We specify and analyse models that capture the geometry of purposeful motion of a collective of mobile agents, with a focus on planar motion, dyadic strategies and attention graphs which are static, directed and cyclic. Strategies are formulated as constraints on joint shape space and are implemented through feedback laws for the actions of individual agents, here modelled as self-steering particles. By reduction to a labelled shape space (using a redundant parametrization to account for cycle closure constraints) and a further reduction through time rescaling, we characterize various special solutions (relative equilibria and pure shape equilibria) for cyclic pursuit with a constant bearing (CB) strategy. This is accomplished by first proving convergence of the (nonlinear) dynamics to an invariant manifold (the CB pursuit manifold), and then analysing the closed-loop dynamics restricted to the invariant manifold. For illustration, we sketch some low-dimensional examples. This formulation—involving strategies, attention graphs and sensor-driven steering laws—and the resulting templates of collective motion, are part of a broader programme to interpret the mechanisms underlying biological collective motion.

1. Introduction

Geometric ideas enter the investigation of collective behaviour from multiple vantage points. In this paper, we specify and analyse models that capture the geometry
of purposeful motion of a collective of mobile agents. To this end, strategies of individual agents are formulated as constraints (via submanifolds) on joint state spaces of two or more interacting individuals. These strategies are implemented through feedback laws for the actions of individual agents, here modelled as self-steering particles. It is natural to prescribe the structure of the interactions by a directed graph, representing the attention given by an individual to particular neighbours. Taken together, these elements determine the dynamics of the collective. This paper is an exploration of how certain strategies, feedback laws and structure, influence the spatio-temporal evolution of collective dynamics.

Here, we focus on planar motion, dyadic strategies (i.e. constraints based on pairs of agents) and static directed cycle graphs, leading to explicit computations and results. It is noteworthy that certain key ideas are not dependent on this specialization, and one can profit from the perspective of interacting particles on a Lie group $G$ (e.g. $SE(2)$, the rigid motion group in the plane) formulated below (in §2). In this broader setting, control laws (and hence collective dynamics) are required to admit a continuous symmetry group (e.g. each particle’s control depends only on relative positions and relative orientations). This enables reduction to a labelled shape space associated with the collective, preserving particle identity. If the interactions are governed by a graph with cycles, then each such cycle results in a closure constraint equation on a redundant set of shape variables. Such redundancy enables transparent expressions for control laws, convenient analysis of collective dynamics in shape space, and other benefits that we exploit below. In this paper, we investigate the dynamics in shape space to achieve an understanding of how a family of feedback laws associated with strategies of pursuit type influence collective behaviour.

In §2, we set up the model and notation of self-steering particles, interaction graph and reduction to shape space. The resulting shape dynamics is cast in a natural redundant parametrization of shape space tailored to the setting of $SE(2)$. This parametrization fits well with anticipated singularities in feedback laws, such as one associated with motion camouflage (MC) or motion parallax nulling [1]. In §3, we explicitly state the dyadic pursuit strategy we call constant bearing (CB)—of central interest in this paper—using a suitable contrast function to fix a constraint on joint state space. We also recall from Wei et al. [2] a feedback law to execute the CB strategy. For a particular choice of parameter, the CB strategy and feedback law specialize to the case of classical pursuit (CP), a topic with a rich mathematical heritage and continuing interest [3]. For comparison, we also note the motion camouflage feedback law [1]. In the biological realm, it is known that the peregrine falcon, a superior aerial predator, displays a prey–capture behaviour known as stoop, distinguished by a spiral trajectory towards prey. Observations of this behaviour suggest that it is a possible result of a CB strategy (direction of flight constrained to maintain a constant angle with the line-of-sight to prey item). It is further argued that the strategy evolved to cope with the conflicting requirements of aerodynamic performance in high-speed flight and visual fixation of the target [4,5].

When the attention graph has out-degree 1 for each node (i.e. each agent pays attention to exactly one other in the collective), there is always a cycle, and if the graph is connected, exactly one cycle. In §4, we focus on connected pure cycle graphs of interaction. A key result of this section is the derivation of an attractive invariant manifold for the closed-loop shape dynamics under the CB feedback law, setting the stage for the analysis of the dynamics restricted to this invariant manifold in shape space. (We note that some aspects of this work can be extended to the setting of dynamic attention graphs—see Remark at the end of §4c.)

In §5, we observe that the structure of the shape dynamics restricted to the invariant manifold suggests a change of coordinates and time rescaling for a further reduction to pure shape dynamics. This additional reduction plays a crucial role in the three-particle setting, where a nine-dimensional state space leads to a two-dimensional pure shape space, thus permitting the use of phase-plane techniques [6] for analysis.

In §6, we examine certain special solutions (on the attractive invariant manifold in shape space) for multi-agent cyclic pursuit systems under CB feedback laws. Under certain (mutually exclusive) constraints on parameter values, the state-space dynamics admit rectilinear and circling relative equilibria, and other pure shape equilibria, such as spirals, are also possible.
The key results of this section give these parameter constraints explicitly. The symmetric case (i.e. all agents employ the same parameter) is further explored as a concrete example. Linearized stability of these special solutions may be investigated using methods as in [7,8] for certain multi-agent cyclic pursuit systems. Nonlinear analysis of stability for certain low-dimensional examples (of evolution of pure shape) is investigated in [6]. Section 7 provides a brief sketch of some of these examples.

Whereas in this paper, we focus on the execution of a single type of pursuit strategy, namely a CB strategy, by all agents in a collective, such uniformity of strategy type need not hold either in natural or artificial settings. Heterogeneity of behaviour in a collective may arise from strategy preferences of agents dictated by differences in their sensory capabilities, and behavioural context. The methods of this paper lend themselves to analysis of such richer settings.

For the reader, a question of interest might be: does nature manifest architectures of collective behaviour such as the one we focus on in this paper (with cycle graphs of attention and CB strategy of pursuit)? Answering this question has qualitative and quantitative aspects. On a qualitative level, the occurrence of toroidal schools of fish [9], foraging circles of dolphins [10] and ant mills [11,12], all with circling patterns of agents as a common feature, elicit comparisons with the circling relative equilibria associated with certain parametric regimes of cyclic CB pursuit (§6). Quantitative assessment of architecture of collective behaviour demands detailed analysis of trajectory data gathered in the field or laboratory, including statistical comparison of individual trajectory properties (such as curvatures) against feedback control laws of the type discussed in this paper (as in §3). With regard to two-agent interactions, we note previous work [13–15] on prey capture behaviour of echolocating bats as illustrative. In a recent study of flocks of starlings [16], and in the work on hierarchical structures in flight behaviour of flocks of (homing) pigeons [17], broad graph structural properties are inferred using statistical analyses. But none of these settings involve cycle graphs. However, see the study of Mischiati & Krishnaprasad [18] on trajectories associated with dynamics of mutual MC for pairs of agents, and a qualitative comparison to patterns of fighting spirals observed in pairs of dragonflies during territorial battles [19]. While there is much work to be done to investigate the relation of biological data to the constructs of this paper, the dynamical understanding here, based on cycle graphs of attention, dyadic strategies and feedback laws, already suggests engineering applications to missions by aerial robotic systems [8].

2. Self-steering particles and labelled shape space

Planar self-steering particles with position vectors $r_i$ are described by the curve and moving frame equations [1]

\[
\begin{align*}
\dot{r}_i &= v_i x_i, \\
\dot{x}_i &= v_i y_i u_i, \\
\dot{y}_i &= -v_i x_i u_i, 
\end{align*}
\]

and

\[
\begin{align*}
\dot{v}_i &= v_i \nu_i (A_1 + A_2 u_i), \quad \nu_i \in \text{se}(2),
\end{align*}
\]

where $v_i > 0$ is the particle speed (held constant), and $u_i$ is a steering (curvature) control. (Here, $x_i$ is the unit tangent vector at $r_i$, and $y_i$ is the unit normal obtained by counterclockwise rotation of $x_i$ by $\pi/2$.)

We can then define the state variables

\[
\begin{bmatrix}
x_i & y_i & r_i \\
0 & 0 & 1 
\end{bmatrix} \in SE(2), \quad i = 1, 2, \ldots, n, \tag{2.2}
\]

along with the state dynamics

\[
\dot{g}_i = \dot{g}_i \xi_i = g_i v_i (A_1 + A_2 u_i), \quad \xi_i \in \text{se}(2), \quad \text{the Lie algebra of } SE(2), \tag{2.3}
\]
\( i = 1, 2, \ldots, n \), where
\[
A_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
A_2 = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Thus, we treat the planar self-steering particles as evolving on a matrix Lie group \( G = SE(2) \) according to the dynamics
\[
\dot{g}_i = g_i \xi_i, \quad i = 1, 2, \ldots, n,
\]
where \( \xi_i \in \mathfrak{g} \), the Lie algebra of \( G \).

(a) Directed graph and shape variables

Given a directed graph \( G = (\mathcal{N}, A) \) with node set \( \mathcal{N} = \{1, 2, \ldots, n\} \) and arc set \( A \), satisfying

(A1) the graph \( G \) is weakly connected, has no ‘self loops’ (arcs that begin and end at the same node), and is static,

we can define the (possibly redundant) shape variables
\[
\tilde{g}_{ij} = g_i^{-1} g_j, \quad (i,j) \in A.
\]

We interpret \( G \) as the ‘attention graph’ governing which particle(s) a particular particle is responsive to (e.g. pursuing).

Consider the ordered \( p \)-tuple \( I = (i_1, i_2, \ldots, i_p) \), \( p \in \mathbb{N} \), defined such that \( (i_{k-1}, i_k) \in A \) for all \( k = 1, 2, \ldots, p-1 \), and suppose that \( (i_p, i_1) \in A \), also. Then, \( I \) determines a closed cycle, and we have
\[
\left( \prod_{k=1}^{p-1} \tilde{g}_{i_k i_{k+1}} \right) \tilde{g}_{i_1 i_p} \triangleq \tilde{g}_{i_1 i_2} \tilde{g}_{i_2 i_3} \cdots \tilde{g}_{i_{p-1} i_p} \tilde{g}_{i_p i_1} \equiv e,
\]
where \( e \) is the group identity element. We refer to (2.7) as a cycle closure constraint.

From (2.5) and (2.6), it follows that the shape dynamics evolve according to
\[
\dot{\tilde{g}}_{ij} = \tilde{g}_{ij} \tilde{\xi}_{ij},
\]
where
\[
\tilde{\xi}_{ij} = \xi_j - \tilde{g}_{ij}^{-1} \xi_i \tilde{g}_{ij} = \xi_j - \text{Ad}_{\tilde{g}_{ij}^{-1}} \xi_i.
\]

From (2.8), if the cycle closure constraints (2.7) are satisfied initially, then they will automatically be satisfied for all future time. We exploit this property below.

(b) Symmetry and reduction

We now impose the requirement on (2.5) that the controls \( \xi_i, \; i = 1, 2, \ldots, n \), depend only on \( \tilde{g}_{ij} \), \( (i,j) \in A \). With this constraint, system (2.5) is \( G \)-invariant, so we can reduce it to a shape space—which we call the labelled shape space because we are only removing the continuous group symmetry and not removing any discrete relabelling symmetry (which may or may not, in fact, be present).

The reduced dynamics are thus (2.8) for \((i,j) \in A\), where the \( \tilde{\xi}_{ij} \) given by (2.9) clearly depend only on the shape variables. If cycles are present in the graph \( G \), then the \( \tilde{g}_{ij} \) are subject to additional cycle closure constraints (2.7) (which, if satisfied initially, will automatically be satisfied for all future time).
Figure 1. Definitions of the shape variables $\kappa_{ij}, \theta_{ij}$ and $\rho_{ij}$. The interpretation of the arc $(i,j) \in \mathcal{A}$ is that particle $i$ is paying attention to particle $j$. (Online version in colour.)

(c) Parametrization of shape space

Given the graph $\mathcal{G}$ with arc set $\mathcal{A}$, we can define graph-dependent shape variables according to equation (2.6). There is a polar parametrization which is convenient when we later exhibit specific interaction laws. The polar singularity (when agents collide) introduced by this choice corresponds to the singularity which will appear later in the particular control laws we consider.

Let $R(\beta)$ denote the $2 \times 2$ rotation matrix

$$R(\beta) = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \in SO(2).$$

(2.10)

We define the shape variables $\kappa_{ij}, \theta_{ij}$, and $\rho_{ij}$ by

$$R(\kappa_{ij})x_i \cdot \frac{r_i - r_j}{|r_i - r_j|} = -1, \quad R(\theta_{ij})x_j \cdot \frac{r_j - r_i}{|r_j - r_i|} = 1 \quad \text{and} \quad \rho_{ij} = |r_i - r_j|,$$

(2.11)

for $(i,j) \in \mathcal{A}$, provided $r_i \neq r_j$ (figure 1). In this parametrization, the shape dynamics (2.8) take the form

$$\dot{\kappa}_{ij} = -v_i u_i + \frac{1}{\rho_{ij}} (v_i \sin \kappa_{ij} + v_j \sin \theta_{ij})$$

$$\dot{\theta}_{ij} = -v_j u_j + \frac{1}{\rho_{ij}} (v_i \sin \kappa_{ij} + v_j \sin \theta_{ij})$$

and

$$\dot{\rho}_{ij} = -v_i \cos \kappa_{ij} - v_j \cos \theta_{ij},$$

(2.12)

provided $\rho_{ij} > 0$. This singularity requires us to impose a non-collocation condition in the state space; specifically, letting $e_3$ denote the standard basis vector in $\mathbb{R}^3$, we define the state space as

$$M_{\text{state}} = \{ (g_1, g_2, \ldots, g_n) \in SE(2) \times SE(2) \times \cdots \times SE(2) | g_i e_3 \neq g_j e_3, (i,j) \in \mathcal{A} \},$$

(2.13)

and note that the condition $g_i e_3 \neq g_j e_3$ is equivalent to $\rho_{ij} > 0$. No claim is being made that the dynamics (2.12) will necessarily ensure that $(g_1, g_2, \ldots, g_n)$ will remain in $M_{\text{state}}$ for all time.

We assume that the steering controls in (2.12) have the form

$$u_i = u_i (\tilde{g}_{i1}, \tilde{g}_{i2}, \ldots, \tilde{g}_{in}), \quad (i,j_1), (i,j_2), \ldots, (i,j_m) \in \mathcal{A},$$

(2.14)

where $(i,j) \in \mathcal{A}$ has the interpretation that particle $i$ is paying attention to particle $j$. This is a further restriction of the assumption (in §2b) that the $\tilde{g}_{ij}$ depend only on shape variables.

The term $(1/\rho_{ij})(v_i \sin \kappa_{ij} + v_j \sin \theta_{ij})$ appearing in (2.12) can be interpreted as a motion parallax term: specifically, it is the angular velocity of the baseline between particles $i$ and $j$. If we set $u_i \equiv 0 \equiv u_j$, then the only contribution to the change in angles $\kappa_{ij}$ and $\theta_{ij}$ is due to the rotation of this baseline. Although the motion parallax term contains a factor $1/\rho_{ij}$, it is important to note that
this factor simply reflects the physical fact that the angular rotation rate of the baseline is inversely proportional to particle separation, and it does not imply that distance sensing is required to measure the motion parallax.

3. Strategies

We distinguish strategies, which are geometrical conditions defined by constraint equations on state space (yielding corresponding constraint equations on shape space), from control laws which achieve (either asymptotically or approximately in finite time) these geometrical conditions. Although in this paper, we are concerned specifically with analysis of CB pursuit, we also briefly mention CP and MC pursuit for comparison.

(a) Constant bearing

Suppose that particle $i$ pursues only particle $j$, i.e. node $i$ of $\mathcal{G}$ has out-degree one, and $(i,j) \in A$. Using the state-space coordinates of (2.1), we can define the CB contrast function for particle $i$ as

$$\Lambda_i(\alpha_i) \equiv R(\alpha_i) x_i \cdot \frac{r_i - r_j}{|r_i - r_j|}.$$  

(3.1)

We say that CB pursuit (by particle $i$ of particle $j$) has been attained if and only if $\Lambda_i = -1$. Thus, the CB pursuit strategy is the specification of a constraint on state space. The constant $\alpha_i \in S^1$, the circle group, is a prescribed bearing angle (which need not be acute). The contrast function (3.1) can be expressed in shape variables as

$$\Lambda_i = -\cos(\kappa_{ij} - \alpha_i).$$  

(3.2)

A feedback control law to achieve CB pursuit of particle $j$ by particle $i$, expressed in state-space coordinates is

$$u_{\text{CB}}(\alpha_i) = -\mu_i \left( R(\alpha_i) y_i \cdot \frac{r_i - r_j}{|r_i - r_j|} - \frac{1}{v_i |r_i - r_j|} \left( \frac{r_i - r_j}{|r_i - r_j|} \cdot \frac{d}{dt} (r_i - r_j) \right) \right),$$  

(3.3)

where $\mu_i$ may be a function of the shape variables, provided $\mu_i > \epsilon > 0$ for all $t$, and $q^\perp = Jq$ with $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This control law was shown to achieve CB pursuit in the sense of finite-time accessibility of the CB pursuit manifold defined by $\Lambda_i = -1$, in the high gain setting (i.e. $\mu_i$ large) with pursuer speed dominance (i.e. $v_i > v_j$), and finite evader maneuverability (i.e. $|u_j|$ bounded) [2]. In §3b, we prove asymptotic convergence to the CB pursuit manifold provided $\mu_i > \epsilon > 0$.

In shape variables, (3.3) becomes

$$u_{\text{CB}}(\alpha_i) = \mu_i \sin(\kappa_{ij} - \alpha_i) + \frac{1}{\rho_{ij}} \left( \sin \kappa_{ij} + \frac{v_i}{v_j} \sin \theta_{ji} \right).$$  

(3.4)

Observe that as a result of the $1/\rho_{ij}$ term in (3.4), we require $\rho_{ij} > 0$ in order for $u_i$ to be well defined. Although we could have chosen an alternative coordinatization of the shape space to avoid the singularity in the coordinatization, we would nevertheless have encountered this singularity in defining the controls according to (3.4). Thus, it is fundamentally the well definition of $u_i$, $i=1,2,\ldots,n$, and not the shape dynamics themselves, which necessitate the non-collocation condition.

Observe also that the second term on the right-hand side of (3.4) is simply the motion parallax term in (2.12) divided by the particle’s own speed $v_i$. The purpose of this term is to cancel the motion parallax term in the expression for $\dot{\kappa}_{ij}$ in (2.12), as will be made explicit in §4.
(b) Classical pursuit

If $\alpha_i = 0$, then the CB strategy becomes the CP strategy, with CP contrast function $\Lambda_i = x_i \cdot (r_i - r_j) / |r_i - r_j| = -\cos \kappa_{ij}$. Furthermore, a pursuit feedback law to achieve CP is given by

$$u_{CP} = \mu_i \sin \kappa_{ij} + \frac{1}{\rho_{ij}} \left( \sin \kappa_{ij} + \frac{v_j}{v_i} \sin \theta_{ji} \right),$$

which is simply (3.4) with $\alpha_i = 0$.

(c) Motion camouflage pursuit

MC pursuit, also referred to as Constant Absolute Target Direction pursuit [13], is closely related to the Pure Proportional Navigation Guidance (PPNG) law in the missile guidance literature [20]. The contrast function associated with MC is

$$\Gamma_i = \frac{r_i - r_j}{|r_i - r_j|} \cdot \frac{(d/\text{d}t)(r_i - r_j)}{|(d/\text{d}t)(r_i - r_j)|},$$

or in shape variables,

$$\Gamma_i = \frac{v_i \cos \kappa_{ij} - v_j \cos \theta_{ji}}{\sqrt{(v_i \cos \kappa_{ij} - v_j \cos \theta_{ji})^2 + (v_i \sin \kappa_{ij} - v_j \sin \theta_{ji})^2}}.$$

We say that MC pursuit has been attained if and only if $\Gamma_i = -1$. A control law to achieve MC pursuit in the sense of finite-time accessibility of the MC pursuit manifold defined by $\Gamma_i = -1$, in the high gain setting (i.e. $\mu_i$ large) with pursuer speed dominance (i.e. $v_i > v_j$), and finite evader maneuverability (i.e. $|u_j|$ bounded) [1] is

$$u_{MC} = -\mu_i \left( \frac{r_i - r_j}{|r_i - r_j|} \cdot \frac{d}{\text{d}t}(r_i - r_j) \right),$$

where $\mu_i$ satisfies the same conditions as in (3.3). In shape variables, (3.8) becomes

$$u_{MC} = \mu_i (v_i \sin \kappa_{ij} + v_j \sin \theta_{ji}),$$

which is seen to have the form of the motion parallax term, but with an additional gain factor. In PPNG, the gain $\mu_i$ is assumed to contain a factor of $1/\rho_{ij}$, so that the overall steering control $u_{MC}$ does not require range sensing. By contrast, in the MC law, we can choose $\mu_i$ differently to permit $u_{MC}$ to be well defined at $\rho_{ij} = 0$, at the expense of requiring relative velocity sensing (e.g. via echolocation) rather than bearing rate sensing.

4. Closed-loop dynamics

In this section, we state the dynamics that result when the feedback loops are closed. Suppose

(A2) the graph $\mathcal{G}$ has the property that every node has an out-degree of one.

Assumptions (A1) and (A2) imply that the graph contains exactly one cycle. Several particle interaction graphs having this property are shown in figure 2. In the context of pursuit, the interpretation is that each particle pursues exactly one other particle. Suppose the steering controls have the form (3.4) for $i = 1, \ldots, n$, where we note that $j = j(i)$ denotes the unique
Figure 2. Several networks of self-steering particles having the property that every node has an out-degree equal to one. (Online version in colour.)

particle serving as the sink of the arc from particle $i$. Substituting (3.4) into (2.12), we obtain the closed-loop dynamics

$$\begin{align*}
\dot{\kappa}_{ij} &= -v_i \mu_i \sin(\kappa_{ij} - \alpha_i), \\
\dot{\theta}_{ji} &= -v_j \mu_j \sin(\kappa_{jk} - \alpha_j) - \frac{1}{\rho_{jk}} (v_j \sin \kappa_{jk} + v_k \sin \theta_{kj}) + \frac{1}{\rho_{ij}} (v_i \sin \kappa_{ij} + v_j \sin \theta_{ji}) \\
\dot{\rho}_{ij} &= -v_i \cos \kappa_{ij} - v_j \cos \theta_{ji},
\end{align*}$$

(4.1)

for $i = 1, 2, \ldots, n$, where $j = j(i)$ satisfies $(i, j) \in \mathcal{A}$, and $k = k(j)$ satisfies $(j, k) \in \mathcal{A}$. The important point is that under control law (3.4), the dynamics for $\kappa_{ij}, (i, j) \in \mathcal{A}, i = 1, 2, \ldots, n$, in (2.12) no longer involve $\theta_{ji}$ and $\rho_{ij}$.

(a) Attractivity of an invariant manifold

We define the CB pursuit manifold by

$$M_{\text{CB}(\alpha)} = \{(\kappa_{ij}, \theta_{ji}, \rho_{ij}) \in S^1 \times S^1 \times \mathbb{R} | (i, j) \in \mathcal{N}, (i, j) \in \mathcal{A}, \rho_{ij} > 0, \text{ and } \Lambda_i = -1\}. \quad (4.2)$$

**Proposition 4.1.** The CB pursuit manifold $M_{\text{CB}(\alpha)}$ is invariant under the dynamics (4.1). Let $\gamma(t)$ be a trajectory of the closed-loop CB shape dynamics (4.1) with initial conditions satisfying the cycle closure constraint. Suppose that the attention graph $\mathcal{G}$ satisfies assumptions (A1) and (A2), and furthermore, that $\gamma(t)$ has no finite escape time (i.e. $\rho_{ij} > 0$ for all $t \geq 0, (i, j) \in \mathcal{A}$). Finally, assume that the initial value of the CB contrast functions satisfy $\Lambda_i(0) \neq 1, i = 1, 2, \ldots, n$. Then

$$\Lambda_i(t) \to -1 \quad \text{as } t \to \infty, \quad i = 1, 2, \ldots, n,$$

(4.3)

i.e. $\gamma(t)$ converges asymptotically to $M_{\text{CB}(\alpha)}$. 
Proof. By definition, \( M_{CB(\alpha)} \) is invariant under the dynamics (4.1) if \( \gamma(0) \in M_{CB(\alpha)} \) implies that 
\( \gamma(t) \in M_{CB(\alpha)} \) for all \( t \geq 0 \). By (3.2) and (4.1), we have 
\[
\dot{A}_i = k_{ij} \sin(\kappa_{ij} - \alpha_i) = -v_i \mu_i \sin^2(\kappa_{ij} - \alpha_i) = -v_i \mu_i (1 - \Lambda_i^2),
\]
(4.4) for \( i = 1, 2, \ldots, n \), and thus \( M_{CB(\alpha)} \) is invariant under (4.1). Furthermore, from (4.4), we conclude that 
\( A_i(0) = \pm 1 \) implies that \( A_i(t) = \pm 1 \), for all \( t \geq 0 \).

So we assume \( A_i(0) \neq \pm 1 \) and, as in [1], write (4.4) as 
\[
\frac{dA_i}{1 - \Lambda_i^2} = -v_i \mu_i \, dt.
\]
(4.5)

Integrating both sides of (4.5) yields 
\[
\int_{A_i(0)}^{A_i} \frac{d\Lambda_i}{1 - \Lambda_i^2} = -v_i \mu_i \int_0^t \, dt = -v_i \mu_i t,
\]
(4.6)

and since 
\[
\int_{A_i(0)}^{A_i} \frac{d\Lambda_i}{1 - \Lambda_i^2} = \int_{A_i(0)}^{\Lambda_i(0)} \, d(\tanh^{-1}(\Lambda_i)) = \tanh^{-1}(A_i) - \tanh^{-1}(A_i(0)),
\]
(4.7)

we have 
\[
A_i(t) = \tanh(\tanh^{-1}(A_i(0)) - v_i \mu_i t), \quad i = 1, 2, \ldots, n.
\]
(4.8)

Thus, since \( \tanh(\cdot) \) is a monotone increasing function, we have \( A_i(t) \to -1 \) as \( t \to \infty \).

(b) Dynamics restricted to the invariant manifold

Thus, under the hypotheses of the above proposition, (i) \( \kappa_{ij} \to \alpha_i, \; i = 1, 2, \ldots, n \); and (ii) the manifold \( M_{CB(\alpha)} \), on which we have \( \kappa_{ij} = \alpha_i, \; i = 1, 2, \ldots, n \), is invariant in the sense that for any initial condition on the manifold, the controlled system (4.1) will stay on the manifold for all future time. The behaviour of the system can then be decomposed into two parts: convergence to the invariant manifold (from arbitrary initial conditions in \( M_{\text{state}} \)) and the behaviour on the invariant manifold. The dynamics on the invariant manifold is given by 
\[
\begin{align*}
\dot{\theta}_{ij} &= -\frac{1}{\rho_{jk}}(v_j \sin \alpha_j + v_k \sin \theta_{kj}) + \frac{1}{\rho_{ij}}(v_i \sin \alpha_i + v_j \sin \theta_{ji}) \\
\dot{\rho}_{ij} &= -v_i \cos \alpha_i - v_j \cos \theta_{ji},
\end{align*}
\]
(4.9)

for \( i = 1, 2, \ldots, n \), where \( j = j(i) \) satisfies \((i, j) \in A\), and \( k = k(j) \) satisfies \((j, k) \in A\).

Equations (4.1) and (4.9) are an interesting family of nonlinear dynamics, and we argue that these merit study.

(c) Cyclic interactions

Assumptions (A1) and (A2) imply that the graph contains exactly one cycle, and if we further assume 

(A3) \( A = \{(i, i + 1) \mid i = 1, 2, \ldots, n, \text{ interpreted mod } n\} \),

then the graph consists entirely of one cycle. This particular type of graph, the cyclic graph, will be the focus of the remainder of the paper. (Henceforth, in the context of cyclic graphs, indices will be interpreted modulo \( n \)).

To further focus the analysis, we also assume 

(A4) \( v_1 = v_2 = \cdots = v_n = 1 \).
Under assumptions (A3) and (A4), the closed-loop cyclic CB shape dynamics (4.1) can then equivalently be written as

\[
\begin{align*}
\dot{k}_i &= -\mu_i \sin(k_i - \alpha_i), \\
\dot{\theta}_i &= -\mu_i \sin(k_i - \alpha_i) - \frac{1}{\rho_i} (\sin k_i + \sin \theta_{i+1}) + \frac{1}{\rho_{i-1}} (\sin k_{i-1} + \sin \theta_i), \\
\dot{\rho}_i &= -\cos k_i - \cos \theta_{i+1}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(4.10)

and

\[
\dot{\rho}_i = -\cos k_i - \cos \theta_{i+1}, \quad i = 1, 2, \ldots, n,
\]

where

\[
k_i \triangleq k_{i,i+1}, \quad \theta_i \triangleq \theta_{i,i-1} \quad \text{and} \quad \rho_i \triangleq \rho_{i,i+1}.
\]

(4.11)

Although notationally (4.10) gives the impression that the shape variables \((k_i, \theta_i, \rho_i)\) are associated with node \(i\), as the foregoing discussion shows, it is more natural to consider \((k_i, \theta_{i+1}, \rho_i)\) as associated with the arc \((i, i+1)\). Also, recall that the initial conditions for (4.10) are required to satisfy the cycle closure constraint (2.7), which can be expressed in terms of the shape variables \((k_i, \theta_i, \rho_i)\) as

\[
R \left( \sum_{i=1}^{n} (\pi + k_i - \theta_i) \right) = \mathbb{1} \quad \text{and} \quad \sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\pi + k_j - \theta_j) \right) = 0,
\]

(4.12)

where \(\mathbb{1}\) is the \(2 \times 2\) identity matrix and 0 represents the \(2 \times 2\) zero matrix. (For a detailed calculation verifying (4.12), see the electronic supplementary material, Supplemental Calculations, section (a).)

Observe that proposition 4.1 still holds under assumptions (A3) and (A4), and from (4.9) we have the corresponding reduced dynamics on \(M_{CB(\alpha)}\) given by

\[
\begin{align*}
\dot{\theta}_i &= \frac{1}{\rho_{i-1}} (\sin \alpha_{i-1} + \sin \theta_i) - \frac{1}{\rho_i} (\sin \alpha_i + \sin \theta_{i+1}), \\
\dot{\rho}_i &= -\cos \alpha_i - \cos \theta_{i+1}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(4.13)

with initial conditions subject to

\[
R \left( \sum_{i=1}^{n} (\pi + \alpha_i - \theta_i) \right) = \mathbb{1} \quad \text{and} \quad \sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\pi + \alpha_j - \theta_j) \right) = 0.
\]

(4.14)

At the start of this section, and for the remainder of the analysis, we chose to have each agent apply a steering law for CB pursuit. Note, however, that heterogeneity of sensing capabilities among agents could instead lead one to consider dyadic interactions which are not common to all agents.

Remark. The set of cyclic graphs with \(n\) nodes is in one-to-one correspondence with a subset of the permutation group \(S_n\) on \(n\) elements. By constructing a dynamics on \(S_n\) (for instance, using a Poisson-counter-driven stochastic differential equation model on \(\mathbb{R}^{n \times n}\) viewed as a space for embedding the group \(S_n\), [21]) it is possible to extend the model of this section to a class of dynamic attention graphs. Analysis of such extensions is of interest.

5. Reduction by rescaling time

In this section, we demonstrate that an alternative parametrization of \(M_{CB(\alpha)}\) and scaling of the time variable yields dynamics which separate into two parts: one part describing the evolution of the size of the particle formation, and the other describing the dynamics of the pure shape of...
the formation, i.e. the formation shape up to geometric similarity. We begin by introducing the change of variables

\[
\begin{align*}
\lambda & \equiv \ln(\rho_1) \\
\tilde{\rho}_i & \equiv \frac{\rho_i}{\rho_1} = \rho_i e^{-\lambda}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(5.1)

for which the dynamics (4.13) can be expressed in the new variables as

\[
\begin{align*}
\dot{\lambda} &= -e^{-\lambda}(\cos \alpha_1 + \cos \theta_2), \\
\dot{\theta}_i &= e^{-\lambda} \left( \frac{1}{\tilde{\rho}_{i-1}}(\sin \alpha_{i-1} + \sin \theta_i) - \frac{1}{\tilde{\rho}_i}(\sin \alpha_i + \sin \theta_{i+1}) \right), \\
\dot{\tilde{\rho}}_i &= e^{-\lambda}(\tilde{\rho}_i(\cos \alpha_1 + \cos \theta_2) - (\cos \alpha_i + \cos \theta_{i+1})), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

(5.2)

(Note that \(\tilde{\rho}_1 \equiv 1\), and therefore (5.2) includes the trivial equation \(\dot{\tilde{\rho}}_1 \equiv 0\).) Then, introducing the time-scaling

\[
\tau = \int_0^t e^{-\lambda(\sigma)} \, d\sigma,
\]

(5.3)

so that

\[
\frac{d\lambda}{d\tau} = \frac{d\lambda}{dt} \frac{dt}{d\tau} = e^{\lambda(t)} \dot{\lambda}, \quad \frac{d\theta_i}{d\tau} = \frac{d\theta_i}{dt} \frac{dt}{d\tau} = e^{\lambda(t)} \dot{\theta}_i, \quad \frac{d\tilde{\rho}_i}{d\tau} = \frac{d\tilde{\rho}_i}{dt} \frac{dt}{d\tau} = e^{\lambda(t)} \dot{\tilde{\rho}}_i,
\]

(5.4)

we have

\[
\begin{align*}
\lambda' &= -(\cos \alpha_1 + \cos \theta_2), \\
\theta'_i &= \frac{1}{\tilde{\rho}_{i-1}}(\sin \alpha_{i-1} + \sin \theta_i) - \frac{1}{\tilde{\rho}_i}(\sin \alpha_i + \sin \theta_{i+1}), \\
\tilde{\rho}'_i &= \tilde{\rho}_i(\cos \alpha_1 + \cos \theta_2) - (\cos \alpha_i + \cos \theta_{i+1}), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(5.5, 5.6, 5.7)

where the prime notation is used to denote differentiation with respect to \(\tau\). Observe that (5.5) describes the evolution of the size (i.e. scale) of the formation, and (5.6) and (5.7) form a self-contained sub-system which we refer to as the pure shape dynamics, corresponding to the shape of the formation up to geometric similarity. The pure shape dynamics (5.6) and (5.7) are subject to the initial condition cycle closure constraints

\[
R \left( \sum_{i=1}^n (\pi + \alpha_i - \theta_i) \right) = 1 \quad \text{and} \quad \sum_{i=1}^n \tilde{\rho}_i R \left( \sum_{j=1}^i (\pi + \alpha_j - \theta_j) \right) = 0.
\]

(5.8)

Note that while (4.13) consists of \(2n\) independent equations, the self-contained system consisting of (5.6) and (5.7) consists of only \(2n-1\) independent equations, since \(\tilde{\rho}_1 \equiv 1\). Thus, introducing the new time scaling has enabled us to reduce the shape dynamics (restricted to \(M_{\text{CB}(\alpha)}\)) by one dimension. This reduction step turns out to be particularly important in three-particle cyclic pursuit, because even though the original system in state space is nine-dimensional, the closure constraint reduces this to six dimensions, restriction to \(M_{\text{CB}(\alpha)}\) removes three equations, and thus the time-rescaling ultimately yields a two-dimensional system. This two-dimensional system can then be analysed using phase-plane techniques [6].

6. Special solutions

We proceed by investigating the existence of certain special solutions in terms of conditions on the CB parameters \(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\). In particular, we consider equilibria for the shape dynamics (4.13), which correspond to relative equilibria for the full system (2.1), and equilibria for the pure shape dynamics (5.6) and (5.7), which we refer to as pure shape equilibria.
(a) Relative equilibria

System dynamics of the form (2.1) with \( v_i \equiv 1 \) for all \( i \) permit only two types of relative equilibria: rectilinear and circling. Rectilinear relative equilibria are characterized by all particles moving in the same direction along parallel straight-line trajectories, while circling relative equilibria correspond to all particles travelling on a common (since speeds are equal) closed circular trajectory separated by fixed chordal distances [22].

From (4.13), we obtain

\[
\dot{\rho}_i = 0 \iff \cos \alpha_i + \cos \theta_{i+1} = 0, \quad i = 1, 2, \ldots, n, \tag{6.1}
\]

and (under our assumption of no sequential colocation)

\[
\dot{\gamma}_i = 0 \iff \begin{cases} \sin \alpha_i + \sin \theta_{i+1} = 0, & i = 1, 2, \ldots, n, \\ \sin \alpha_i + \sin \theta_{i+1} \neq 0, & \frac{\rho_i}{\rho_{i-1}} = \frac{\sin \alpha_i + \sin \theta_{i+1}}{\sin \alpha_{i-1} + \sin \theta_i} > 0, & i = 1, 2, \ldots, n. \end{cases} \tag{6.2}
\]

To see that the right-hand side of (6.2) exhausts all possibilities, let

\[
\gamma_i \triangleq \frac{1}{\rho_i} [\sin \alpha_i + \sin \theta_{i+1}], \tag{6.3}
\]

so that \( \dot{\gamma}_i = \gamma_{i-1} - \gamma_i \). Then \( \dot{\gamma}_i = 0, \quad i = 1, 2, \ldots, n \) if and only if \( \gamma_{i-1} = \gamma_i, \quad i = 1, 2, \ldots, n \). Therefore, if there exists \( k \in \{1, 2, \ldots, n\} \) such that \( \gamma_k = 0 \) and it holds that \( \dot{\gamma}_i = 0, \quad i = 1, 2, \ldots, n \), then we must have \( \gamma_i = 0, \quad i = 1, 2, \ldots, n \).

Thus, taking (6.1) and (6.2) together, we have

\[
\dot{\rho}_i = \dot{\gamma}_i = 0 \iff \begin{cases} \theta_{i+1} = \pi + \alpha_i, & i = 1, 2, \ldots, n, \\ \theta_{i+1} = \pi - \alpha_i, & \sin \alpha_i \neq 0, & \frac{\rho_i}{\rho_{i-1}} = \frac{\sin \alpha_i}{\sin \alpha_{i-1}} > 0, & i = 1, 2, \ldots, n. \end{cases} \tag{6.4}
\]

Thus, relative equilibria exist on \( M_{\text{CB}(\alpha)} \) if and only if there exists a choice of shape variables \( \{\theta_1, \rho_1, \theta_2, \rho_2, \ldots, \theta_n, \rho_n\} \) which simultaneously satisfy the closure constraint equations (4.14) as well as one of the equilibrium conditions in (6.4). In fact, one can show that the first condition in (6.4) (i.e. \( \theta_{i+1} = \pi + \alpha_i \)) corresponds to a rectilinear relative equilibrium, while the second condition (i.e. \( \theta_{i+1} = \pi - \alpha_i \)) corresponds to a circling relative equilibrium. We then have the following result governing existence of relative equilibria.

**Proposition 6.1.** Consider an \( n \)-particle cyclic CB pursuit system evolving on \( M_{\text{CB}(\alpha)} \) according to the shape dynamics (4.13) parametrized by \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \).

(i) A rectilinear relative equilibrium exists if and only if there exists a set of constants \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) such that \( \sigma_i > 0, \quad i = 1, 2, \ldots, n, \) and

\[
\sum_{i=1}^{n} \sigma_i e^{\alpha_i} = 0, \tag{6.5}
\]

(where \( j = \sqrt{-1} \)), in which case the corresponding equilibrium angles \( \dot{\theta}_i \) and equilibrium side lengths \( \dot{\rho}_i \) are given by

\[
\dot{\theta}_i = \pi + \alpha_{i-1} \quad \text{and} \quad \dot{\rho}_i = \sigma_i, \quad i = 1, 2, \ldots, n. \tag{6.6}
\]

(ii) A circling relative equilibrium exists if and only if

(i) \( \sin(\alpha_i) \sin(\alpha_{i-1}) > 0, \quad i = 1, 2, \ldots, n \) \tag{6.7}

and

(ii) \( \sin\left(\sum_{i=1}^{n} \alpha_i\right) = 0, \tag{6.8} \)
in which case the corresponding equilibrium angles \( \hat{\theta}_i \) and equilibrium side ratios \( \hat{\rho}_i / \hat{\rho}_{i-1} \) are given by

\[
\hat{\theta}_i = \pi - \alpha_{i-1} \quad \text{and} \quad \frac{\hat{\rho}_i}{\hat{\rho}_{i-1}} = \frac{\sin \alpha_i}{\sin \alpha_{i-1}}, \quad i = 1, 2, \ldots, n.
\] (6.9)

**Proof.** First, suppose that the rectilinear relative equilibrium condition is satisfied, i.e. \( \theta_{i+1} = \pi + \alpha_i \), \( i = 1, 2, \ldots, n \). This holds if and only if (cf. equation (4) in the electronic supplementary material, Supplemental Calculations)

\[
x_i \cdot x_{i+1} = \cos(\pi + \alpha_i - \theta_{i+1}) = 1, \quad \theta_{i+1} = \pi + \alpha_i,
\] (6.10)

from which we conclude that what we have called the ‘rectilinear relative equilibrium condition’ in (6.4) does indeed correspond to a rectilinear relative equilibrium. Substituting \( \theta_{i+1} = \pi + \alpha_i \) into (4.14), we see that the left-hand side equation holds without additional conditions on the \( \alpha_i \), while the right-hand side equation yields

\[
\sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\pi + \alpha_j - \theta_j) \right) = \sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\alpha_j - \alpha_{j-1}) \right)
\]

\[
= \sum_{i=1}^{n} \rho_i R (\alpha_i - \alpha_{n})
\]

\[
= R(-\alpha_n) \sum_{i=1}^{n} \rho_i R(\alpha_i).
\] (6.11)

Thus, (4.14) holds if and only if

\[
\sum_{i=1}^{n} \rho_i R(\alpha_i) = 0,
\] (6.12)

which is equivalent to (6.5) with (6.6).

Now suppose instead that the circling relative equilibrium conditions are satisfied, i.e.

\[
\theta_{i+1} = \pi - \alpha_i, \quad \sin \alpha_i \neq 0 \quad \frac{\rho_i}{\rho_{i-1}} = \frac{\sin \alpha_i}{\sin \alpha_{i-1}}, \quad i = 1, 2, \ldots, n.
\] (6.13)

This is equivalent to the existence of a point \( r_{cc} \in \mathbb{R}^2 \) (the circumcentre) such that

1. \( |r_{cc} - r_i| = |r_{cc} - r_{i-1}|, \quad i = 1, 2, \ldots, n \) (i.e. all particles are equidistant from the circumcentre),
2. \( x_i \cdot (r_{cc} - r_i) = 0, \quad i = 1, 2, \ldots, n \) (i.e. each particle’s velocity vector is perpendicular to the associated radial vector) and
3. \( (x_{i-1} \cdot (r_{cc} - r_{i-1}))(x_i \cdot (r_{cc} - r_i)) > 0, \quad i = 1, 2, \ldots, n \) (i.e. all particles are moving in the same direction, clockwise or counter-clockwise, around the circle).

This equivalence is demonstrated as follows. As is derived in the electronic supplementary material, Supplemental Calculations (section (b)), under (6.13) and the shape variable definitions (2.11), on \( M_{CB(a)} \) we have

\[
r_i + \frac{\rho_i}{2 \sin \alpha_i} x_i = r_{i-1} + \frac{\rho_{i-1}}{2 \sin \alpha_{i-1}} x_{i-1}, \quad i = 1, 2, \ldots, n,
\] (6.14)

so that the assignment

\[
r_{cc} \triangleq r_i + \frac{\rho_i}{2 \sin \alpha_i} x_i
\] (6.15)

is consistent for all \( i = 1, 2, \ldots, n \). Then using \( (\rho_i / \sin \alpha_i) = (\rho_{i-1} / \sin \alpha_{i-1}), \quad i = 1, 2, \ldots, n \), we obtain

\[
|r_{cc} - r_i| = \frac{1}{2} \left| \frac{\rho_i}{\sin \alpha_i} \right| = \frac{1}{2} \left| \frac{\rho_{i-1}}{\sin \alpha_{i-1}} \right| = |r_{cc} - r_{i-1}|, \quad i = 1, 2, \ldots, n,
\] (6.16)
establishing that all particles are equidistant from \( r_{cc} \). It follows from (6.15) that \( x_i \cdot (r_{cc} - r_i) = 0, i = 1, 2, \ldots, n \), and

\[
(x_{i-1}^+ \cdot (r_{cc} - r_{i-1})) (x_i^+ \cdot (r_{cc} - r_i)) = \left( \frac{\rho_{i-1}}{2 \sin \alpha_{i-1}} \right) \left( \frac{\rho_i}{2 \sin \alpha_i} \right) = \frac{\rho_i^2}{4 \sin^2 \alpha_i} > 0. \quad (6.17)
\]

Conversely, if the three conditions listed above are satisfied, then (6.16) can be straightforwardly derived from the Law of Cosines, and (6.13) follows. This establishes that what we have called the ‘circling relative equilibrium conditions’ in (6.4) (equivalently, (6.13)) do indeed correspond to a circling relative equilibrium.

Substituting our circling relative equilibrium conditions into the left-hand side equation in (4.14), we obtain

\[
R \left( \sum_{i=1}^{n} (\pi + \alpha_i - \theta_i) \right) = R \left( \sum_{i=1}^{n} (\alpha_i + \alpha_{i-1}) \right) = R \left( 2 \sum_{i=1}^{n} \alpha_i \right). \quad (6.18)
\]

A straightforward calculation (see the electronic supplementary material, Supplemental Calculations, section (c)) yields

\[
R \left( 2 \sum_{i=1}^{n} \alpha_i \right) = 1 \iff \sin \left( \sum_{i=1}^{n} \alpha_i \right) = 0, \quad (6.19)
\]

which establishes (6.8).

Substituting our circling relative equilibrium conditions into the right-hand side equation in (4.14), we obtain

\[
\sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\pi + \alpha_j - \theta_j) \right) = \rho_n \sum_{i=1}^{n} \rho_i R \left( \sum_{j=1}^{i} (\pi + \alpha_j - \theta_j) \right)
\]

\[
= \rho_n \sum_{i=1}^{n} \frac{\sin \alpha_i}{\sin \alpha_n} R \left( \sum_{j=1}^{i} (\pi + \alpha_j - (\pi - \alpha_{j-1})) \right)
\]

\[
= \frac{\rho_n}{\sin \alpha_n} \sum_{i=1}^{n} \sin \alpha_i R \left( \sum_{j=1}^{i} (\alpha_j + \alpha_{j-1}) \right). \quad (6.20)
\]

It can be shown (see the electronic supplementary material, Supplemental Calculations, section (d)) that

\[
\sin \alpha_n \sum_{i=1}^{n-1} \sin \alpha_i R \left( \sum_{j=1}^{i} (\alpha_j + \alpha_{j-1}) \right) = \sin \left( \sum_{i=1}^{n} \alpha_i \right) R \left( \sum_{i=1}^{n-1} \alpha_i \right), \quad (6.21)
\]

and therefore, making use of (6.8) we conclude that the right-hand side equation in (4.14) requires no additional conditions on the \( \alpha_i \). It is also clear from (6.17) that (6.7) is satisfied.

Conversely, if (6.7) and (6.8) are satisfied, then clearly \( \sin \alpha_i \neq 0, i = 1, 2, \ldots, n \), and it is clear from the above calculations that a solution of the form shown in (6.9) exists and is a circling relative equilibrium. □

Remark 6.2. Observe that the existence condition for rectilinear equilibria (6.5) and the condition (6.7) for circling equilibria are mutually exclusive, i.e. if a particular choice of CB parameters \( \alpha_1, \alpha_2, \ldots, \alpha_n \) admits rectilinear equilibria, then circling equilibria do not exist for those CB parameters (and vice versa). Also, note that (6.7) is satisfied if and only if \( \sin \alpha_i > 0, i = 1, 2, \ldots, n \) (which corresponds to counter-clockwise circling equilibria) or \( \sin \alpha_i < 0, i = 1, 2, \ldots, n \) (which corresponds to clockwise circling equilibria).
Figure 3. The planar trajectories corresponding to the four types of pure shape equilibria: (a) spirals, (b) expansion/contraction without rotation, (c) circling, and (d) rectilinear motion. For each trajectory, the base of the black arrow denotes the position at the end of the simulation time, and the direction of the arrow indicates the tangent direction to the particle’s motion at that time. (This convention is also used in the subsequent figures showing simulated particle trajectories.) (Online version in colour.)

(b) Pure shape equilibria

Equilibria for the pure shape dynamics (5.6) and (5.7), which we refer to as pure shape equilibria, correspond to planar trajectories which preserve the geometric shape of the particle formation while allowing for possible evolution of the formation size, as depicted in figure 3. We note that circling and rectilinear relative equilibria (figure 3c,d) are actually special cases of pure shape equilibria.

From (5.6) and (5.7), following analogous reasoning to that in §6a for rectilinear and circling relative equilibria, we conclude that

\[
\begin{align*}
\theta'_i &= 0 \iff \\
& \begin{cases}
(Aa) \sin \alpha_i + \sin \theta_{i+1} = 0, \ i = 1, 2, \ldots, n, \\
(Ba) \sin \alpha_i + \sin \theta_{i+1} \neq 0, \ \tilde{\rho}_i = \frac{\sin \alpha_i + \sin \theta_{i+1}}{\sin \alpha_i + \sin \theta_i} > 0, \ i = 1, 2, \ldots, n
\end{cases} \\
\tilde{\rho}'_i &= 0 \iff \\
& \begin{cases}
(Ba) \cos \alpha_i + \cos \theta_{i+1} = 0, \ i = 1, 2, \ldots, n, \\
(Bb) \cos \alpha_i + \cos \theta_{i+1} \neq 0, \ \tilde{\rho}_i = \frac{\cos \alpha_i + \cos \theta_{i+1}}{\cos \alpha_i + \cos \theta_i} > 0, \ i = 1, 2, \ldots, n
\end{cases}
\end{align*}
\]

and therefore the four possible cases corresponding to \(\theta'_i = 0, \ \tilde{\rho}'_i = 0, \ i = 1, 2, \ldots, n\), are described by the four possible combinations of an element from the first pair of constraints (Aa and Ab) with an element from the second pair of constraints (Ba and Bb). Analogously to §6a, it is relatively straightforward to show that (Aa,Ba) corresponds to rectilinear relative equilibria and (Ab,Ba) corresponds to circling relative equilibria.
Now consider \((Ab,Bb)\). It is a straightforward exercise (see the electronic supplementary material, Supplemental Calculations, section (e)) to show that constraints \((Ab)\) and \((Bb)\) both hold if and only if
\[
\alpha_1 + \theta_2 = \alpha_2 + \theta_3 = \ldots = \alpha_{n-1} + \theta_n = \psi, \tag{6.23}
\]
for some \(\psi \in [0, 2\pi)\). But the \(\theta_i\) must also satisfy the closure constraints in (5.8). Substituting (6.23) into the first equation from (5.8), we obtain
\[
1 = R \left( \sum_{j=1}^{n} \left( \pi + \alpha_j - \theta_j \right) \right) = R \left( n \left( \pi - \psi + 2 \sum_{j=1}^{n} \frac{\alpha_j}{n} \right) \right), \tag{6.24}
\]
which holds if and only if \((\pi - \psi + 2 \sum_{j=1}^{n} \frac{\alpha_j}{n})\) is one of the \(n\) roots of unity, i.e.
\[
\pi - \psi + 2 \sum_{j=1}^{n} \frac{\alpha_j}{n} = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n - 1. \tag{6.25}
\]
There are therefore \(n\) possible solutions for \(\psi\), given by
\[
\psi^{(k)} = \left( \frac{n - 2k}{n} \right) \pi + 2 \sum_{j=1}^{n} \frac{\alpha_j}{n}, \quad k = 0, 1, \ldots, n - 1, \tag{6.26}
\]
where the superscript \(k\) is used to explicitly denote the association with a particular root of unity. As will be further explained below, the quantity \((\psi^{(k)} - \pi)/2\) has an appealing geometric property (figure 4), and therefore we denote
\[
\tau_k \triangleq \frac{\psi^{(k)} - \pi}{2} = \sum_{j=1}^{n} \frac{\alpha_j}{n} - \frac{k}{n} \pi. \tag{6.27}
\]
Thus, for \((Ab,Bb)\), using (6.23), the first closure constraint from (5.8) implies that, at equilibrium,
\[
\theta_i = \hat{\theta}_i^{(k)} = \pi - \alpha_{i-1} + 2\tau_k, \tag{6.28}
\]
for every \(i \in \{1, 2, \ldots, n\}\), and for a particular \(k \in \{0, 1, 2, \ldots, n - 1\}\). Substituting (6.28) into constraint (Bb) from (6.22) and simplifying, we have, at equilibrium,

\[
\hat{\rho}_i = \hat{\rho}_i^{(k)} = \frac{\sin(\alpha_i - \tau_k)}{\sin(\alpha_i - \tau_k)}.
\]

(6.29)

Since \(\hat{\rho}_i\) must be strictly positive for every \(i = 1, 2, \ldots, n\), we incur an additional condition which the \(\alpha_i\) parameters must satisfy, namely

\[
\sin(\alpha_i - \tau_k) \sin(\alpha_{i-1} - \tau_k) > 0, \quad i = 1, 2, \ldots, n.
\]

(6.30)

One can show (see the electronic supplementary material, Supplemental Calculations, section (f) for details) that under (6.28) and (6.29), the second closure constraint in (5.8) is also satisfied without further conditions on the \(\alpha_i\) parameters. Furthermore, with some effort, it can be shown (see the electronic supplementary material, Supplemental Calculations, section (g)) that the final combination of constraints from (5.8), namely (Ab,Bb), does not lead to any additional solutions beyond those obtained for (Ab,Bb). We are thus led to the following proposition.

**Proposition 6.3.** Pure shape equilibria exist if and only if the conditions of proposition 6.1 are met or there exists an integer \(k \in \{0, 1, 2, \ldots, n - 1\}\) such that (6.30) holds for \(\tau_k = \left(\sum_{i=1}^{n}(\alpha_i/n)\right) - (k/n)\pi\). If (6.30) holds for a particular value of \(k\), then the corresponding equilibrium values for \(\theta_i\) and \(\hat{\rho}_i\) are given by

\[
\hat{\theta}_i^{(k)} = \pi - \alpha_{i-1} + 2\tau_k \quad \text{and} \quad \hat{\rho}_i^{(k)} = \frac{\sin(\alpha_i - \tau_k)}{\sin(\alpha_1 - \tau_k)}, \quad i = 1, 2, \ldots, n.
\]

(6.31)

**Proof.** The proof follows from the discussion above.

(c) **Example: the symmetric case**

For the symmetric case \(\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \in [0, 2\pi]\), we can apply the results obtained above to fully characterize the existence of relative equilibria and pure shape equilibria in terms of the single parameter \(\alpha\). We can state a proposition.

**Proposition 6.4.** Consider an \(n\)-particle cyclic CB pursuit system evolving on \(M_{\text{CB}(a)}\) according to the shape dynamics (4.13) parametrized by \(\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \in [0, 2\pi]\). Then

(i) No rectilinear relative equilibria exist;

(ii) Circling relative equilibria exist if and only if \(\alpha = \ell \pi / n\), for \(\ell = 1, 2, \ldots, n - 1, n + 1, \ldots, 2n - 1\), in which case the relative equilibria satisfy \(\hat{\theta}_i = (n - \ell)\pi / n\) and \(\hat{\rho}_i / \hat{\rho}_{i-1} = 1, i = 1, 2, \ldots, n\); and

(iii) There exist exactly \(n - 1\) unique pure shape equilibria, each associated with a unique value of \(k \in \{1, 2, \ldots, n - 1\}\). The pure shape equilibrium values satisfy \(\hat{\theta}^{(k)} = \alpha + (n - 2k)\pi / n\) and \(\hat{\rho}_i^{(k)} / \hat{\rho}_{i-1}^{(k)} = 1, i = 1, 2, \ldots, n\).
Proof. First, consider the rectilinear relative equilibrium existence condition (6.5) in proposition 6.1, which becomes $e^{\mu}(\sum_{i=1}^{n} \sigma_i) = 0$ for some $\{\sigma_1, \ldots, \sigma_n\}$ all positive. Since at least one $\sigma_i$ must be non-positive for the sum to be zero, we conclude that the symmetric case admits no rectilinear relative equilibria.

As for circling relative equilibria, note that condition (6.7) in proposition 6.1 requires $\sin(\alpha_{i-1}) \sin(\alpha_i) = \sin^2 \alpha > 0$, which holds provided $\alpha \neq 0, \pi$. The other existence condition for circling relative equilibria, (6.8), requires

$$0 = \sin \left( \sum_{i=1}^{n} \alpha_i \right) = \sin(n\alpha), \quad (6.33)$$

from which we conclude that circling relative equilibria exist if and only if $\alpha = \ell \pi/n$, $\ell = 1, 2, \ldots, n-1, n+1, \ldots, 2n-1$. If such circling equilibria exist, then by (6.9) we have the equilibrium values $\dot{\theta}_i = \pi - \alpha = (n - \ell)\pi/n$ and $\dot{\rho}_i/\dot{\rho}_{i-1} = 1$, i.e. the relative equilibrium formation shape is equilateral.

Next, we turn to pure shape equilibria (other than circling relative equilibria) for the symmetric case. Note that (6.27) becomes

$$\tau_k = \sum_{j=1}^{n} \frac{\alpha_k - k\pi}{n} = \frac{\alpha_k}{n} - \frac{k\pi}{n}. \quad (6.34)$$

Thus, $\sin(\alpha_i - \tau_k) = \sin(\alpha - (\alpha - k\pi/n)) = \sin(k\pi/n)$, $i = 1, 2, \ldots, n$, and by proposition 6.3, pure shape equilibria (other than relative equilibria) exist if and only if there exists $k \in \{0, 1, 2, \ldots, n-1\}$ such that $\sin^2(k\pi/n) > 0$. But then since $\sin(k\pi/n) > 0$, $k \in \{1, 2, \ldots, n-1\}$, we have a family of pure shape equilibria satisfying (6.31), i.e.

$$\hat{\theta}_i^{(k)} = \pi + \alpha - \frac{2\pi k}{n} \quad \text{and} \quad \hat{\rho}_i = 1. \quad (6.35)$$

Thus, the pure shape equilibrium formations are also equilateral.

In figure 5, we display trajectories corresponding to the four unique pure shape equilibria which exist for the particular case $n = 5$, $\alpha_i = \pi/2$. Observe that both outward spirals (top plots) and inward spirals (bottom plots) are possible, with initial conditions dictating system behaviour.

7. Examples

For small numbers of particles, and special choices of parameters $\{\alpha_i\}$, further analysis is possible. For purposes of illustration, we present two such examples here.

(a) Mutual constant bearing pursuit

For $n = 2$, cyclic pursuit becomes mutual pursuit. In this setting, the closure constraint equations (4.12) yield

$$\rho_1 R(\pi + \kappa_1 - \theta_1) + \rho_2 1 = 0, \quad (7.1)$$

or equivalently,

$$\rho_1 \cos(\pi + \kappa_1 - \theta_1) + \rho_2 = 0 \quad \text{and} \quad \rho_1 \sin(\pi + \kappa_1 - \theta_1) = 0. \quad (7.2)$$

Since $\rho_1$ and $\rho_2$ must be positive, the only valid solution for (7.2) is given by

$$\theta_1 = \kappa_1 \quad \text{and} \quad \rho_1 = \rho_2, \quad (7.3)$$

which also implies $\theta_2 = \kappa_2$. Defining $\rho \triangleq \rho_1 = \rho_2$, the mutual CB pursuit shape dynamics are thus

$$\begin{cases} \dot{\kappa}_1 = -\mu_1 \sin(\kappa_1 - \alpha_1), \\ \dot{\kappa}_2 = -\mu_2 \sin(\kappa_2 - \alpha_2) \end{cases} \quad (7.4)$$

and

$$\dot{\rho} = -\cos \kappa_1 - \cos \kappa_2,$$
subject only to the constraint $\rho > 0$. In fact, these shape dynamics can be integrated to obtain closed-form solutions.

**Proposition 7.1.** The mutual CB pursuit shape dynamics (7.4) are integrable and yield the closed-form solutions

\[
\kappa_i(t) = \alpha_i + 2 \arctan(C_i e^{-\mu_i t}), \quad i = 1, 2
\]

and

\[
\rho(t) = \rho(0) - (\cos \alpha_1 + \cos \alpha_2) t + \sum_{i=1}^{2} \frac{1}{\mu_i} \left[ \cos \alpha_i \ln \left( \frac{1 + C_i^2}{1 + C_i^2 e^{-2\mu_i t}} \right) \right. \\
\left. - 2 \sin \alpha_i \arctan \left( \frac{C_i (e^{-\mu_i t} - 1)}{1 + C_i^2 e^{-\mu_i t}} \right) \right],
\]  

for $\kappa_i(0) = \kappa_i^0 \neq \alpha_i + \pi$, $C_i = \tan(\frac{1}{2}(\kappa_i^0 - \alpha_i))$, $\rho(0) = \rho_0 > 0$, and $t < t_c$, where $t_c$ is the minimum time such that $\rho(t_c) = 0$, with $t_c = \infty$ if $\rho(t) > 0$ for all finite $t$.

For $\kappa_i(0) = \alpha_i + \pi$, it holds that $\kappa_i(t) = \alpha_i + \pi$, $i = 1, 2$ and $\rho(t) = \rho_0 + (\cos \alpha_1 + \cos \alpha_2) t$ for $t < t_c$, where $t_c = \rho_0 / (\cos \alpha_1 + \cos \alpha_2)$ for $\cos \alpha_1 + \cos \alpha_2 < 0$ and $t_c = \infty$ otherwise.

**Proof.** See the electronic supplementary material, Supplemental Calculations, section (h). □

**Figure 6** shows some different types of mutual CB pursuit trajectories.

A result corresponding to proposition 7.1 can also be obtained in the three-dimensional setting [23]. Parameters in the closed-form expressions determine whether the centroid of the pair of particles follows a linear, circling, helical or spiral trajectory.
Figure 6. Different types of mutual CB pursuit trajectories: (a) rectilinear, (b) linearly expanding, (c) circling and (d) spiralling. The moving particles are denoted by \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), and the centroid by \( \mathbf{z} \). In (a) and (b), the centroid follows a linear trajectory, in (c) it follows a circling trajectory, and in (d) it follows a spiralling trajectory. The centre of the circle in (c) and of the spiral in (d) is denoted by \( \mathbf{r}_c \), and \( \sigma \) denotes the distance between the centroid and the centre of the circle or spiral. (Online version in colour.)

(b) Three-particle cyclic constant bearing pursuit

Setting \( n = 3 \), from (4.13) and (4.14) we can derive the following system of equations for the shape dynamics on the CB manifold \( M_{\text{CB}}(\alpha_1, \alpha_2, \alpha_3) [6] \):

\[
\begin{align*}
\dot{\theta}_2 &= \frac{1}{\rho_1} (\sin \alpha_1 + \sin \theta_2) - \frac{1}{\rho_2} \left[ \sin \alpha_2 + \frac{\sin(\theta_2 - \alpha_2 - \alpha_3) + (\rho_2/\rho_1) \sin \alpha_3}{P(\rho_1, \rho_2, \theta_2)} \right], \\
\dot{\rho}_1 &= -\cos \alpha_1 - \cos \theta_2 \\
\dot{\rho}_2 &= -\cos \alpha_2 - \frac{-\cos(\theta_2 - \alpha_2 - \alpha_3) + (\rho_2/\rho_1) \cos \alpha_3}{P(\rho_1, \rho_2, \theta_2)},
\end{align*}
\]

where

\[
P(\rho_1, \rho_2, \theta_2) = \sqrt{\left( \frac{\rho_2}{\rho_1} \right)^2 - 2 \left( \frac{\rho_2}{\rho_1} \right) \cos(\theta_2 - \alpha_2) + 1},
\]

and we note that \( P(\rho_1, \rho_2, \theta_2) = 0 \) only for sequential colocation.

To place (7.6) into context, recall that the original three-particle system lives in a nine-dimensional state space (i.e. three dimensions for each planar particle). Applying the CB pursuit law and descending to shape space yields (4.10), which for \( n = 3 \), still consists of nine equations, but the cycle closure conditions (4.12) (which introduce three constraints in shape space), bring the total number of degrees of freedom down to 6. Next, restricting to the CB manifold (motivated by proposition 4.1, which informs us that the CB manifold is both invariant and attractive),
we obtain (4.13), the shape space dynamics restricted to the CB manifold. The dynamics (4.13) are six-dimensional (for \( n = 3 \)), but the cycle closure constraints (4.14) remain, cutting the total number of degrees of freedom for the closed-loop system on the CB manifold to 3. In passing to (7.6), we have incorporated the cycle closure constraints, yielding just three equations.

Following the procedure of §5, we next introduce the change of variables \( \tilde{\lambda} = \ln(\rho_2/\rho_1) \) which under the condition \( \alpha_1 = \alpha_2 = \alpha, \alpha_3 = \alpha + \pi \), yields [6]:

\[
\begin{align*}
\theta_2' &= P[e^{\tilde{\lambda}}(\sin \alpha + \sin \theta_2) - \sin(\theta_2 - 2\alpha + \pi) + e^{\tilde{\lambda}} \sin \alpha] - \sin(\theta_2 - 2\alpha + \pi) + e^{\tilde{\lambda}} \sin \alpha \\
\tilde{\lambda}' &= P[e^{\tilde{\lambda}}(\cos \alpha + \cos \theta_2) - \cos(\theta_2 - 2\alpha + \pi) + e^{\tilde{\lambda}} \cos \alpha],
\end{align*}
\]

(7.8)

where \( P(\tilde{\lambda}, \theta_2) = \sqrt{e^{2\tilde{\lambda}} - 2e^{\tilde{\lambda}} \cos(\theta_2 - \alpha) + 1} \). Thus, from (7.8), we see that it is indeed possible to write down explicitly the two-dimensional pure shape equations associated with three-particle cyclic CB pursuit, restricted to the CB manifold. Phase plane methods can then be employed to study the stability of solutions for (7.8). For example, with \( \alpha = \pi/2 \), it is demonstrated in [6] that (7.8) exhibits periodic solutions, corresponding to precessing solutions for the planar particles, as illustrated in figure 7.

8. Conclusion

Through an in-depth study of cyclic CB pursuit, we have illustrated certain conceptual and analytic techniques for exploring collective motion. Using curves and moving frames to describe
particle motion subject to steering control leads to a geometric perspective that captures essential physical considerations. Strategies are identified with constraint manifolds, and suitable contrast functions measure the ‘distance’ from such constraint manifolds. An attention graph specifies which neighbours the agents respond to via their steering actions, designed to drive the closed-loop system towards an invariant manifold, thus realizing a collective strategy. Each cycle in the attention graph introduces a closure constraint on the dynamics, and although cyclic graphs can result in circling motion, it is important to note that these concepts are distinct: cyclic graphs can also be associated with rectilinear motion (or more complicated motions, as we show).

There are various directions in which these ideas may be extended, while retaining key aspects of the conceptual framework described above. Different agents may employ different steering laws and some may be virtual agents, as in work on control laws for following the boundary of an obstacle [24]. Stabilizing collective motion through local interactions, and particularly through steering laws based on perception of relative quantities, is important in both nature and engineering. While one might emphasize stabilizing particular spatio-temporal patterns, there are also situations where the imperative is to avoid certain patterns—for example, the ant mill, where circling behaviour becomes a death march. Strategies, attention graphs and (sensor-driven) steering laws give rise to templates of collective motion. Understanding these in a broader context in nature—how they arise, how they are reinforced through natural selection, social communication and environmental interaction—and how such templates may be employed in artificial settings, is an appealing agenda for the future.

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References


