right)^k \left[\ln \left| \frac{R^2(r)}{4} - \frac{R^2(\rho)}{4} \right| - G(r) \right] \frac{dr}{r} - \pi \cos(\alpha k) \left(\frac{\rho}{a}\right)^k, \\ &a < \rho < \infty, \quad \text{Re } k < \frac{\pi}{2\alpha}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} F_3(\varphi, k) &= -k \int_{-\alpha}^{\alpha} \sin(k\theta) \ln \left| \sin\left(\frac{\pi\varphi}{2\alpha}\right) + \sin\left(\frac{\pi\theta}{2\alpha}\right) \right| d\theta \\ &+ k \cos(\alpha k) \int_a^{\infty} \left(\frac{r}{a}\right)^k \ln \left| \frac{R(r) + 2 \sin\left(\frac{\pi\varphi}{2\alpha}\right)}{R(r) - 2 \sin\left(\frac{\pi\varphi}{2\alpha}\right)} \right| \frac{dr}{r} - \pi \sin(k\varphi), \\ &- \alpha < \varphi < \alpha, \quad \text{Re } k < \frac{\pi}{2\alpha}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \text{and} \quad F_4(\rho, k) &= k \int_{-\alpha}^{\alpha} \sin(k\theta) \ln \left| \frac{R(\rho)}{2} + \sin\left(\frac{\pi\theta}{2\alpha}\right) \right| d\theta + k \cos(\alpha k) \int_a^{\infty} \left(\frac{r}{a}\right)^k \ln \left| \frac{R(r) - R(\rho)}{R(r) + R(\rho)} \right| \frac{dr}{r} \\ &+ \pi \sin(\alpha k) \left(\frac{\rho}{a}\right)^k, \quad a < \rho < \infty, \quad \text{Re } k < \frac{\pi}{2\alpha}. \end{aligned} \quad (3.10)$$

In the above equations, principal-value integrals are assumed if needed.

The function $g_-(\rho) - g_+(\rho)$ is related to $g(\varphi)$ through the equation $S = 0$, whereas the function $g_-(\rho) + g_+(\rho)$ is independent of $g(\varphi)$. Hence, equations (3.5) and (3.6) imply the following basic identities:

$$F_1(\varphi, k) = f(k), \quad F_2(\rho, k) = f(k) \quad (3.11)$$

and

$$F_3(\varphi, k) = 0, \quad F_4(\rho, k) = 0, \quad (3.12)$$

where

$$a < \rho < \infty, \quad -\alpha < \varphi < \alpha, \quad \text{Re } k < \frac{\pi}{2\alpha}$$

and $f(k)$ is some function of k .

The above equations will be verified explicitly in the next section, where it will also be shown that the function $f(k)$ is given by

$$f(k) = -\sin(\alpha k) \left[2 \ln 2 + \frac{\pi}{\alpha k} \right]. \quad (3.13)$$

Remark 3.1. Using the estimates

$$\ln \left| \frac{R^2(r)}{4} - \left(\sin \frac{\pi \varphi}{2\alpha} \right)^2 \right| - G(r) = O(r^{-\pi/\alpha}), \quad |\varphi| < \frac{\pi}{2}, \quad r \rightarrow \infty,$$

and

$$\ln \left| \frac{R^2(r)}{4} - \frac{R^2(\rho)}{\alpha} \right| - G(r) = O(r^{-\pi/\alpha}), \quad 0 < \rho < \infty, \quad r \rightarrow \infty,$$

it follows that the first two integrals on the r.h.s. of equation (3.7) involving the above terms are well defined for $\operatorname{Re} k < \pi/2\alpha$. If k satisfies the stronger restriction $\operatorname{Re} k < 0$, then it is *not* necessary to subtract the term involving $G(r)$. Actually in this case, we find

$$k \sin(\alpha k) \int_a^\infty \left(\frac{r}{a} \right)^k G(r) \frac{dr}{r} = f(k), \quad \operatorname{Re} k < 0. \quad (3.14)$$

Remark 3.2. If $\operatorname{Re} k < 0$, then equations (3.11) and (3.12) are replaced by the equations

$$\tilde{F}_1(\varphi, k) = \tilde{F}_2(\rho, k) = F_3(\varphi, k) = F_4(\rho, k) = 0, \quad |\varphi| < \frac{\pi}{2}, \quad 0 < \rho < \infty, \quad (3.15)$$

where \tilde{F}_1 and \tilde{F}_2 denote the expressions obtained from F_1 and F_2 by neglecting the terms involving $G(r)$.

Equation (3.15) can be obtained *directly* as follows: the function

$$q(r, \theta) = \operatorname{Re}(z^k) = r^k \cos(k\theta), \quad \operatorname{Re} k < 0, \quad (3.16)$$

is a solution of the Neumann boundary-value problem with

$$g_\pm(r) = \mp k r^k \sin(k\alpha) \quad \text{and} \quad g(\theta) = k a^{k-1} \cos(k\theta). \quad (3.17)$$

Substituting $q(r, \alpha) = r^k \cos(k\alpha)$ and expressions (3.17) in equation (2.22)⁺, we find

$$\begin{aligned} r^k \cos(k\alpha) &= \frac{k \sin(k\alpha)}{\pi} \int_a^\infty \rho^k \ln \left| \frac{R^2(\rho)}{4} - \frac{R^2(r)}{4} \right| \frac{d\rho}{\rho} \\ &\quad + \frac{k a^k}{\pi} \int_{-\alpha}^\alpha \cos(k\varphi) \ln \left| \frac{R(r)}{2} - \sin \left(\frac{\pi \varphi}{2\alpha} \right) \right| d\varphi. \end{aligned} \quad (3.18)$$

Replacing φ with $-\varphi$ and dividing by a^k/π , we find $\tilde{F}_2 = 0$.

Similarly, substituting $q(a, \theta) = a^k \cos(k\theta)$ and expressions (3.17) in equation (2.21), we find $\tilde{F}_1 = 0$.

The function

$$q(r, \theta) = \operatorname{Im}(z^k) = r^k \sin(k\theta), \quad \operatorname{Re} k < 0,$$

is also a solution of the Neumann boundary-value problem with

$$g_\pm(r) = \pm k r^k \sin(k\alpha) \quad \text{and} \quad g(\theta) = k a^{k-1} \cos(k\theta).$$

The above solution implies $F_3 = F_4 = 0$.

Remark 3.3. The first integral in the definition of F_1 involves a singularity at $\theta = -\varphi$. Making the change of variables $\theta = -\varphi + x$, the relevant singularity is mapped at $x = 0$ and the associated integral is $\ln |x|$. This singularity can be handled by Cauchy principal-value integrals. In this case,

the contribution of this singularity vanishes. Indeed, this contribution involves

$$\lim_{\epsilon \rightarrow 0} \left(\int_{-\alpha}^{-\epsilon} + \int_{\epsilon}^{\alpha} \right) \ln |x| \, dx = 2 \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\alpha} \ln x \, dx,$$

and using integration by parts, it follows that the contribution from $x = \epsilon$ vanishes.

If, instead of the principal-value integral, we use the limit from above and below, the relevant contribution is

$$\lim_{\epsilon \rightarrow 0} i \int_0^{\pi} \ln(\epsilon) e^{i\theta} \, d\theta,$$

which clearly does not exist.

4. Verification of the four identities

In order to simplify the functions $\{F_j\}_1^4$ defined by (3.7)–(3.10), we introduce a change of variables,

$$\left. \begin{aligned} \frac{r}{a} &= x^{-2\alpha/\pi}, & \frac{\rho}{a} &= y^{-2\alpha/\pi}, & k &= -\frac{\pi}{2\alpha}\kappa, \\ u &= e^{i\pi\theta/2\alpha} & \text{and } v &= e^{i\pi\varphi/2\alpha}. \end{aligned} \right\} \quad (4.1)$$

Then, $\{F_j\}_1^4$ can be written as

$$F_1(\kappa, v) = -\kappa h_1(\kappa, v) + \kappa \sin\left(\frac{\pi\kappa}{2}\right) [h_3(\kappa, v) + h_3(\kappa, -v)] - \frac{\pi}{2}(v^\kappa + v^{-\kappa}), \quad (4.2)$$

$$F_2(\kappa, v) = -\kappa h_6(\kappa, y) + \kappa \sin\left(\frac{\pi\kappa}{2}\right) [h_4(\kappa, y) + h_5(\kappa, y)] - \pi y^\kappa \cos\left(\frac{\pi\kappa}{2}\right), \quad (4.3)$$

$$F_3(\kappa, v) = \kappa h_2(\kappa, v) + \kappa \cos\left(\frac{\pi\kappa}{2}\right) [h_3(\kappa, v) - h_3(\kappa, -v)] + \frac{\pi}{2i}(v^\kappa - v^{-\kappa}) \quad (4.4)$$

and $F_4(\kappa, v) = \kappa h_7(\kappa, y) - \kappa \cos\left(\frac{\pi\kappa}{2}\right) [h_4(\kappa, y) - h_5(\kappa, y)] - \pi y^\kappa \sin\left(\frac{\pi\kappa}{2}\right), \quad (4.5)$

where the functions $\{h_j\}_1^7$ are defined by

$$h_1(\kappa, v) = \int_{\gamma_1} \frac{u^{\kappa-1} + u^{-\kappa-1}}{2i} \ln \left| \frac{v - v^{-1}}{2i} + \frac{u - u^{-1}}{2i} \right| du, \quad (4.6)$$

$$h_2(\kappa, v) = \int_{\gamma_1} \frac{u^{\kappa-1} - u^{-\kappa-1}}{2} \ln \left| \frac{v - v^{-1}}{2i} + \frac{u - u^{-1}}{2i} \right| du, \quad (4.7)$$

$$h_3(\kappa, v) = \int_0^1 x^{\kappa-1} \left[\ln \left| \frac{x + x^{-1}}{2} - \frac{v - v^{-1}}{2i} \right| + \ln x + \ln 2 \right] dx, \quad (4.8)$$

$$h_4(\kappa, y) = \int_0^1 x^{\kappa-1} \left[\ln \left| \frac{x + x^{-1}}{2} - \frac{y + y^{-1}}{2} \right| + \ln x + \ln 2 \right] dx, \quad (4.9)$$

$$h_5(\kappa, y) = \int_0^1 x^{\kappa-1} \left[\ln \left| \frac{x + x^{-1}}{2} + \frac{y + y^{-1}}{2} \right| + \ln x + \ln 2 \right] dx, \quad (4.10)$$

$$h_6(\kappa, y) = \int_{\gamma_1} \frac{u^{\kappa-1} + u^{-\kappa-1}}{2i} \ln \left| \frac{y + y^{-1}}{2} + \frac{u - u^{-1}}{2i} \right| du, \quad (4.11)$$

$$h_7(\kappa, y) = \int_{\gamma_1} \frac{u^{\kappa-1} - u^{-\kappa-1}}{2} \ln \left| \frac{y + y^{-1}}{2} + \frac{u - u^{-1}}{2i} \right| du \quad (4.12)$$

and γ_1 is depicted in figure 2.

In order to compute $\{h_j\}_1^7$, the following lemma will be useful.

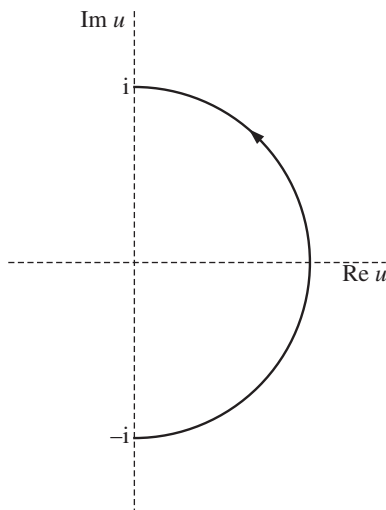


Figure 2. The oriented contour γ_1 .

Lemma 4.1. *Let γ be a smooth curve from z_1 to z_2 in the z -complex plane, which passes through z_0 . Then,*

$$I(p, z_0, \gamma) = \text{PV} \int_{\gamma} \frac{z^p}{z - z_0} dz = \frac{1}{(p+1)z_0} \left[z_1^{p+1} {}_2F_1 \left(1, \frac{p+1}{p+2}; \frac{z_1}{z_0} \right) - z_2^{p+1} {}_2F_1 \left(1, \frac{p+1}{p+2}; \frac{z_2}{z_0} \right) \right] - \pi i z_0^p, \quad \text{Re } p > -1, \quad (4.13)$$

where PV denotes the principal-value integral. If the singularity lies off the path γ , then the last term in (4.13) is absent.

Proof. Letting $x = z/z_0$, we find

$$I = z_0^p \text{PV} \int_{\gamma_0} \frac{x^p}{x - 1} dx,$$

where γ_0 runs from z_1/z_0 to z_2/z_0 , avoids $x = 0$ and passes through $x = 1$. We recall that

$$\frac{1}{p+1} {}_2F_1 \left(1, \frac{p+1}{p+2}; x \right) = \int_0^1 \frac{\rho^p}{1 - \rho x} d\rho, \quad \text{Re } p > -1.$$

Letting $\rho x = v$, we find

$$\frac{x^{p+1}}{p+1} {}_2F_1 \left(1, \frac{p+1}{p+2}; x \right) = \int_0^x \frac{v^p}{1 - v} dv,$$

so that

$$\frac{x^p}{x - 1} = -\frac{1}{p+1} \frac{d}{dx} \left[x^{p+1} {}_2F_1 \left(1, \frac{p+1}{p+2}; x \right) \right], \quad (4.14)$$

which immediately gives (4.13), the principal part at z_0 being included by subtracting half the residue at this point. ■

The hypergeometric functions in (4.13) are related to the Lerch transcendent, a generalization of the Riemann zeta function

$${}_2F_1 \left(1, \frac{p}{p+1}; z \right) = p\Phi(z, 1, p); \quad (4.15)$$

the Lerch transcendent is analytic in the z -plane for fixed $p \neq 1$.

Proposition 4.2. For $\text{Re } \kappa > -1$,

$$F_1(\varphi, k) = F_2(\rho, k) = -\sin(\alpha k) \left[2 \ln 2 + \frac{\pi}{\alpha k} \right]. \quad (4.16)$$

Proof. If $\kappa = 0$ (i.e. $k = 0$), then $F_1 = F_2 = -\pi$, which agrees with (4.16) in the limit $k \rightarrow 0$. Thus, we consider the case of $\kappa \neq 0$. Using integration by parts and a partial fraction decomposition, we find

$$h_1(\kappa, v) = \frac{1}{\kappa} \sin\left(\frac{\pi\kappa}{2}\right) \left[\ln \left| \frac{v - v^{-1}}{2i} + 1 \right| + \ln \left| \frac{v - v^{-1}}{2i} - 1 \right| \right] \\ + \frac{1}{2i\kappa(v + v^{-1})} \text{PV} \int_{\gamma_1} (u^\kappa - u^{-\kappa})(u + u^{-1}) \left(\frac{1}{u + v} - \frac{1}{u - v^{-1}} \right) du, \quad (4.17)$$

where the contour γ_1 is depicted in figure 2 and we have used

$$(i)^\kappa - (-i)^\kappa = 2i \sin \frac{\pi\kappa}{2}.$$

Using the change of variables $u \rightarrow 1/u$ for the term involving $u^{-\kappa}$, the above principal-value integral becomes

$$\frac{1}{2i\kappa(v + v^{-1})} \text{PV} \int_{\gamma_1} u^\kappa (u + u^{-1}) \left(\frac{1}{u + v} - \frac{1}{u - v^{-1}} - \frac{1}{u - v} + \frac{1}{u + v^{-1}} \right) du. \quad (4.18)$$

The integral in (4.18) can be split into two parts, and each resulting integral can be computed by using lemma 4.1. Noting that $-\alpha < \varphi < \alpha$, the definition of v (see (4.1)) implies that v and v^{-1} lie on the contour γ_1 . The relevant residue contributions yield the term

$$\frac{\pi}{2} (v^\kappa + v^{-\kappa}). \quad (4.19)$$

Hence, we find

$$\frac{1}{2i\kappa(v + v^{-1})} \text{PV} \int_{\gamma_1} u^\kappa (u + u^{-1}) \left(\frac{1}{u + v} - \frac{1}{u - v^{-1}} - \frac{1}{u - v} + \frac{1}{u + v^{-1}} \right) du \\ = -\frac{1}{\kappa(v + v^{-1})} \sin\left(\frac{\pi\kappa}{2}\right) \left[\frac{1}{v} \tilde{F}\left(\kappa + 1; -\frac{i}{v}\right) + \frac{1}{v} \tilde{F}\left(\kappa + 1; \frac{i}{v}\right) + v \tilde{F}(\kappa + 1; -iv) \right. \\ \left. + v \tilde{F}(\kappa + 1; iv) - \frac{1}{v} \tilde{F}\left(\kappa - 1; -\frac{i}{v}\right) - \frac{1}{v} \tilde{F}\left(\kappa - 1; \frac{i}{v}\right) - v \tilde{F}\left(\kappa - 1; -\frac{i}{v}\right) - v \tilde{F}(\kappa - 1; iv) \right] \\ - \frac{\pi}{2\kappa} (v^\kappa + v^{-\kappa}), \quad (4.20)$$

where

$$\tilde{F}(\kappa; u_0) = \frac{1}{\kappa + 1} {}_2F_1\left(1, \begin{matrix} \kappa + 1 \\ \kappa + 2 \end{matrix}; u_0\right). \quad (4.21)$$

Regarding $h_3(\kappa, v)$, using integration by parts and a partial fraction decomposition, we find

$$h_3(\kappa, v) = \frac{1}{\kappa} \left\{ \ln \left| \frac{v - v^{-1}}{2i} - 1 \right| - \frac{1}{\kappa} + \ln 2 - \frac{i}{v + v^{-1}} \int_0^1 x^\kappa (x - x^{-1}) \left(\frac{1}{x + iv} - \frac{1}{x - iv^{-1}} \right) dx \right\}. \quad (4.22)$$

The integral in (4.22) can be split into two parts, and each resulting integral can be computed using lemma 4.1. The definition of v implies that the integrand in (4.22) does not have any singularities



Figure 3. The deformed contour γ_2 .

on $[0, 1]$. Hence, we find

$$h_3(\kappa, v) = \frac{1}{\kappa} \left\{ \ln \left| \frac{v - v^{-1}}{2i} - 1 \right| - \frac{1}{\kappa} + \ln 2 - \frac{1}{v + v^{-1}} \right. \\ \left. \times \left[\frac{1}{v} \tilde{F} \left(\kappa + 1; \frac{i}{v} \right) - \frac{1}{v} \tilde{F} \left(\kappa - 1; \frac{i}{v} \right) + v \tilde{F}(\kappa + 1; -iv) - v \tilde{F}(\kappa - 1; -iv) \right] \right\}. \quad (4.23)$$

Substituting (4.20) into (4.17) and using (4.23), we find that the terms involving the hypergeometric functions and the logarithmic functions cancel. Hence, recalling $\kappa = -(2\alpha/\pi)k$, we obtain (4.16) for F_1 .

Regarding F_2 , we first deform the contour γ_1 to γ_2 , which is the segment $(-i, i)$ of the imaginary axis (figure 3). Then, using integration by parts and a partial fraction decomposition, we find

$$h_6(\kappa, y) = \frac{1}{\kappa} \sin \left(\frac{\pi \kappa}{2} \right) \left[\ln \left| \frac{1}{2}(y + y^{-1}) + 1 \right| + \ln \left| \frac{1}{2}(y + y^{-1}) - 1 \right| + i\pi y^\kappa \right] \\ - \frac{1}{2\kappa(y - y^{-1})} \text{PV} \int_{\gamma_2} u^\kappa (u + u^{-1}) \left(\frac{1}{u + iy} - \frac{1}{u + iy^{-1}} - \frac{1}{u - iy} + \frac{1}{u - iy^{-1}} \right) du. \quad (4.24)$$

Using the fact that the integrand of the above principal-value integral has poles at $x = \pm iy$ and employing lemma 4.1, we find that the principal-value integral of the r.h.s. of (4.24) is given by

$$\frac{1}{\kappa(y - y^{-1})} \sin \left(\frac{\pi \kappa}{2} \right) \left[\frac{1}{y} \tilde{F} \left(\kappa + 1; -\frac{1}{y} \right) - y \tilde{F}(\kappa + 1; -y) + \frac{1}{y} \tilde{F} \left(\kappa + 1; \frac{1}{y} \right) - y \tilde{F}(\kappa + 1; y) \right. \\ \left. - \frac{1}{y} \tilde{F} \left(\kappa - 1; -\frac{1}{y} \right) + y \tilde{F}(\kappa - 1; -y) - \frac{1}{y} \tilde{F} \left(\kappa - 1; \frac{1}{y} \right) + y \tilde{F}(\kappa - 1; y) \right] - \frac{\pi}{\kappa} y^\kappa \cos \left(\frac{\pi \kappa}{2} \right). \quad (4.25)$$

For $h_4(\kappa, y)$ and $h_5(\kappa, y)$, using the fact that the integrands have a pole at $x = y$, we find

$$h_4(\kappa, y) = \frac{1}{\kappa} \left\{ \ln \left| \frac{y + y^{-1}}{2} - 1 \right| - \frac{1}{\kappa} + \ln 2 + \frac{1}{y - y^{-1}} \right. \\ \left. \times \left[\frac{1}{y} \tilde{F} \left(\kappa + 1; \frac{1}{y} \right) - \frac{1}{y} \tilde{F} \left(\kappa - 1; \frac{1}{y} \right) - y \tilde{F}(\kappa + 1; y) + y \tilde{F}(\kappa - 1; y) \right] + i\pi y^\kappa \right\} \quad (4.26)$$

and

$$h_5(\kappa, y) = \frac{1}{\kappa} \left\{ \ln \left| \frac{y + y^{-1}}{2} + 1 \right| - \frac{1}{\kappa} + \ln 2 + \frac{1}{y - y^{-1}} \right. \\ \left. \times \left[\frac{1}{y} \tilde{F} \left(\kappa + 1; -\frac{1}{y} \right) - \frac{1}{y} \tilde{F} \left(\kappa - 1; -\frac{1}{y} \right) - y \tilde{F}(\kappa + 1; -y) + y \tilde{F}(\kappa - 1; -y) \right] \right\}. \quad (4.27)$$

Combining (4.24), (4.26) and (4.27) with (4.25), we find that the terms involving the hypergeometric functions and the logarithmic functions cancel and hence we find (4.16) for F_2 . ■

Proposition 4.3. For $\text{Re } \kappa > -1$,

$$F_3(\varphi, k) = F_4(\rho, k) = 0. \quad (4.28)$$

Proof. If $\kappa = 0$, it is obvious that $F_3 = F_4 = 0$. Thus, we consider the case $\kappa \neq 0$. Using integration by parts and a partial fraction decomposition, $h_2(\kappa, v)$ can be written as

$$h_2(\kappa, v) = \frac{1}{\kappa} \cos \left(\frac{\pi \kappa}{2} \right) \left[\ln \left| \frac{v - v^{-1}}{2i} + 1 \right| - \ln \left| \frac{v - v^{-1}}{2i} - 1 \right| \right] \\ + \frac{1}{2\kappa(v + v^{-1})} \text{PV} \int_{\gamma_1} u^\kappa (u + u^{-1}) \left(\frac{1}{u + v} - \frac{1}{u - v^{-1}} + \frac{1}{u - v} - \frac{1}{u + v^{-1}} \right) du, \quad (4.29)$$

where we have used

$$(i)^\kappa + (-i)^\kappa = 2 \cos \frac{\pi \kappa}{2}.$$

The principal-value integral in (4.29) can be evaluated by lemma 4.1,

$$- \frac{1}{\kappa(v + v^{-1})} \cos \left(\frac{\pi \kappa}{2} \right) \left[\frac{1}{v} \tilde{F} \left(\kappa + 1; -\frac{i}{v} \right) - \frac{1}{v} \tilde{F} \left(\kappa + 1; \frac{i}{v} \right) - v \tilde{F} \left(\kappa + 1; -\frac{i}{v} \right) + v \tilde{F}(\kappa + 1; iv) \right. \\ \left. - \frac{1}{v} \tilde{F} \left(\kappa - 1; -\frac{i}{v} \right) + \frac{1}{v} \tilde{F} \left(\kappa - 1; \frac{i}{v} \right) + v \tilde{F}(\kappa - 1; -iv) - v \tilde{F}(\kappa - 1; iv) \right] \\ - \frac{\pi}{2i\kappa} (v^\kappa - v^{-\kappa}). \quad (4.30)$$

Substituting (4.30) into (4.29) and combining the resulting expression with (4.23), we find $F_3 = 0$.

Regarding F_4 , deforming the contour γ_1 to γ_2 and then using integration by parts, we find

$$h_7(\kappa, y) = \frac{1}{\kappa} \cos \left(\frac{\pi \kappa}{2} \right) \left[\ln \left| \frac{1}{2}(y + y^{-1}) + 1 \right| - \ln \left| \frac{1}{2}(y + y^{-1}) - 1 \right| - i\pi y^\kappa \right] \\ + \frac{1}{2i\kappa(y - y^{-1})} \text{PV} \int_{\gamma_2} u^\kappa (u + u^{-1}) \left(\frac{1}{u + iy} - \frac{1}{u + iy^{-1}} + \frac{1}{u - iy} - \frac{1}{u - iy^{-1}} \right) du. \quad (4.31)$$

Using the fact that the integrand of the above principal-value integral has poles at $x = \pm iy$ and employing lemma 4.1, we find that the principal-value integral of the r.h.s. of (4.31) is given by

$$- \frac{1}{\kappa(y - y^{-1})} \cos \left(\frac{\pi \kappa}{2} \right) \left[\frac{1}{y} \tilde{F} \left(\kappa + 1; -\frac{1}{y} \right) - y \tilde{F}(\kappa + 1; -y) - \frac{1}{y} \tilde{F} \left(\kappa + 1; \frac{1}{y} \right) + y \tilde{F}(\kappa + 1; y) \right. \\ \left. - \frac{1}{y} \tilde{F} \left(\kappa - 1; -\frac{1}{y} \right) + y \tilde{F}(\kappa - 1; -y) + \frac{1}{y} \tilde{F} \left(\kappa - 1; \frac{1}{y} \right) - y \tilde{F}(\kappa - 1; y) \right] - \frac{\pi}{\kappa} y^\kappa \sin \left(\frac{\pi \kappa}{2} \right). \quad (4.32)$$

Using (4.26) and (4.27), together with (4.32), we find $F_4 = 0$. ■

5. Functional identities

The functions h_3 , h_5 , h_6 and h_7 do not involve principal-value integrals. We note that principal-value integrals always yield a term involving i in accordance with the Plemelj formula. Thus, for

those real values of k for which h_3, h_5, h_6 and h_7 are well defined, it should be possible to express these functions in terms of real functions. We have succeeded in doing this for the last three functions, but not for h_3 . Indeed, the following expressions are valid for $\text{Re } k > -1$. Although the formulae in this section are valid for complex y , we shall restrict attention to $0 < y < 1$,

$$h_5(k, y) = \frac{1}{k} \left\{ \ln \left| \frac{1}{2}(y + y^{-1}) + 1 \right| - \frac{1}{k} + \ln 2 \right. \\ \left. + \frac{1}{k(y^{-2} - 1)} \left[y^{-2} {}_2F_1 \left(\begin{matrix} 1, & k \\ & k+1 \end{matrix}; -y^{-1} \right) - {}_2F_1 \left(\begin{matrix} 1, & k \\ & k+1 \end{matrix}; -y \right) \right] \right. \\ \left. - \frac{1}{(k+2)(y^{-2} - 1)} \left[y^{-2} {}_2F_1 \left(\begin{matrix} 1, & k+2 \\ & k+3 \end{matrix}; -y^{-1} \right) - {}_2F_1 \left(\begin{matrix} 1, & k+2 \\ & k+3 \end{matrix}; -y \right) \right] \right\}, \quad (5.1)$$

$$h_6(k, y) = \frac{\sin(\pi k/2)}{k(y - y^{-1})^2} \left\{ (y - y^{-1})^2 \ln \left| \frac{1}{2}(y - y^{-1}) \right| - \frac{1}{k-2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ & 2 - \frac{k}{2}, & \frac{k}{2} \end{matrix}; -\frac{4}{(y - y^{-1})^2} \right) \right. \\ \left. + \frac{1}{k+2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ & -\frac{k}{2}, & 2 + \frac{k}{2} \end{matrix}; -\frac{4}{(y - y^{-1})^2} \right) \right\} \quad (5.2)$$

$$\text{and } h_7(k, y) = \frac{2}{k} \cos \left(\frac{\pi k}{2} \right) \left\{ \frac{1}{2} \left[\ln \left| \frac{y + y^{-1}}{2} - 1 \right| - \ln \left| \frac{y + y^{-1}}{2} + 1 \right| \right] \right. \\ \left. + \frac{y + y^{-1}}{(y - y^{-1})^2} \left[\frac{1}{k+1} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ & \frac{1-k}{2}, & \frac{3+k}{2} \end{matrix}; -\frac{4}{(y - y^{-1})^2} \right) \right. \right. \\ \left. \left. - \frac{1}{k-1} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ & \frac{3-k}{2}, & \frac{1+k}{2} \end{matrix}; -\frac{4}{(y - y^{-1})^2} \right) \right] \right\}. \quad (5.3)$$

In addition, for k real, the functions h_1 and h_2 possess the following more complicated real forms:

$$h_1(k, v) = \frac{1}{k} \left\{ 2 \sin \left(\frac{\pi k}{2} \right) \ln \left| \frac{v + v^{-1}}{2} \right| + \frac{\pi^{3/2}}{(v + v^{-1})^2} \left[G_{33}^{21} \left(\frac{(v + v^{-1})^2}{4} \middle| \begin{matrix} 1 & -\frac{k}{2} & 2 + \frac{k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) \right. \right. \\ \left. \left. - G_{33}^{21} \left(\frac{(v + v^{-1})^2}{4} \middle| \begin{matrix} 1 & 2 - \frac{k}{2} & \frac{k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) \right] \right\} \quad (5.4)$$

and

$$h_2(k, v) = -\frac{1}{k} \left\{ \cos \left(\frac{\pi k}{2} \right) \left[\ln \left| \frac{v - v^{-1}}{2i} - 1 \right| - \ln \left| \frac{v - v^{-1}}{2i} + 1 \right| \right] \right. \\ \left. - \frac{2\pi^{3/2}(v - v^{-1})}{i(v + v^{-1})} \left[G_{33}^{21} \left(\frac{(v + v^{-1})^2}{4} \middle| \begin{matrix} 1 & \frac{1-k}{2} & \frac{3+k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) \right. \right. \\ \left. \left. + G_{33}^{21} \left(\frac{(v + v^{-1})^2}{4} \middle| \begin{matrix} 1 & \frac{3-k}{2} & \frac{1+k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) \right] \right\}, \quad (5.5)$$

where G_{pq}^{mn} denotes the Meijer G-function.

The basic identities (3.11) and (3.12) yield a plethora of novel identities involving the generalized hypergeometric and related functions.

Example 5.1. Replacing in the definition (4.5) of F_4 , h_7 by (5.3), h_4 by (4.26) and h_5 by (5.1), the identity $F_4 = 0$ yields, apart from $k = 1, 3, 5, \dots$, where both sides are singular,

$$\begin{aligned} & \frac{1}{k+1} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ \frac{1-k}{2}, & \frac{3+k}{2} \end{matrix}; -\frac{4}{(y-y^{-1})^2} \right) + \frac{1}{1-k} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ \frac{1+k}{2}, & \frac{3-k}{2} \end{matrix}; -\frac{4}{(y-y^{-1})^2} \right) \\ &= \frac{\pi(y^2-1)^2 y^{k-1}}{2(y^2+1)} \left(i + \tan \left(\frac{\pi k}{2} \right) \right) + \frac{y^2-1}{2(y^2+1)} \left\{ \frac{y}{k} \left[{}_2F_1 \left(\begin{matrix} 1, & k \\ 1+k \end{matrix}; y \right) - {}_2F_1 \left(\begin{matrix} 1, & k \\ 1+k \end{matrix}; -y \right) \right] \right. \\ & \quad \left. - \frac{y}{k+2} \left[{}_2F_1 \left(\begin{matrix} 1, & k+2 \\ k+3 \end{matrix}; y \right) - {}_2F_1 \left(\begin{matrix} 1, & k+2 \\ k+3 \end{matrix}; -y \right) \right] - (y \rightarrow y^{-1}) \right\}, \\ & \operatorname{Re} k > -1, \quad 0 < y < 1. \end{aligned} \quad (5.6)$$

The term $(y \rightarrow y^{-1})$ in (5.6) represents all the terms within the curly brackets where y is replaced by y^{-1} ; the i appears as a consequence of the fact that the analytic continuation of Gauss' hypergeometric function acquires an imaginary part for real arguments exceeding unity. Note that the limit $k \rightarrow 0$ of both sides of (5.6) exist.

Letting $k = 2$ with $z = 2/(y - y^{-1})$, equation (5.6) yields the following novel identity:

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ -\frac{1}{2}, & \frac{5}{2} \end{matrix}; -z^2 \right) = \frac{3}{z^2} \left\{ \frac{4 + 3z^2}{\sqrt{1+z^2}} \frac{\sinh^{-1} z}{z} - 4 \right\}, \quad (5.7)$$

which, by analytic continuation, is valid for all z .

For $k = \frac{1}{2}$, equation (5.6) yields the curious identity

$$\begin{aligned} & \left(\frac{4}{(y-y^{-1})^2} \right) \left[{}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ \frac{1}{4}, & \frac{7}{4} \end{matrix}; -\frac{4}{(y-y^{-1})^2} \right) + 3 {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ \frac{3}{4}, & \frac{5}{4} \end{matrix}; -\frac{4}{(y-y^{-1})^2} \right) \right] \\ &= \frac{3(1+i)\pi y^{3/2}}{y^2+1} + 6 \frac{\sqrt{y}}{y^2+1} \left[y \left(\tanh^{-1} \frac{1}{\sqrt{y}} - \tan^{-1} \frac{1}{\sqrt{y}} \right) + \left(\tanh^{-1} \sqrt{y} - \tan^{-1} \sqrt{y} \right) \right], \\ & 0 < y < 1. \end{aligned} \quad (5.8)$$

Letting

$$z = \frac{2}{y - y^{-1}}, \quad y = \frac{1}{z} + \sqrt{1 + \frac{1}{z^2}},$$

it follows that equation (5.8) can be rewritten in terms of the variable z .

Similarly, for $k = 4$, we find the novel identity for $y^2 \leq 1$,

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ -\frac{3}{2}, & \frac{7}{2} \end{matrix}; -\frac{4}{(y-y^{-1})^2} \right) \\ &= \frac{5}{6y^5} \frac{1-y^2}{1+y^2} \left\{ 3(1-y^{10}) \sinh^{-1} \left(\frac{2y}{1-y^2} \right) - 2y(1-y^4)(3y^4+y^2+3) \right\}. \end{aligned} \quad (5.9)$$

Example 5.2. Replacing in the definition (4.3) of F_2 , h_6 by (5.2), h_4 by (4.26) and h_5 by (5.1), the identity $F_2 = f(k)$ yields the following identity:

$$\begin{aligned} & \frac{4}{(k+2)(y-y^{-1})^2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ -\frac{k}{2}, & 2+\frac{1}{2}k, & \end{matrix} ; -\frac{4}{(y-y^{-1})^2} \right) \\ & + \frac{4}{(2-k)(y-y^{-1})^2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, & 1, & 1 \\ \frac{k}{2}, & 2-\frac{1}{2}k, & \end{matrix} ; -\frac{4}{(y-y^{-1})^2} \right) \\ & = 2\pi y^k \left(i - \cot \left(\frac{\pi k}{2} \right) \right) + \frac{2y^{-1}}{y^{-1}-y} \left\{ \frac{1}{k} \left[{}_2F_1 \left(\begin{matrix} 1, & k \\ k+1, & \end{matrix} ; y^{-1} \right) \right. \right. \\ & \quad \left. \left. + {}_2F_1 \left(\begin{matrix} 1, & k \\ k+1, & \end{matrix} ; -y^{-1} \right) \right] \right. \\ & \quad \left. - \frac{1}{k+2} \left[{}_2F_1 \left(\begin{matrix} 1, & k+2 \\ k+3, & \end{matrix} ; y^{-1} \right) + {}_2F_1 \left(\begin{matrix} 1, & 2+k \\ 3+k, & \end{matrix} ; -y^{-1} \right) \right] \right\} \\ & + (y \rightarrow y^{-1}), \quad \operatorname{Re} k > -1, \quad |\ln y| < \infty. \end{aligned} \quad (5.10)$$

In this case, the term $(y \rightarrow y^{-1})$ represents the expression in curly brackets with y replaced by y^{-1} .

Replacing in the definition (4.4) of F_4 , h_2 by (5.5), we find the novel identity, valid for $\operatorname{Re} k > -1$ and all real v ,

$$\begin{aligned} & G_{33}^{21} \left(\frac{(v+v^{-1})^2}{4} \middle| \begin{matrix} 1 & \frac{1+k}{2} & \frac{3-k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) + G_{33}^{21} \left(\frac{(v+v^{-1})^2}{4} \middle| \begin{matrix} 1 & \frac{1-k}{2} & \frac{3+k}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) \\ & = \frac{(v+v^{-1})^2}{i\pi^{1/2}(v-v^{-1})} \left\{ \frac{2}{\pi} \cos \left(\frac{\pi k}{2} \right) \left[\frac{1}{k+2} \operatorname{Re} \left[\frac{2v}{v+v^{-1}} {}_2F_1 \left(\begin{matrix} 1, & 2+k \\ 3+k, & \end{matrix} ; iv \right) \right] \right. \right. \\ & \quad \left. - \frac{2v^{-1}}{v+v^{-1}} {}_2F_1 \left(\begin{matrix} 1, & 2+k \\ 3+k, & \end{matrix} ; iv^{-1} \right) - \frac{1}{k+1} \operatorname{Re} \left[\frac{v^2-1}{i(v-v^{-1})} {}_2F_1 \left(\begin{matrix} 1, & k+1 \\ k+2, & \end{matrix} ; iv^{-1} \right) \right. \right. \\ & \quad \left. \left. - \frac{v^{-2}-1}{i(v^{-1}-v)} {}_2F_1 \left(\begin{matrix} 1, & k+1 \\ k+2, & \end{matrix} ; iv \right) \right] - \frac{i}{2} (v^k - v^{-k}) \right\}. \end{aligned} \quad (5.11)$$

In particular, for $k=0$ and all real b ,

$$G_{33}^{21} \left(\cos^2 b \middle| \begin{matrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & 1 & \frac{1}{2} \end{matrix} \right) = \frac{2}{\pi^{3/2}} \cos b \cot b \ln \left(\frac{\cos b}{1 + \sin b} \right). \quad (5.12)$$

The above identities appear to be new and different in character from any in the standard literature. For example, consider equations (5.7) and (5.9) that are two particular cases stemming from the identity (5.6). These two equations present two hypergeometric identifications that are *not* contained in the most extensive compilation available, that of Prudnikov *et al.* [33]. It is true that MATHEMATICA is able to reduce the l.h.s. of (5.7) to an expression containing a ${}_2F_1$ whose indefinite integral is listed in [33], but some algebraic manipulation is required to produce the r.h.s. of (5.7) (for the casual user, MATHEMATICA provides no description as to the derivation of its results). Furthermore, MATHEMATICA is silent on (5.9). All the hypergeometric identities required for the manipulations in this section are contained in table 7.3 of [33]. It should be noted that k is *by no means restricted to integer values*. It is restricted solely by the requirement $\operatorname{Re} k > -1$. Thus, our results encompass functions such as ${}_3F_2(\frac{1}{2}, 1, 1; -i, 2+i; z)$, for which MATHEMATICA

and the literature are silent. We maintain that identities such as (5.6) and (5.11), which are only illustrations of what our methods are capable of providing, are *completely new* to hypergeometric theory. Thus, it seems that by employing the global relation to the solution of certain boundary-value problems, it is possible to construct new formulae in the area of special functions, although it is not yet clear how the form of these identities can be predicted in advance.

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