We implement the unified transform method to the
initial-boundary value (IBV) problem of the Sasa–
Satsuma equation on the half line. In addition to
presenting the basic Riemann–Hilbert formalism,
which linearizes this IBV problem, we also analyse the
associated general Dirichlet to Neumann map using
the so-called global relation.

1. Introduction

Several of the most important PDEs in mathematics and
physics are integrable (in this paper, integrable means the
PDEs admit Lax pair). Integrable PDEs can be analysed
by means of the inverse scattering transform (IST)
formalism. Until the 1990s, the IST methodology was
pursued almost entirely for pure initial value problems.
However, in many laboratory and field situations, the
wave motion is initiated by what corresponds to the
imposition of boundary conditions rather than initial
conditions. This naturally leads to the formulation of
initial-boundary value (IBV) problems instead of a pure
initial value problem.

In 1997, Fokas announced a new unified approach for
the analysis of IBV problems for linear and nonlinear
integrable PDEs [1,2] (see also [3]). The Fokas method
provides a generalization of the IST formalism from
initial value to IBV problems, and over the last 15
years, this method has been used to analyse boundary
value problems for several of the most important
integrable equations possessing $2 \times 2$ Lax pairs, such
as the Korteweg–de Vries, the nonlinear Schrödinger
(NLS), the sine-Gordon and the stationary axisymmetric
Einstein equations, e.g. [4–11]. Just like the IST on the
line, the unified method yields an expression for the
solution of an IBV problem in terms of the solution of a
Riemann–Hilbert problem. In particular, the asymptotic behaviour of the solution can be analysed in an effective way by using this Riemann–Hilbert problem and by employing the nonlinear version of the steepest descent method introduced by Deift & Zhou [12]. However, for the formulation of this Riemann–Hilbert problem, both the $x$- and the $t$-parts of the Lax pair play an important role. Actually, it is the $t$-part which determines where, in the complex $k$-plane, the jumps occur. It differs significantly from the IST formalism because it just needs the $x$-part of the Lax pair to formulate a Riemann–Hilbert problem, the $t$-part will just be used to determine the time evolution of scattering data.

It is well known that the NLS equation

$$i q_T + \frac{1}{2} q_{XX} + |q|^2 q = 0 \quad (1.1)$$

describes slowly varying wave envelopes in dispersive media and arises in various physical systems including water waves, plasma physics, solid-state physics and nonlinear optics. One of the most successful related applications is the description of optical solitons in fibres. However, several phenomena have been observed by experimental which cannot be explained by equation (1.1). In order to understand such phenomena, Kodama and Hasegawa proposed the higher order NLS equation

$$i q_T + \frac{1}{2} q_{XX} + |q|^2 q + i \varepsilon (\beta_1 q_{xxx} + \beta_2 |q|^2 q_X + \beta_3 q(|q|^2)_X) = 0. \quad (1.2)$$

In general, equation (1.2) is not completely integrable, unless certain restrictions are imposed on the real parameters $\beta_1$, $\beta_2$ and $\beta_3$. In particular, the following four cases, including the NLS equation, are known to be solvable:

— the derivative NLS equation-type I ($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1$),
— the derivative NLS equation-type II ($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0$),
— the Hirota equation ($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 0$),
— the Sasa–Satsuma (SS) equation ($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$),

$$i q_T + \frac{1}{2} q_{XX} + |q|^2 q + i \varepsilon (q_{XXX} + 6|q|^2 q_X + 3q(|q|^2)_X) = 0. \quad (1.3)$$

Recently, Lenells [13] implemented the Fokas method to IBV problems for integrable evolution equations with Lax pairs involving $3 \times 3$ matrices. He also used this method to analyse the Degasperis–Procesi eqn in [14]. In this paper, following this approach, we analyse the IBV problem of the SS equation on the half-line by using this method. The IST formalism for the initial value problem of this equation has been obtained in [15].

According to [15], we introduce variable transformations,

$$u(x, t) = q(X, T) \exp \left\{ -\frac{i}{6\varepsilon} \left( X - \frac{T}{18\varepsilon} \right) \right\}, \quad (1.4a)$$

$$t = T \quad (1.4b)$$

and

$$x = X - \frac{T}{12\varepsilon}. \quad (1.4c)$$

Then equation (1.2) reduces to a complex modified KdV-type equation

$$u_t + \varepsilon (u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x) = 0. \quad (1.5)$$

(a) Organization of the paper

In §2, we perform the spectral analysis of the associated Lax pair. We formulate the main Riemann–Hilbert problem in §3 and we analyse the associated general Dirichlet to Neumann map using the so-called global relation in §4.
2. Spectral analysis

The Lax pair of equation (1.5) is [15]

\[ \Psi_x = U\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \] (2.1a)

and

\[ \Psi_t = V\Psi, \] (2.1b)

where

\[ U = -ik\Lambda + V_1 \] (2.2)

and

\[ V = -4i\varepsilon k^3 \Lambda + V_2, \] (2.3)

where

\[ \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \bar{u} \\ -\bar{u} & -u & 0 \end{pmatrix} \text{ and } \quad V_2 = k^2 V_2^{(2)} + kV_2^{(1)} + V_2^{(0)}, \] (2.4)

with

\[ V_2^{(2)} = 4\varepsilon \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \bar{u} \\ -\bar{u} & -u & 0 \end{pmatrix}, \]
\[ V_2^{(1)} = 2i\varepsilon \begin{pmatrix} |u|^2 & u^2 & ux \\ \bar{u}^2 & |\bar{u}|^2 & \bar{u}x \\ \bar{u}x & ux & -2|u|^2 \end{pmatrix} \]

and

\[ V_2^{(0)} = -4|u|^2\varepsilon \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \bar{u} \\ -\bar{u} & -u & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 & 0 & u_{xx} \\ 0 & 0 & \bar{u}_{xx} \\ -\bar{u}_{xx} & -u_{xx} & 0 \end{pmatrix} + \varepsilon(u\bar{u}_x - u_x\bar{u}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (2.5)

In the following, we let \( \varepsilon = 1 \) for the convenience of the analysis.

(a) The closed one-form

Suppose that \( u(x,t) \) is a sufficiently smooth function of \((x,t)\) in the half-line domain \( \Omega = \{0 < x < \infty, 0 < t < T\} \) which decay as \( x \to \infty \). Introducing a new eigenfunction \( \mu(x,t,k) \) by

\[ \Psi = \mu e^{-ik\Lambda x - 4i\varepsilon k^3 t} \] (2.6)

we then find the Lax pair equations

\[ \mu_x + [ik\Lambda, \mu] = V_1 \mu \] \quad (2.7)

and

\[ \mu_t + [4ik^3 \Lambda, \mu] = V_2 \mu. \]

Equations (2.7) can be written in differential form as

\[ d(e^{(ikx+4ik^3t)\Lambda} \mu) = W, \] (2.8)
We have the following inequalities on the contours:
The first, second and third column of matrix equation (2.10) involves the exponentials
\[ L_i \text{ where } \lambda_i \in k \text{ belongs to } k \]
We define three eigenfunctions \( \mu_j \) of (2.7) by the Volterra integral equations
\[ \mu_j(x, t, k) = 1 + \int_{\gamma_j} e^{-ikx+4ik^3 t} \hat{A}(V_1 \, dx + V_2 \, dt) \mu, \quad j = 1, 2, 3, \] (2.10)
where \( W_j \) is given by (2.9) with \( \mu \) replaced with \( \mu_j \), and the contours \( \gamma_j^3 \) are shown in figure 1. The first, second and third column of matrix equation (2.10) involves the exponentials
\[
\begin{align*}
[\mu_j]_1 & : e^{ik(x-x')+4ik^3(t-t')}, \\
[\mu_j]_2 & : e^{ik(x-x')+4ik^3(t-t')}, \\
[\mu_j]_3 & : e^{-ik(x-x')-8ik^3(t-t')}, e^{-2ik(x-x')-8ik^3(t-t')},
\end{align*}
\] (2.11)
and
\[
\begin{align*}
\gamma_1 & : x - x' \geq 0, t - t' \leq 0, \\
\gamma_2 & : x - x' \geq 0, t - t' \geq 0, \\
\gamma_3 & : x - x' \leq 0.
\end{align*}
\] (2.12)
So, these inequalities imply that the functions \( \mu_j \) are bounded and analytical for \( k \in \mathbb{C} \) such that \( k \) belongs to\[
\begin{align*}
\mu_1 : (D_2, D_2, D_3), \\
\mu_2 : (D_1, D_1, D_4), \\
\mu_3 : (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2),
\end{align*}
\] (2.13)
where \( \{D_n\}^4 \) denote four open, pairwise disjointed subsets of the Riemann k-sphere shown in figure 2. And the sets \( \{D_n\}^4 \) have the following properties:
\[
\begin{align*}
D_1 & : \{ k \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 > \text{Rel}_3, \text{Rez}_1 = \text{Rez}_2 > \text{Rez}_3 \}, \\
D_2 & : \{ k \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 > \text{Rel}_3, \text{Rez}_1 = \text{Rez}_2 < \text{Rez}_3 \}, \\
D_3 & : \{ k \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 < \text{Rel}_3, \text{Rez}_1 = \text{Rez}_2 > \text{Rez}_3 \}, \\
D_4 & : \{ k \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 < \text{Rel}_3, \text{Rez}_1 = \text{Rez}_2 < \text{Rez}_3 \},
\end{align*}
\]
where \( l_i(k) \) and \( z_i(k) \) are the diagonal entries of matrices \(-ikA\) and \(-4ik^3 A\), respectively.
Figure 2. The sets $D_n, n = 1, \ldots, 4$, which decompose the complex $k$-plane.

In fact, for $x = 0$, $\mu_1(0, t, k)$ has enlarged the domain of boundedness: $(D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3)$ and $\mu_2(0, t, k)$ have enlarged the domain of boundedness: $(D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4)$.

(c) The $M_n$’s

For each $n = 1, \ldots, 4$, a solution $M_n(x, t, k)$ of (2.7) is defined by the following system of integral equations:

$$(M_n)_{ij}(x, t, k) = \delta_{ij} + \int_{\gamma_{ij}^n} (e^{-ikx - 4ik^3t})^{\Lambda} W_n(x', t', k)_{ij}, \quad k \in D_n, \; i, j = 1, 2, 3,$$

(2.14)

where $W_n$ is given by (2.9) with $\mu$ replaced with $M_n$, and the contours $\gamma_{ij}^n, n = 1, \ldots, 4, i, j = 1, 2, 3$ are defined by

$$\gamma_{ij}^n = \begin{cases} \gamma_1 & \text{if } \text{Re}l_i(k) < \text{Re}l_j(k) \text{ and } \text{Re}z_i(k) \geq \text{Re}z_j(k), \\ \gamma_2 & \text{if } \text{Re}l_i(k) < \text{Re}l_j(k) \text{ and } \text{Re}z_i(k) < \text{Re}z_j(k), \\ \gamma_3 & \text{if } \text{Re}l_i(k) \geq \text{Re}l_j(k). \end{cases} \quad (2.15)$$

According to the definition of the $\gamma^n$, we find that

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_2 & \gamma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_1 & \gamma_3 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix} \quad \text{and} \quad \gamma^4 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}. \quad (2.16)$$

The following proposition ascertains that the $M_n$’s defined in this way have the properties required for the formulation of a Riemann–Hilbert problem.

**Proposition 2.1.** For each $n = 1, \ldots, 4$, the function $M_n(x, t, k)$ is well defined by equation (2.14) for $k \in \bar{D}_n$ and $(x, t) \in \Omega$. For any fixed point $(x, t)$, $M_n$ is bounded and analytical as a function of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. Moreover, $M_n$ admits a bounded and continuous extension to $\bar{D}_n$ and

$$M_n(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad k \in D_n. \quad (2.17)$$
Proof. The boundedness and analyticity properties are established in appendix B in [13]. Substituting the expansion

\[ M = M_0 + \frac{M^{(1)}}{k} + \frac{M^{(2)}}{k^2} + \cdots, \quad k \to \infty. \]

into the Lax pair (2.7) and comparing the terms of the same order of \( k \) yields equation (2.17). ■

Remark 2.2. We have defined two sets of eigenfunctions: \( \{\mu_j\}_{j=1}^3 \) and \( \{M_n\}_{n=1}^4 \). The unified approach of Fokas [1] for Lax pairs involving \( 2 \times 2 \) matrices also implicitly uses two types of eigenfunctions: the \( \mu_j \)'s are used for the spectral analysis, whereas the Riemann–Hilbert problem is formulated in terms of another set of eigenfunctions; our \( M_n \)'s are the analogues of this latter set of eigenfunctions, see §2d.

(d) The jump matrices

We define spectral functions \( S_n(k), n = 1, \ldots, 4, \) and

\[ S_n(k) = M_n(0, 0, k), \quad k \in D_n, \quad n = 1, \ldots, 4. \] (2.18)

Let \( M \) denote the sectionally analytical function on the Riemann \( k \)-sphere which equals \( M_n \) for \( k \in D_n \). Then, \( M \) satisfies the jump conditions

\[ M_n = M_m J_{m,n} \quad \text{and} \quad k \in \bar{D}_n \cap \bar{D}_m, \quad n, m = 1, \ldots, 4, \quad n \neq m, \] (2.19)

where the jump matrices \( J_{m,n}(x, t, k) \) are defined by

\[ J_{m,n} = e^{-ikx - 4ik^3t} \hat{\Lambda} (S_m^{-1} S_n). \] (2.20)

Remark 2.3. As the integral equations (2.14) that define \( M_n(0, 0, k) \) involve only integration along the initial half-line \( \{0 < x < \infty, t = 0\} \) and along the boundary \( \{x = 0, 0 < t < T\} \), the \( S_n \)'s (and hence also the \( J_{m,n} \)'s) can be computed from the initial and boundary data alone. Thus, relation (2.19) provides the jump condition for a Riemann–Hilbert problem, which, in the absence of singularities, can be used to reconstruct the solution \( u(x, t) \) from the initial and boundary data. However, if the \( M_n \)'s have pole singularities at some points \( \{k_j\}, \quad k_j \in \mathbb{C} \), the Riemann–Hilbert problem needs to include the residue conditions at these points. For the purpose of determining the correct residue conditions (and also for the purposes of analysing the nonlinearizable boundary conditions in §4), it is convenient to introduce three eigenfunctions \( \{\mu_j(x, t, k)\}_{j=1}^3 \) in addition to the \( M_n \)'s.

(e) The adjugated eigenfunctions

We will also need the analyticity and boundedness properties of the minors of the matrices \( \{\mu_j(x, t, k)\}_{j=1}^3 \). We recall that the cofactor matrix \( X^A \) of a \( 3 \times 3 \) matrix \( X \) is defined by

\[
X^A = \begin{pmatrix}
m_{11}(X) & -m_{12}(X) & m_{13}(X) \\
-m_{21}(X) & m_{22}(X) & -m_{23}(X) \\
m_{31}(X) & -m_{32}(X) & m_{33}(X)
\end{pmatrix},
\]

where \( m_{ij}(X) \) denote the \((ij)th minor of X.

It follows from (2.7) that the adjugated eigenfunction \( \mu^A \) satisfies the Lax pair

\[
\begin{align*}
\mu^A_x - [ik \hat{\Lambda}, \mu^A] &= -V^T_1 \mu^A, \\
\mu^A_t - [4ik^3 \hat{\Lambda}, \mu^A] &= -V^T_2 \mu^A,
\end{align*}
\] (2.21)
where $V^T$ denotes the transform of a matrix $V$. Thus, the eigenfunctions $|\mu_j^A|_1^3$ are solutions of the integral equations

$$
|\mu_j^A(x, t, k) = | \int_{\gamma} e^{ik(x-x')} + 4ik^3(t-r) \hat{A} (V^T \, dx + V^T) \mu^A, j = 1, 2, 3.
$$

(2.22)

Then we can get the following analyticity and boundedness properties:

$$
\begin{align*}
\mu_1^A & : (D_3, D_3, D_2), \\
\mu_2^A & : (D_4, D_4, D_1) \\
\mu_3^A & : (D_1 \cup D_2, D_1 \cup D_2, D_3 \cup D_4).
\end{align*}
$$

(2.23)

In fact, for $x = 0$, $\mu_1^A(0, t, k)$ has enlarged the domain of boundedness: $(D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4)$ and $\mu_2^A(0, t, k)$ have enlarged the domain of boundedness: $(D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3)$.

(f) The $J_{m,n}$’s computation

Let us define the $3 \times 3$ matrix value spectral functions $s(k)$ and $S(k)$ by

$$
\begin{align*}
\mu_3(x, t, k) & = \mu_2(x, t, k) e^{-ikx - 4ik^3(t)]} \hat{A} s(k) \\
\mu_1(x, t, k) & = \mu_2(x, t, k) e^{-ikx - 4ik^3(t)]} \hat{A} S(k).
\end{align*}
$$

(2.24)

Thus,

$$
\begin{align*}
s(k) & = \mu_3(0, 0, k) \quad \text{and} \quad S(k) = \mu_1(0, 0, k).
\end{align*}
$$

(2.25)

We deduce from the properties of $\mu_j$ and $\mu_j^A$ that $s(k)$ and $S(k)$ have the following boundedness properties:

$$
\begin{align*}
s(k) & : (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2), \\
S(k) & : (D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3), \\
s^A(k) & : (D_1 \cup D_2, D_1 \cup D_2, D_3 \cup D_4), \\
S^A(k) & : (D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4).
\end{align*}
$$

and

Moreover,

$$
\begin{align*}
M_{m,n}(x, t, k) & = \mu_2(x, t, k) e^{-ikx - 4ik^3(t)]} \hat{A} S_m(n)(k), \quad k \in D_n.
\end{align*}
$$

(2.26)

Proposition 2.4. The $S_n$ can be expressed in terms of the entries of $s(k)$ and $S(k)$ as follows:

$$
\begin{align*}
S_1 & = \left( \begin{array}{ccc}
m_{22}(s) & m_{21}(s) & s_{13} \\
m_{12}(s) & s_{33} & s_{23} \\
0 & 0 & s_{33}
\end{array} \right), \\
S_2 & = \left( \begin{array}{ccc}
m_{22}(s)m_{33}(S) - m_{32}(s)m_{23}(S) & m_{21}(s)m_{33}(S) - m_{31}(s)m_{23}(S) & s_{13} \\
m_{12}(s)m_{33}(S) - m_{32}(s)m_{13}(S) & m_{11}(s)m_{33}(S) - m_{31}(s)m_{13}(S) & s_{23} \\
m_{12}(s)m_{23}(S) - m_{22}(s)m_{13}(S) & m_{11}(s)m_{23}(S) - m_{21}(s)m_{13}(S) & s_{33}
\end{array} \right).
\end{align*}
$$

(2.27)

and

\[
S_3 = \left( \begin{array}{ccc}
\frac{S_{13}}{(ST^A)_{33}} & \frac{S_{23}}{(ST^A)_{33}} & \frac{S_{33}}{(ST^A)_{33}} \\
\frac{S_{12}}{(ST^A)_{33}} & \frac{S_{22}}{(ST^A)_{33}} & \frac{S_{32}}{(ST^A)_{33}} \\
\frac{S_{11}}{(ST^A)_{33}} & \frac{S_{21}}{(ST^A)_{33}} & \frac{S_{31}}{(ST^A)_{33}}
\end{array} \right), \quad S_4 = \left( \begin{array}{ccc}
\frac{s_{11}}{m_{33}} & \frac{s_{12}}{m_{33}} & 0 \\
\frac{s_{21}}{m_{33}} & \frac{s_{22}}{m_{33}} & 0 \\
\frac{s_{31}}{m_{33}} & \frac{s_{32}}{m_{33}} & 1
\end{array} \right). \tag{2.27b}
\]

**Proof.** Let \( \gamma_3^{X_0} \) denote the contour \( (X_0, 0) \to (x, t) \) in the \((x, t)\)-plane, here \( X_0 > 0 \) is a constant. We introduce \( \mu_3(x, t; k; X_0) \) as the solution of (2.10) with \( j = 3 \) and with the contour \( \gamma_3 \) replaced by \( \gamma_3^{X_0} \). Similarly, we define \( M_n(x, t; k; X_0) \) as the solution of (2.14) with \( \gamma_3 \) replaced by \( \gamma_3^{X_0} \). We will first derive expression for \( S_n(k; X_0) = M_n(0, 0, k; X_0) \) in terms of \( S(k) \) and \( s(k; X_0) = \mu_3(0, 0, k; X_0) \). Then (2.27) will follow by taking the limit \( X_0 \to \infty \).

First, we have the following relations:

\[
M_n(x, t; k; X_0) = \mu_1(x, t, k) e^{(-i k x - 4 i k^3 t) \hat{A}} R_n(k; X_0),
\]

\[
M_n(x, t; k; X_0) = \mu_2(x, t, k) e^{(-i k x - 4 i k^3 t) \hat{A}} S_n(k; X_0),
\]

and

\[
M_n(x, t; k; X_0) = \mu_3(x, t, k) e^{(-i k x - 4 i k^3 t) \hat{A}} T_n(k; X_0).
\]

Then we get \( R_n(k; X_0) \) and \( T_n(k; X_0) \) are defined as follows:

\[
R_n(k; X_0) = e^{4 i k^3 T \hat{A}} M_n(0, T, k; X_0) \tag{2.29a}
\]

and

\[
T_n(k; X_0) = e^{i k x \hat{A}} M_n(X_0, 0, k; X_0). \tag{2.29b}
\]

Relations (2.28) imply that

\[
s(k; X_0) = S_n(k; X_0) T_n^{-1}(k; X_0) \quad \text{and} \quad S(k) = S_n(k; X_0) R_n^{-1}(k; X_0). \tag{2.30}
\]

These equations constitute a matrix factorization problem which, given \( \{s, S\} \) can be solved for the \( \{R_n, S_n, T_n\} \). Indeed, integral equations (2.14) together with the definitions of \( \{R_n, S_n, T_n\} \) imply that

\[
(R_n(k; X_0))_{ij} = 0 \quad \text{if} \quad \gamma''_{ij} = \gamma_1,
\]

\[
(S_n(k; X_0))_{ij} = 0 \quad \text{if} \quad \gamma''_{ij} = \gamma_2, \tag{2.31}
\]

and

\[
(T_n(k; X_0))_{ij} = 0 \quad \text{if} \quad \gamma''_{ij} = \gamma_3.
\]

It follows that (2.30) are 18 scalar equations for 18 unknowns. By computing the explicit solution of this algebraic system, we find that \( \{S_n(k; X_0)\}_{ij}^{4} \) are given by the equation obtained from (2.27) by replacing \( \{S_n(k, s(k))\} \) with \( \{S_n(k; X_0), s(k; X_0)\} \). Taking \( X_0 \to \infty \) in this equation, we arrive at (2.27). \( \blacksquare \)

(g) The global relation

The spectral functions \( S(k) \) and \( s(k) \) are not independent but satisfy an important relation. Indeed, it follows from (2.24) that

\[
\mu_1(x, t, k) e^{(-i k x - 4 i k^3 t) \hat{A}} S^{-1}(k)s(k) = \mu_3(x, t, k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). \tag{2.32}
\]

As \( \mu_1(0, T, k) = 1 \), evaluation at \((0, T)\) yields the following global relation:

\[
S^{-1}(k) s(k) = e^{4 i k^3 T \hat{A}} c(T, k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2), \tag{2.33}
\]

where \( c(T, k) = \mu_3(0, T, k) \).
(h) The residue conditions

As \( \mu_2 \) is an entire function, it follows from (2.26) that \( M \) can only have singularities at the points where the \( S_i \)'s have singularities. We infer from the explicit formulae (2.27) that the possible singularities of \( M \) are as follows:

- [\( M \)]_1 could have poles in \( D_1 \cup D_2 \) at the zeros of \( s_{33}(k) \),
- [\( M \)]_1 could have poles in \( D_2 \) at the zeros of \( (s^T S^A)_{33} \),
- [\( M \)]_2 could have poles in \( D_1 \cup D_2 \) at the zeros of \( s_{33}(k) \),
- [\( M \)]_2 could have poles in \( D_2 \) at the zeros of \( (s^T S^A)_{33} \),
- [\( M \)]_3 could have poles in \( D_3 \) at the zeros of \( (S^T S^A)_{33} \) and
- [\( M \)]_3 could have poles in \( D_3 \cup D_4 \) at the zeros of \( m_{33}(s)(k) \).

We denote the above possible zeros by \([k_j]_1^N\) and assume that they satisfy the following assumption.

**Assumption 2.5.** We assume that

- \( s_{33}(k) \) has \( n_0 \) possible simple zeros in \( D_1 \) denoted by \([k_j]_{1}^{n_0}\),
- \( s_{33}(k) \) has \( n_1 - n_0 \) possible simple zeros in \( D_2 \) denoted by \([k_j]_{n_0+1}^{n_1}\),
- \( (s^T S^A)_{33}(k) \) has \( n_2 - n_1 \) possible simple zeros in \( D_2 \) denoted by \([k_j]_{n_1+1}^{n_2}\),
- \( (S^T S^A)_{33}(k) \) has \( n_3 - n_2 \) possible simple zeros in \( D_3 \) denoted by \([k_j]_{n_2+1}^{n_3}\),
- \( m_{33}(s)(k) \) has \( n_4 - n_3 \) possible simple zeros in \( D_3 \) denoted by \([k_j]_{n_3+1}^{n_4}\),
- \( m_{33}(s)(k) \) has \( n_5 - n_4 \) possible simple zeros in \( D_3 \) denoted by \([k_j]_{n_4+1}^{n_5}\), and
- \( m_{33}(s)(k) \) has \( N - n_5 \) possible simple zeros in \( D_4 \) denoted by \([k_j]_{n_5+1}^{N}\).

and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the \( D_n \)'s.

We determine the residue conditions at these zeros in the following:

**Proposition 2.6.** Let \([M_i]_1^N\) be the eigenfunctions defined by (2.14) and assume that the set \([k_j]_1^N\) of singularities are as the above assumption. Then the following residue conditions hold:

- \( \text{Res}_{k_j}[M_1] = \frac{m_{12}(s)(k_j)}{s_{33}(k_j)s_{23}(k_j)} e^{\theta_2(k_j)}[M(k_j)]_3, \quad 1 \leq j \leq n_0, k_j \in D_1 \) (2.34a)
- \( \text{Res}_{k_j}[M_2] = \frac{m_{12}(s)(k_j)}{s_{33}(k_j)s_{13}(k_j)} e^{\theta_2(k_j)}[M(k_j)]_3, \quad n_0 + 1 \leq j \leq n_1, k_j \in D_1 \) (2.34b)
- \( \text{Res}_{k_j}[M_1] = \frac{m_{21}(s)(k_j)m_{33}(S)(k_j) - m_{31}(s)(k_j)m_{23}(S)(k_j)}{(S^T S^A)_{33}(k_j)s_{23}(k_j)} e^{\theta_2(k_j)}[M(k_j)]_3 \quad n_1 + 1 \leq j \leq n_2, k_j \in D_2 \) (2.34c)
- \( \text{Res}_{k_j}[M_2] = \frac{m_{21}(s)(k_j)m_{33}(S)(k_j) - m_{31}(s)(k_j)m_{23}(S)(k_j)}{(S^T S^A)_{33}(k_j)s_{13}(k_j)} e^{\theta_2(k_j)}[M(k_j)]_3 \quad n_1 + 1 \leq j \leq n_2, k_j \in D_2 \) (2.34d)
- \( \text{Res}_{k_j}[M_3] = \frac{S_{13}(k_j)s_{32}(k_j) - S_{33}(k_j)s_{12}(k_j)}{(S^T S^A)_{33}(k_j)m_{23}(s)(k_j)} e^{\theta_2(k_j)}[M(k_j)]_1 + \frac{S_{33}(k_j)s_{11}(k_j) - S_{13}(k_j)s_{31}(k_j)}{(S^T S^A)_{33}(k_j)m_{23}(s)(k_j)} e^{\theta_2(k_j)}[M(k_j)]_2, \quad n_2 + 1 \leq j \leq n_3, k_j \in D_3 \) (2.34e)
- \( \text{Res}_{k_j}[M_3] = \frac{s_{12}(k_j)}{m_{33}(s)(k_j)m_{23}(s)(k_j)} e^{\theta_2(k_j)}[M(k_j)]_1 - \frac{s_{11}(k_j)}{m_{33}(s)(k_j)m_{23}(s)(k_j)} e^{\theta_2(k_j)}[M(k_j)]_2, \quad n_4 + 1 \leq j \leq N, k_j \in D_4 \) (2.34f)
where \( \dot{f} = \frac{df}{dk} \), and \( \theta_{ij} \) is defined by
\[
\theta_{ij}(x, t, k) = (i - j)x + (z_i - z_j)t, \quad i, j = 1, 2, 3,
\]
which implies that
\[
\theta_{ij} = 0, i, j = 1, 2; \quad \theta_{13} = \theta_{23} = -\theta_{32} = -2ikx - 8ik^3 t.
\]

Proof. We will prove (2.34a), (2.34c), (2.34e) and (2.34f), the other conditions follow by similar arguments. Equation (2.26) implies the relation
\[
M_1 = \mu_2 e^{(-ikx - 4ik^3 t)} S_1,
\]
\[
M_2 = \mu_2 e^{(-ikx - 4ik^3 t)} S_2,
\]
\[
M_3 = \mu_2 e^{(-ikx - 4ik^3 t)} S_3,
\]
and
\[
M_4 = \mu_2 e^{(-ikx - 4ik^3 t)} S_4,
\]
In view of the expressions for \( S_1 \) and \( S_2 \) given in (2.27), the three columns of (2.36a) read
\[
[M_1]_1 = [\mu_2]_1 \frac{m_22(s)}{s_{33}} + [\mu_2]_2 e^{\theta_{i1}} \frac{m_{12}(s)}{s_{33}},
\]
\[
[M_1]_2 = [\mu_2]_1 e^{\theta_{i2}} \frac{m_{21}(s)}{s_{33}} + [\mu_2]_2 \frac{m_{11}(s)}{s_{33}},
\]
and
\[
[M_1]_3 = [\mu_2]_1 e^{\theta_{i3}} s_{13} + [\mu_2]_2 e^{\theta_{i2}} s_{23} + [\mu_2]_3 s_{33},
\]
while the three columns of (2.36b) read
\[
[M_2]_1 = [\mu_2]_1 \frac{m_{22}(s) m_{33}(S) - m_{32}(s) m_{23}(S)}{(s^{T} S A)_{33}} + [\mu_2]_2 \frac{m_{12}(s) m_{33}(S) - m_{32}(s) m_{13}(S)}{(s^{T} S A)_{33}} e^{\theta_{i1}}
\]
\[+ [\mu_2]_3 \frac{m_{12}(s) m_{23}(S) - m_{22}(s) m_{13}(S)}{(s^{T} S A)_{33}} e^{\theta_{i1}},
\]
\[
[M_2]_2 = [\mu_2]_1 \frac{m_{21}(s) m_{33}(S) - m_{31}(s) m_{23}(S)}{(s^{T} S A)_{33}} e^{\theta_{i2}} + [\mu_2]_2 \frac{m_{11}(s) m_{33}(S) - m_{31}(s) m_{13}(S)}{(s^{T} S A)_{33}}
\]
\[+ [\mu_2]_3 \frac{m_{11}(s) m_{23}(S) - m_{21}(s) m_{13}(S)}{(s^{T} S A)_{33}} e^{\theta_{i2}},
\]
and
\[
[M_2]_3 = [\mu_2]_1 s_{13} e^{\theta_{i3}} + [\mu_2]_2 s_{23} e^{\theta_{i2}} + [\mu_2]_3 s_{33},
\]
the three columns of (2.36c) read
\[
[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{\theta_{i1}} + [\mu_2]_3 s_{31} e^{\theta_{i1}},
\]
\[
[M_3]_2 = [\mu_2]_1 s_{12} e^{\theta_{i2}} + [\mu_2]_2 s_{22} e^{\theta_{i2}} + [\mu_2]_3 s_{32} e^{\theta_{i2}},
\]
and
\[
[M_3]_3 = [\mu_2]_1 \frac{s_{13}}{(s^{T} S A)_{33}} e^{\theta_{i3}} + [\mu_2]_2 \frac{s_{23}}{(s^{T} S A)_{33}} e^{\theta_{i2}} + [\mu_2]_3 \frac{s_{33}}{(s^{T} S A)_{33}},
\]
and the three columns of (2.36d) read
\[
[M_4]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{\theta_{i1}} + [\mu_2]_3 s_{31} e^{\theta_{i1}},
\]
\[
[M_4]_2 = [\mu_2]_1 s_{12} e^{\theta_{i2}} + [\mu_2]_2 s_{22} e^{\theta_{i2}} + [\mu_2]_3 s_{32} e^{\theta_{i2}},
\]
and
\[
[M_4]_3 = [\mu_2]_3 \frac{1}{m_{33}(s)},
\]
We first suppose that \( k_i \in D_1 \) is a simple zero of \( s_{33}(k) \). Solving (2.37c) for \( [\mu_2]_2 \) and substituting the result into (2.37a), we find
\[
[M_1]_1 = \frac{m_{12}(s)}{s_{33} s_{23}} e^{\theta_{i1}} [M_1]_3 + \frac{m_{32}(s)}{s_{23}} [\mu_2]_2 - \frac{m_{12}(s)}{s_{23}} e^{\theta_{i1}} [\mu_2]_3.
\]
Taking the residue of this equation at \( k_j \), we find condition (2.34a) in the case when \( k_j \in D_1 \).

Similarly, solving (2.38c) for \([\mu_2]^2\) and substituting the result into (2.38a), we find

\[
[M_2]_1 = \frac{m_{12}(s)m_{33}(s) - m_{32}(s)m_{13}(s)}{(s^TS^A)_{33}s_{23}} e^{\theta_3}[M_1]_3 - \frac{m_{32}(s)}{s_{23}}[\mu_2]_1 - \frac{m_{12}(s)}{s_{23}} e^{\theta_3}[\mu_2]_3.
\]

Taking the residue of this equation at \( k_j \), we find condition (2.34c) in the case when \( k_j \in D_2 \).

In order to prove (2.34e), we solve (2.39a) and (2.39b) for \([\mu_2]^1\) and \([\mu_2]^3\), respectively, then substituting the result into (2.39c), we find

\[
[M_3]_3 = \frac{s_{12} - s_{32}}{m_{33}(s)m_{23}(s)} e^{\theta_3}[M_4]_1 + \frac{s_{32}s_{13} - s_{12}s_{32}}{(s^TS^A)_{33}(s)m_{23}(s)} e^{\theta_3}[M_4]_2 + \frac{1}{m_{23}(s)}[\mu_2]_3.
\]

Taking the residue of this equation at \( k_j \), we find condition (2.34e) in the case when \( k_j \in D_3 \).

Similarly, solving (2.40a) and (2.40b) for \([\mu_2]^1\) and \([\mu_2]^3\), respectively, then substituting the result into (2.40c), we find

\[
[M_4]_3 = \frac{s_{12}}{m_{33}(s)m_{23}(s)} e^{\theta_3}[M_4]_1 - \frac{s_{11}}{m_{33}(s)m_{23}(s)} e^{\theta_3}[M_4]_2 - \frac{1}{m_{23}(s)}[\mu_2]_3.
\]

Taking the residue of this equation at \( k_j \), we find the condition (2.34f) in the case when \( k_j \in D_4 \). ■

3. The Riemann–Hilbert problem

The sectionally analytical function \( M(x, t, k) \) defined in §2 satisfies a Riemann–Hilbert problem which can be formulated in terms of the initial and boundary values of \( u(x, t) \). By solving this Riemann–Hilbert problem, the solution of (1.5) (then (1.3)) can be recovered for all values of \( x, t \).

**Theorem 3.1.** Suppose that \( u(x, t) \) is a solution of (1.5) in the half-line domain \( \Omega \) with sufficient smoothness and decays as \( x \to \infty \). Then \( u(x, t) \) can be reconstructed from the initial value \( u_0(x) \) and boundary values \( (g_0(t), g_1(t), g_2(t)) \) defined as follows,

\[
u_0(x) = u(x, 0), \quad g_0(t) = u(0, t), \quad g_1(t) = \frac{\partial u}{\partial x}(0, t) \quad \text{and} \quad g_2(t) = u_{2x}(0, t).
\]

Use the initial and boundary data to define the jump matrices \( J_{m,n}(x, t, k) \) as well as the spectral \( s(k) \) and \( S(k) \) by equation (2.24). Assume that the possible zeros \( k_j \) of the functions \( s_{33}(k), (s^TS^A)_{33}(k), (s^TS^A)_{33}(k) \) and \( m_{33}(s)(k) \) are as in assumption 2.3.

Then the solution \( u(x, t) \) is given by

\[
u(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{13},
\]

where \( M(x, t, k) \) satisfies the following \( 3 \times 3 \) matrix Riemann–Hilbert problem:

- \( M \) is sectionally meromorphic on the Riemann \( k \)-sphere with jumps across the contours \( \hat{D}_n \cap \hat{D}_m, n, m = 1, \ldots, 4 \) (figure 2).
- Across the contours \( \hat{D}_n \cap \hat{D}_m, M \) satisfies the jump condition

\[
M_n(x, t, k) = M_n(x, t, k)J_{m,n}(x, t, k), \quad k \in \hat{D}_n \cap \hat{D}_m, n, m = 1, 2, 3, 4.
\]

- \( M(x, t, k) = \mathbb{I} + O(1/k), k \to \infty \).
- The residue condition of \( M \) is shown in proposition 2.6.

**Proof.** It remains only to prove (3.2) and this equation follows from the large \( k \) asymptotics of the eigenfunctions, see appendix A. ■
Remark 3.2. According to the coordinate transformation (1.4), we in fact, consider the so-called ‘complex modified KdV-type’ equation in this paper.

4. Non-linearizable boundary conditions

A major difficulty of IBV problems is that some of the boundary values are unknown for a well-posed problem. All boundary values are needed for the definition of $S(k)$, and hence for the formulation of the Riemann–Hilbert problem. Our main result expresses the spectral function $S(k)$ in terms of the prescribed boundary data and the initial data via the solution of a system of nonlinear integral equations.

(a) Asymptotics

An analysis of (2.7) shows that the eigenfunctions $\{\mu_j\}_1^3$ have the following asymptotics as $k \to \infty$ (see appendix A):

$$
\mu_j(x, t, k) = 1 + \frac{1}{k} \begin{pmatrix}
\frac{i}{2} \int_{(x_j,t)} \Delta \Delta_1 \frac{1}{2i} u \\
\frac{i}{2} \int_{(x_j,t)} \Delta_2 \frac{1}{2i} u \\
\frac{1}{2i} \bar{u} \\
\end{pmatrix} + \frac{1}{k^2} \begin{pmatrix}
\frac{1}{4} \int_{(x_j,t)} (\eta + \nu_1) \\
\frac{1}{4} \int_{(x_j,t)} \eta_1 \\
\frac{1}{4} \int_{(x_j,t)} (\eta + \nu_2) \\
\end{pmatrix} + \frac{1}{k^3} \begin{pmatrix}
\mu_{11}^{(3)} \mu_{12}^{(3)} \mu_{13}^{(3)} \\
\mu_{21}^{(3)} \mu_{22}^{(3)} \mu_{23}^{(3)} \\
\mu_{31}^{(3)} \mu_{32}^{(3)} \mu_{33}^{(3)} \\
\end{pmatrix} + O \left( \frac{1}{k^4} \right),
$$

(4.1a)

where

$$
\Delta = -|u|^2 \, dx + (u \ddot{u}_{xx} + u_{xx} \ddot{u} - u_x \ddot{u}_x + 6|u|^4) \, dt,
$$

$$
\Delta_1 = -u^2 \, dx + (uu_{xx} + u_{xx}u - (u_x)^2 + 6u^2u^2) \, dt,
$$

and

$$
\Delta_2 = -\bar{u}^2 \, dx + (\bar{u} \ddot{u}_{xx} + \bar{u}_{xx} \ddot{u} - (\bar{u}_x)^2 + 6|\bar{u}|^2|\bar{u}|^2) \, dt.
$$

(4.2a)

and

$$
\mu_{13}^{(2)} = -\frac{1}{2} u \int_{(x_j,t)} \Delta + \frac{1}{4} u_{xx},
$$

$$
\mu_{23}^{(2)} = -\frac{1}{2} \bar{u} \int_{(x_j,t)} \Delta + \frac{1}{4} \bar{u}_{xx},
$$

$$
\mu_{31}^{(2)} = \frac{1}{4} \left( \bar{u} \int_{(x_j,t)} \Delta + u \int_{(x_j,t)} \Delta_2 \right) - \frac{1}{4} \bar{u}_x,
$$

(4.2b)

and

$$
\mu_{32}^{(2)} = \frac{1}{4} \left( \bar{u} \int_{(x_j,t)} \Delta_1 + u \int_{(x_j,t)} \Delta \right) - \frac{1}{4} \bar{u}_x.
$$
The functions \( \Phi \) are analytic and bounded in \( D_1 \cup D_2 \) away from the possible zeros of \( s_{33}(k) \) and of order \( O(1/k) \) as \( k \to \infty \).

From the asymptotic of \( \mu_j(x, t) \) in (4.1a), we have

\[
\begin{pmatrix}
\frac{s_{13}(k)}{s_{23}(k)} \\
\frac{s_{23}(k)}{s_{33}(k)}
\end{pmatrix}
\approx
\begin{pmatrix}
0 & 1 \\
0 & 2 \frac{u(0, 0)}{k^2}
\end{pmatrix}
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\begin{pmatrix}
\Delta
\end{pmatrix}
+ O\left(\frac{1}{k^2}\right)
\]

(4.6)
and
\[
\Phi_{\beta}(t,k) = \Phi_{\beta}^{(1)}(t) + \frac{\Phi_{\beta}^{(2)}(t)}{k} + \frac{\Phi_{\beta}^{(3)}(t)}{k^3} + O\left(\frac{1}{k^4}\right), \quad (4.7a)
\]
\[
\Phi_{33}(t,k) = 1 + \frac{\Phi_{33}^{(1)}(t)}{k} + \frac{\Phi_{33}^{(2)}(t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty, k \in D_1 \cup D_2, \quad (4.7b)
\]
where
\[
\Phi_{\beta}^{(1)}(t) = \frac{1}{2i}g_0(t)^T, \quad \Phi_{\beta}^{(2)}(t) = \frac{1}{4}g_1(t)^T - \frac{1}{2}g_0^T \int_{(0,0)}^{(x,t)} \Delta
\]
\[
\Phi_{\beta}^{(3)}(t) = \frac{1}{2i}g_0^T \Phi_{33}^{(1)} + \frac{1}{4}g_1^T \Phi_{33}^{(2)} + \frac{i}{4}|u|^2 \Phi_{33}^T + \frac{i}{8}g_0^T
\]

Here, the definition of \( \Phi_{\beta}(t,k) \) can be found in appendix A.

In particular, we find the following expressions for the boundary values:
\[
\begin{align*}
\Phi_0^T &= 2i\Phi_{\beta}^{(1)}(t), \\
\Phi_1^T &= 2i\Phi_{33}^{(1)}(t) + 4\Phi_{\beta}^{(2)}(t)
\end{align*}
\]
and
\[
\Phi_2^T = -2g_0^T \Phi_0^T + 2i\Phi_{33}^{(1)}(t) + 4\Phi_{33}^{(2)}(t) - 8i\Phi_{\beta}^{(3)}(t).
\]

We will also need the asymptotic of \( c_j(t,k) \).

**Lemma 4.1.** Global relation (4.5) implies that the large \( k \) behaviour of \( c_j(t,k) \) satisfies
\[
c_j(t,k) = \frac{\Phi_{\beta}^{(1)}(t)}{k} + \frac{\Phi_{\beta}^{(2)}(t)}{k^2} + \frac{\Phi_{\beta}^{(3)}(t)}{k^3} + O\left(\frac{1}{k^4}\right), \quad k \to \infty, k \in D_1. \quad (4.9)
\]

**Proof.** See appendix B. \( \blacksquare \)

(b) The Dirichlet and Neumann problems

We can now derive effective characterizations of spectral function \( S(k) \) for the Dirichlet (\( g_0 \) prescribed), the first Neumann (\( g_1 \) prescribed) and the second Neumann (\( g_2 \) prescribed) problems.

Define \( \alpha \) by \( \alpha = e^{2\pi i/3} \) and let \( \{ \Pi_j(t,k), \tilde{\Pi}_j(t,k), \tilde{\tilde{\Pi}}_j(t,k) \}^3 \) denote the following combinations formed from \( \{\Phi_{\beta}(t,k)\}^3 \):

\[
\begin{align*}
\Pi_j(t,k) &= \Phi_{\beta}(t,k) + \alpha \Phi_{\beta}(t,\alpha k) + \alpha^2 \Phi_{\beta}(t,\alpha^2 k), \quad j = 1, 2, 3, \\
\tilde{\Pi}_j(t,k) &= \Phi_{\beta}(t,k) + \alpha^2 \Phi_{\beta}(t,\alpha k) + \alpha \Phi_{\beta}(t,\alpha^2 k), \quad j = 1, 2, 3 \\
\tilde{\tilde{\Pi}}_j(t,k) &= \Phi_{\beta}(t,k) + \Phi_{\beta}(t,\alpha k) + \Phi_{\beta}(t,\alpha^2 k), \quad j = 1, 2, 3
\end{align*}
\]

And let \( R(k) = \Phi_{11}(s_{13}/s_{33}) + \Phi_{12}(s_{23}/s_{33}) \).

Let \( D_1 = D'_1 \cup D''_1 \), where \( D'_1 = D_1 \cap \{\text{Re} > 0\} \) and \( D''_1 = D_1 \cap \{\text{Re} < 0\} \). Similarly, let \( D_4 = D'_4 \cup D''_4 \), where \( D'_4 = D_4 \cap \{\text{Re} > 0\} \) and \( D''_4 = D_4 \cap \{\text{Re} < 0\} \).

**Theorem 4.2.** Let \( T < \infty \). Let \( u_0(x), u \geq 0 \), be a function of Schwartz class.

For the Dirichlet problem, it is assumed that the function \( g_0(t), 0 \leq t < T, \) has sufficient smoothness and is compatible with \( u_0(x) \) at \( x = t = 0 \).

For the first Neumann problem, it is assumed that the function \( g_1(t), 0 \leq t < T, \) has sufficient smoothness and is compatible with \( u_0(x) \) at \( x = t = 0 \).

Similarly, for the second Neumann problem, it is assumed that the function \( g_2(t), 0 \leq t < T, \) has sufficient smoothness and is compatible with \( u_0(x) \) at \( x = t = 0 \).

Suppose that \( s_{33}(k) \) has a finite number of simple zeros in \( D_3 \).
Then the spectral function \( S(k) \) is given by
\[
S(k) = \begin{pmatrix}
A(k) & B(k) & e^{i k^3 T C(k)} \\
D(k) & E(k) & e^{i k^3 T F(k)} \\
e^{-i k^3 T G(k)} & e^{-i k^3 T H(k)} & I(k)
\end{pmatrix},
\tag{4.11}
\]
where
\[
A(k) = \Phi_{22}(k)\Phi_{33}(k) - \Phi_{23}(k)\Phi_{32}(k), \quad B(k) = \Phi_{13}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{33}(k),
\]
\[
C(k) = \Phi_{12}(k)\Phi_{23}(k) - \Phi_{13}(k)\Phi_{22}(k), \quad D(k) = \Phi_{23}(k)\Phi_{31}(k) - \Phi_{21}(k)\Phi_{33}(k),
\]
\[
E(k) = \Phi_{11}(k)\Phi_{33}(k) - \Phi_{13}(k)\Phi_{31}(k), \quad F(k) = \Phi_{21}(k)\Phi_{13}(k) - \Phi_{11}(k)\Phi_{23}(k),
\]
\[
G(k) = \Phi_{21}(k)\Phi_{32}(k) - \Phi_{22}(k)\Phi_{31}(k), \quad H(k) = \Phi_{12}(k)\Phi_{31}(k) - \Phi_{11}(k)\Phi_{32}(k),
\]
\[
I(k) = \Phi_{11}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{21}(k)
\]
and the complex-value functions \( \{\Phi_{i3}(t,k)\}_{i=1}^3 \) satisfy the following system of integral equations:
\[
\begin{align*}
\Phi_{13}(t,k) &= \int_0^t e^{-i k^3(t-t')}[(2i k|g_0|^2 + (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{13} \\
&+ \bar{g}_0^2\Phi_{23} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - g_2)\Phi_{33}](t',k) \, dt', \quad \tag{4.12a}
\Phi_{23}(t,k) &= \int_0^t e^{-i k^3(t-t')}[(2i k|g_0|^2 - (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{13} \\
&+ \bar{g}_0^2\Phi_{23} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - g_2)\Phi_{33}](t',k) \, dt', \quad \tag{4.12b}
\Phi_{33}(t,k) &= 1 + \int_0^t [(-4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + \bar{g}_2)\Phi_{13} \\
&+ (-4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + g_2)\Phi_{23} + -4ik|g_0|^2\Phi_{33}](t',k) \, dt' \quad \tag{4.12c}
\end{align*}
\]
and \( \{\Phi_{i1}(t,k)\}_{i=1}^3, \{\Phi_{i2}(t,k)\}_{i=1}^3 \) satisfy the following system of integral equations:
\[
\begin{align*}
\Phi_{11}(t,k) &= 1 + \int_0^t [(2i k|g_0|^2 + (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{11} \\
&+ \bar{g}_0^2\Phi_{21} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - g_2)\Phi_{31}](t',k) \, dt', \quad \tag{4.13a}
\Phi_{21}(t,k) &= \int_0^t [(2i k|g_0|^2 - (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{11} \\
&+ \bar{g}_0^2\Phi_{21} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - \bar{g}_2)\Phi_{31}](t',k) \, dt', \quad \tag{4.13b}
\Phi_{31}(t,k) &= \int_0^t e^{i k^3(t-t')}[(4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + \bar{g}_2)\Phi_{11} \\
&+ (-4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + g_2)\Phi_{21} + -4ik|g_0|^2\Phi_{31}](t',k) \, dt' \quad \tag{4.13c}
\end{align*}
\]
\[
\begin{align*}
\Phi_{12}(t,k) &= \int_0^t [(2i k|g_0|^2 + (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{12} \\
&+ \bar{g}_0^2\Phi_{22} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - g_2)\Phi_{32}](t',k) \, dt', \quad \tag{4.14a}
\Phi_{22}(t,k) &= 1 + \int_0^t [(2i k|g_0|^2 - (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{12} \\
&+ \bar{g}_0^2\Phi_{22} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - \bar{g}_2)\Phi_{32}](t',k) \, dt' \quad \tag{4.14b}
\end{align*}
\]
and
\[
\begin{align*}
\Phi_{32}(t,k) &= \int_0^t e^{i k^3(t-t')}[(4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + \bar{g}_2)\Phi_{12} \\
&+ (-4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + g_2)\Phi_{22} + -4ik|g_0|^2\Phi_{32}](t',k) \, dt' \quad \tag{4.14c}
\end{align*}
\]
(i) For the Dirichlet problem, the unknown Neumann boundary values \( g_1(t) \) and \( g_2(t) \) are given by

\[
g_1(t) = \frac{2g_0(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) \, dk + \frac{2}{\pi i} \int_{\partial D_3} \left[ kP_1(t, k) - \frac{3g_0(t)}{2i} \right] \, dk
\]

\[
- \frac{2}{\pi i} \int_{\partial D_3} k e^{-8ik_1 t} [(\alpha^2 - \alpha)R(\alpha k) + (\alpha - \alpha^2)R(\alpha^2 k)] \, dk
\]

\[
+ 4 \left\{ (1 - \alpha^2) \sum_{k_j \in D_1^\ell} + (1 - \alpha^2) \sum_{k_j \in D_1^r} \right\} k_j e^{-8ik_j t} \text{Res}_j R(k)
\]

\[
(4.15a)
\]

and

\[
g_2(t) = g_0(t)^3 - \frac{4}{\pi} \int_{\partial D_3} \left[ k^2 P_1(t, k) - \frac{3k g_0(t)}{2i} \right] \, dk
\]

\[
+ \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8ik_1 t} [(1 - \alpha)R(\alpha k) + (1 - \alpha^2)R(\alpha^2 k)] \, dk
\]

\[
- 8i \left\{ (1 - \alpha) \sum_{k_j \in D_1^\ell} + (1 - \alpha^2) \sum_{k_j \in D_1^r} \right\} k_j^2 e^{-8ik_j t} \text{Res}_j R(k)
\]

\[
+ \frac{4g_0(t)}{\pi i} \int_{\partial D_3} k \hat{N}_3(t, k) \, dk + \frac{2g_1(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) \, dk.
\]

\[
(4.15b)
\]

(ii) For the first Neumann problem, the unknown boundary values \( g_0(t) \) and \( g_2(t) \) are given by

\[
g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \hat{N}_1(t, k) \, dk - \frac{1}{\pi} \int_{\partial D_3} e^{-8ik_1 t} [(\alpha - \alpha^2)R(\alpha k) + (\alpha^2 - \alpha)R(\alpha^2 k)] \, dk
\]

\[
+ 2i \left\{ (1 - \alpha) \sum_{k_j \in D_1^\ell} + (1 - \alpha^2) \sum_{k_j \in D_1^r} \right\} e^{-8ik_j t} \text{Res}_j R(k)
\]

\[
(4.16a)
\]

and

\[
g_2(t) = g_0(t)^3 - \frac{4}{\pi} \int_{\partial D_3} \left[ k^2 \hat{N}_1(t, k) - \frac{3}{\pi i} \int_{\partial D_3} \hat{N}_1(t, l) \, dl \right] \, dk
\]

\[
+ \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8ik_1 t} [(1 - \alpha^2)R(\alpha k) + (1 - \alpha)R(\alpha^2 k)] \, dk
\]

\[
- 8i \left\{ (1 - \alpha) \sum_{k_j \in D_1^\ell} + (1 - \alpha^2) \sum_{k_j \in D_1^r} \right\} k_j^2 e^{-8ik_j t} \text{Res}_j R(k)
\]

\[
+ \frac{4g_0(t)}{\pi i} \int_{\partial D_3} k \hat{N}_3(t, k) \, dk + \frac{2g_1(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) \, dk.
\]

\[
(4.16b)
\]

(iii) For the second Neumann problem, the unknown boundary values \( g_0(t) \) and \( g_1(t) \) are given by

\[
g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \hat{N}_1(t, k) \, dk - \frac{1}{\pi} \int_{\partial D_3} e^{-8ik_1 t} [(\alpha - \alpha^2)R(\alpha k) + (\alpha^2 - \alpha)R(\alpha^2 k)] \, dk
\]

\[
+ 2i \left\{ (1 - \alpha) \sum_{k_j \in D_1^\ell} + (1 - \alpha^2) \sum_{k_j \in D_1^r} \right\} e^{-8ik_j t} \text{Res}_j R(k)
\]

\[
(4.17a)
\]
and
\[
g_1(t) = \frac{2\xi_0(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) \, dk + \frac{2}{\pi i} \int_{\partial D_3} k\bar{I}_1(t, k) \, dk \\
- \frac{2}{\pi i} \int_{\partial D_3} k e^{8i\xi t} [(\alpha^2 - 1) R(\alpha k) + (\alpha - 1) R(\alpha^2 k)] \, dk \\
+ 4 \left\{ (1 - \alpha) \sum_{k \in D'_1} + (1 - \alpha^2) \sum_{k \in D'_2} \right\} k_i e^{8i\xi t} \text{Res}_k R(k). \tag{4.17b}
\]

**Proof.** Representations (4.11) follow from the relation \( S(t) = e^{8i\xi T} \mu_2^A(0, T, k)^T \), and system (4.12) is the direct result of the Volterra integral equations of \( \mu_2(0, t, k) \).

(i) In order to derive (4.15a), we note that equation (4.8b) expresses \( g_1 \) in terms of \( \Phi_{33}^{(1)} \) and \( \Phi_{13}^{(2)} \). Furthermore, equation (4.7) and Cauchy theorem imply
\[
- \frac{2\pi i}{3} \Phi_{33}^{(1)}(t) = 2 \int_{\partial D_2} [\Phi_{33}(t, k) - 1] \, dk = \int_{\partial D_3} [\Phi_{33}(t, k) - 1] \, dk
\]
and
\[
- \frac{2\pi i}{3} \Phi_{13}^{(2)}(t) = 2 \int_{\partial D_2} k \Phi_{13}(t) - \frac{g_0(t)}{2i} \, dk = \int_{\partial D_3} \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] \, dk.
\]
Thus,
\[
i\pi \Phi_{33}^{(1)}(t) = - \left( \int_{\partial D_2} + \int_{\partial D_4} \right) [\Phi_{33}(t, k) - 1] \, dk = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) [\Phi_{33}(t, k) - 1] \, dk
\]
\[
= \int_{\partial D_3} [\Phi_{33}(t, k) - 1] \, dk + \alpha \int_{\partial D_3} [\Phi_{33}(t, k) - 1] \, dk + \alpha^2 \int_{\partial D_3} [\Phi_{33}(t, k) - 1] \, dk
\]
\[
= \int_{\partial D_3} \Pi_3(t, k) \, dk. \tag{4.18}
\]
Similarly,
\[
i\pi \Phi_{13}^{(2)}(t) = \left( \int_{\partial D_3} + \int_{\partial D_4} \right) \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] \, dk
\]
\[
= \left( \int_{\partial D_3} + \alpha^2 \int_{\partial D'_1} + \alpha \int_{\partial D'_2} \right) \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] \, dk + I(t)
\]
\[
= \int_{\partial D_3} \left[ k I_1(t, k) - \frac{3g_0(t)}{2i} \right] \, dk + I(t), \tag{4.19}
\]
where \( I(t) \) is defined by
\[
I(t) = \left( 1 - \alpha^2 \right) \int_{\partial D'_1} + (1 - \alpha) \int_{\partial D'_2} \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] \, dk.
\]
The last step involves using the global relation to compute \( I(t) \)
\[
I(t) = \left( 1 - \alpha^2 \right) \int_{\partial D'_1} + (1 - \alpha) \int_{\partial D'_2} \left[ k c_1(t, k) - \frac{g_0(t)}{2i} \right] \, dk
\]
\[
- \left( 1 - \alpha^2 \right) \int_{\partial D'_1} + (1 - \alpha) \int_{\partial D'_2} k e^{-8i\xi t} R(k) \, dk. \tag{4.20}
\]
Using asymptotic (4.9) and Cauchy theorem to compute the first term on the right-hand side of equation (4.20) and using the transformation \( k \to \alpha k \) and \( k \to \alpha^2 k \) in the second
term on the right-hand side of (4.20), we find
\[
I(t) = -i\pi \Phi_{13}^{(2)}(t) - \int_{\partial D_3} k e^{-8ik^3 t} [(\alpha^2 - \alpha) R(\alpha k) + (\alpha - \alpha^2) R(\alpha^2 k)] \, dk
\]
\[
+ 2\pi i \left\{ (1 - \alpha^2) \sum_{k_j \in D_1^i} + (1 - \alpha) \sum_{k_j \in D_1^j} \right\} k e^{-8ik^3 t} \text{Res}_{k_j} R(k).
\] (4.21)

Equations (4.19) and (4.21) imply
\[
\Phi_{13}^{(2)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} \left[ k^2 \Pi_1(t, k) - \frac{3\xi_0(t)}{2i} \right] \, dk
\]
\[
- \frac{1}{2\pi i} \int_{\partial D_3} k^2 e^{-8ik^3 t} [(1 - \alpha) R(\alpha k) + (1 - \alpha^2) R(\alpha^2 k)] \, dk
\]
\[
\times \left\{ (1 - \alpha^2) \sum_{k_j \in D_1^i} + (1 - \alpha) \sum_{k_j \in D_1^j} \right\} k_j e^{-8ik^3 t} \text{Res}_{k_j} R(k).
\] (4.22a)

This equation together with (4.8b) and (4.18) yields (4.15a).

In order to derive (4.15b), we note that (4.8c) expresses \( g_2 \) in terms of \( \Phi_{13}^{(3)} \), \( \Phi_{33}^{(2)} \) and \( \Phi_{13}^{(1)} \).

Equation (4.15b) follows from expression (4.18) for \( \Phi_{33}^{(1)} \) and the following formulae:
\[
\Phi_{33}^{(2)}(t) = \frac{1}{\pi i} \int_{\partial D_3} k \hat{f}_3 \, dk
\] (4.22b)

(ii) In order to derive representations (4.16) relevant to the first Neumann problem, we use (4.8) together with (4.18), (4.22a) and the following formulae:
\[
\Phi_{13}^{(1)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} \hat{f}_1(t, k) \, dk
\]
\[
- \frac{1}{2\pi i} \int_{\partial D_3} e^{-8ik^3 t} [(\alpha - \alpha^2) R(\alpha k) + (\alpha^2 - \alpha) R(\alpha^2 k)] \, dk
\]
\[
\times \left\{ (1 - \alpha^2) \sum_{k_j \in D_1^i} + (1 - \alpha) \sum_{k_j \in D_1^j} \right\} e^{-8ik^3 t} \text{Res}_{k_j} R(k),
\] (4.23a)

\[
\Phi_{13}^{(2)}(t) = \frac{1}{\pi i} \int_{\partial D_3} k \hat{f}_1 \, dk
\] (4.23b)

and
\[
\Phi_{13}^{(3)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} [k^2 \hat{f}_1(t, k) - 3\Phi_{13}^{(2)}] \, dk
\]
\[
- \frac{1}{2\pi i} \int_{\partial D_3} k^2 e^{-8ik^3 t} [(1 - \alpha^2) R(\alpha k) + (1 - \alpha) R(\alpha^2 k)] \, dk
\]
\[
\times \left\{ (1 - \alpha^2) \sum_{k_j \in D_1^i} + (1 - \alpha) \sum_{k_j \in D_1^j} \right\} k_j e^{-8ik^3 t} \text{Res}_{k_j} R(k).
\] (4.23c)
(iii) In order to derive representations (4.17) relevant for the second Neumann problem, we use (4.8) together with (4.18) and the following formulae:

\[
\Phi_{13}^{(1)}(t) = \frac{1}{2\pi i} \int_{aD_3} \hat{f}_1(t, k) \, dk \\
- \frac{1}{2\pi i} \int_{aD_3} e^{-8ik^3t}[(\alpha - 1)R(\alpha k) + (\alpha^2 - 1)R(\alpha^2 k)] \, dk \\
\times \left\{ (1 - \alpha^2) \sum_{k_j \in D'_1} + (1 - \alpha) \sum_{k_j \in D'_2} \right\} e^{-8ik^3t} \text{Res}_{k_j} R(k) \tag{4.24a}
\]

and

\[
\Phi_{13}^{(2)}(t) = \frac{1}{2\pi i} \int_{aD_3} k\hat{f}_1(t, k) \, dk \\
- \frac{1}{2\pi i} \int_{aD_3} ke^{-8ik^3t}[(\alpha^2 - 1)R(\alpha k) + (\alpha - 1)R(\alpha^2 k)] \, dk \\
\times \left\{ (1 - \alpha) \sum_{k_j \in D'_1} + (1 - \alpha^2) \sum_{k_j \in D'_2} \right\} e^{-8ik^3t} \text{Res}_{k_j} R(k). \tag{4.24b}
\]

(c) Effective characterizations

Substituting into system (4.12), the expressions

\[
\Phi_{ij} = \Phi_{ij}^{(0)} + \varepsilon \Phi_{ij}^{(1)} + \varepsilon^2 \Phi_{ij}^{(2)} + \cdots, \quad i,j = 1, 2, 3, \tag{4.25a}
\]

\[
g_0 = \varepsilon g_0 + \varepsilon^2 g_0 + \cdots, \tag{4.25b}
\]

\[
g_1 = \varepsilon g_1 + \varepsilon^2 g_1 + \cdots \tag{4.25c}
\]

and

\[
g_2 = \varepsilon g_2 + \varepsilon^2 g_2 + \cdots \tag{4.25d}
\]

where \(\varepsilon > 0\) is a small parameter, we find that the terms of \(O(1)\) give \(\Phi_{ij}^{(0)} = \Phi_{13}^{(0)} = \Phi_{23}^{(0)} = 0\) and \(\Phi_{33}^{(0)} = 1\). Moreover, the terms of \(O(\varepsilon)\) give \(\Phi_{13}^{(1)} = 0\) and

\[
O(\varepsilon): \quad \Phi_{13}^{(1)}(t, k) = \int_0^t e^{-8ik^3(t-t')}(4k^2 g_0 + 2ikg_1 - g_2)(t', k) \, dt'. \tag{4.26}
\]

From equation (4.26), we can get

\[
\hat{f}_1^{(1)}(t, k) = 12k^2 \int_0^t e^{-8ik^3(t-t')}g_0(t') \, dt', \tag{4.27a}
\]

\[
\hat{f}_1^{(1)}(t, k) = 6ik \int_0^t e^{-8ik^3(t-t')}g_1(t') \, dt' \tag{4.27b}
\]

and

\[
\hat{f}_1^{(1)}(t, k) = -3 \int_0^t e^{-8ik^3(t-t')}g_1(t') \, dt'. \tag{4.27c}
\]

The Dirichlet problem can now be solved perturbatively as follows: assuming for simplicity that \(s_{33}(k)\) has no zeros and expanding (4.15a) and (4.15b), we find

\[
s_{11} = \frac{2}{\pi i} \int_{aD_3} \left[ k\hat{f}_1^{(1)}(t, k) - \frac{3g_0(t)}{2i} \right] \, dk \\
- \frac{2}{\pi i} \int_{aD_3} ke^{-8ik^3t}[(\alpha^2 - \alpha)s_{131}(\alpha k) + (\alpha - \alpha^2)s_{131}(\alpha^2 k)] \, dk \tag{4.28a}
\]
and

\[ g_{21} = -\frac{4}{\pi} \int_{\partial D_3} \left[ k^2 P_1^{(1)}(t, k) - \frac{3k g_{01}(t)}{2i} \right] dk + \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8iki^3t} \left[ (1 - \alpha_1) s_{131}(\alpha k) + (1 - \alpha_2^2) s_{131}(\alpha^2 k) \right] dk. \]  

(4.28b)

Using equation (4.27a) to determine \( \Pi^{(1)}_1 \), we can determine \( g_{11}, g_{21} \) from (4.28), then \( \Phi^{(1)}_{13} \) can be found from (4.26). And these arguments can be extended to higher orders and also can be extended to systems (4.13a) and (4.14a), thus yields a constructive scheme for computing \( S(k) \) to all orders.

Similarly, these arguments also can be used to the first Neumann problem and the second Neumann problem. That is to say, in all cases, the system can be solved perturbatively to all orders.

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**Appendix A. The asymptotic behaviour of the functions \( \{\mu_j(x, t, k)\}_{j=1}^3 \)**

We denote some symbols as follows:

\[ \Lambda = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -1 \end{pmatrix} \]  

(A 1a)

and

\[ V_1 = \begin{pmatrix} 0 & U^T \\ -\tilde{U} & 0 \end{pmatrix}, \]

\[ V_2^{(2)} = \begin{pmatrix} 0 & U^T \\ -\tilde{U} & 0 \end{pmatrix}, \]

\[ V_2^{(1)} = 2i \begin{pmatrix} U^T \tilde{U} & U_x^T \\ -\tilde{U}_x & -2|u|^2 \end{pmatrix}, \]

\[ V_2^{(0)} = -4|u|^2 \begin{pmatrix} 0 & U^T \\ -\tilde{U} & 0 \end{pmatrix} - \begin{pmatrix} 0 & U_x^T \\ -\tilde{U}_x & 0 \end{pmatrix} + (u\bar{u}_x - u_x \bar{u})(\sigma_3 \begin{pmatrix} 0 \\ 0 \end{pmatrix}), \]

(A 1b)

where \( I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( U = (u, \bar{u}) \).

From the Lax pair of \( \mu \)

\[ \mu_x + [ik\Lambda, \mu] = V_1 \mu \]

and

\[ \mu_t + [4ik^3\Lambda, \mu] = V_2 \mu. \]  

(A 2)

Suppose that

\[ \mu(x, t, k) = D_0 + \frac{D_1}{k} + \frac{D_2}{k^2} + \frac{D_3}{k^3} + \cdots. \]  

(A 3)
We substitute equation (A 3) into the Lax pair (A 2), and compare the order of $k$, we find that

$$
\begin{align*}
O(k) : [i\Lambda, D_0] &= 0, \\
O(1) : D_{0x} + [i\Lambda, D_1] &= V_1 D_0, \\
O(k^{-1}) : D_{1x} + [i\Lambda, D_2] &= V_1 D_1
\end{align*}
$$

and

$$
\begin{align*}
O(k^2) : [i\Lambda, D_0] &= 0, \\
O(k^2) : [i\Lambda, D_1] &= V_2 D_0, \\
O(k^2) : [i\Lambda, D_2] &= V_2 D_1 + V_1 D_0,
\end{align*}
$$

and

$$
\begin{align*}
O(k^3) : [i\Lambda, D_0] &= 0, \\
O(k^3) : [i\Lambda, D_1] &= V_2^2 D_0, \\
O(k^3) : [i\Lambda, D_2] &= V_2^2 D_1 + V_1 D_0, \\
O(k^3) : [i\Lambda, D_3] &= V_2^2 D_2 + V_1 D_1 + V_0 D_0.
\end{align*}
$$

We denote the $D_l$ by $D_l = \begin{pmatrix} D_l^{(2)} & D_l^{(0)} \\ D_{lj}^{(0)} & D_{lj}^{(0)} \end{pmatrix}$, $j = 1, 2$.

Then, from $O(k^3)$, we have

$$
D_{j3}^{(0)} = 0, \quad D_{3j}^{(0)} = 0.
$$

$O(k^2)$, we get

$$
4i \begin{pmatrix} 0 & 2D_{j3}^{(1)} \\ -2D_{3j}^{(1)} & 0 \end{pmatrix} = 4 \begin{pmatrix} 0 & U^T D_{33}^{(0)} \\ -\bar{U}D_{2x2}^{(0)} & 0 \end{pmatrix},
$$

this implies that

$$
D_{j3}^{(1)} = -\frac{i}{2} U^T D_{33}^{(0)}
$$

and

$$
D_{3j}^{(1)} = -\frac{i}{2} \bar{U}D_{2x2}^{(0)}.
$$

$O(k)$, we find

$$
4i \begin{pmatrix} 0 & 2D_{j3}^{(2)} \\ -2D_{3j}^{(2)} & 0 \end{pmatrix} = 4 \begin{pmatrix} U^T D_{j3}^{(1)} & U^T D_{33}^{(1)} \\ -\bar{U}D_{2x2}^{(1)} & -\bar{U}D_{j3}^{(1)} \end{pmatrix} + 2i \begin{pmatrix} U^T \bar{U}D_{2x2}^{(0)} & U^T \bar{U}D_{33}^{(0)} \\ -\bar{U}x D_{2x2}^{(0)} & -2\bar{u}^2 D_{33}^{(0)} \end{pmatrix},
$$

this implies that

$$
D_{j3}^{(2)} = -\frac{i}{2} U^T D_{33}^{(1)} + \frac{1}{4} U^T D_{33}^{(0)}
$$

and

$$
D_{3j}^{(2)} = -\frac{i}{2} \bar{U}D_{2x2}^{(1)} - \frac{1}{4} \bar{U}x D_{2x2}^{(0)}.
$$
(A 8a)

this implies that

\[
\begin{aligned}
D^{(0)}_{2 \times 2} &= 0, \quad D^{(0)}_{3 \times 3} = 0, \\
D^{(3)}_{\beta j} &= -\frac{i}{2} U T D^{(2)}_{3 j} + \frac{1}{4} U T D^{(1)}_{3 j} + \frac{i}{4} |u|^2 U T D^{(0)}_{3 j} + \frac{i}{8} U T D^{(0)}_{3 j} \\
\end{aligned}
\]

(A 8b)

and

\[
\begin{aligned}
D^{(3)}_{\beta j} &= -\frac{i}{2} \bar{u} D^{(2)}_{2 \times 2} - \frac{1}{4} \bar{u} D^{(1)}_{2 \times 2} + \frac{i}{4} |u|^2 \bar{u} D^{(0)}_{2 \times 2} + \frac{i}{8} \bar{u} D^{(0)}_{2 \times 2}. \\
\end{aligned}
\]

(A 9a)

this implies that

\[
\begin{aligned}
D^{(1)}_{2 \times 2} &= \frac{i}{2} (U T \bar{U} x + U x \bar{U} x - U T \bar{U} x + 6 |u|^2 U T \bar{U}) D^{(0)}_{2 \times 2}, \\
D^{(1)}_{3 \times 3} &= -i(u \bar{u} x + u x \bar{u} - u x \bar{u} x + 6 |u|^4) D^{(0)}_{3 \times 3}, \\
D^{(4)}_{\beta j} &= \frac{1}{16} U T D^{(0)}_{3 j} - \frac{i}{2} U T D^{(3)}_{3 j} + \frac{1}{4} U T D^{(2)}_{3 j} + \frac{i}{4} |u|^2 U T D^{(1)}_{3 j} + \frac{i}{8} U T D^{(1)}_{3 j} + \frac{i}{8} |u|^2 U T D^{(0)}_{3 j} \\
\end{aligned}
\]

(A 9b)
O(k^−2), we get

\[
\begin{pmatrix}
D_{2×2t}^{(2)} & D_{3j}^{(2)} \\
D_{3j}^{(2)} & D_{33}^{(2)}
\end{pmatrix} + 4i \begin{pmatrix}
0 & 2D_{j3}^{(5)} \\
-2D_{j3}^{(5)} & 0
\end{pmatrix} =
\begin{pmatrix}
U^T D_{3j}^{(4)} - \bar{U} D_{3j}^{(4)} \\
-\bar{U} D_{2×2}^{(4)} - \bar{U} D_{j3}^{(4)}
\end{pmatrix} + 2i \begin{pmatrix}
U^T \bar{U} D_{2×2}^{(3)} + U_x^2 D_{3j}^{(3)} & U^T \bar{U} D_{j3}^{(3)} + U_x^2 D_{j3}^{(3)} \\
-\bar{U}_x D_{2×2}^{(3)} - 2|u|^2 D_{3j}^{(3)} & \bar{U}_x D_{j3}^{(3)} - 2|u|^2 D_{j3}^{(3)}
\end{pmatrix}
- 4|u|^2 \begin{pmatrix}
U^T D_{3j}^{(2)} - \bar{U} D_{3j}^{(2)} \\
-\bar{U} D_{2×2}^{(2)} - \bar{U} D_{j3}^{(2)}
\end{pmatrix} - \begin{pmatrix}
U^T U D_{3j}^{(2)} & U^T U D_{j3}^{(2)} \\
-\bar{U}_x U D_{2×2}^{(2)} - \bar{U}_x U D_{j3}^{(2)}
\end{pmatrix}
+ (u\tilde{u}_x - u_x \tilde{u})(\sigma_3 D_{2×2}^{(2)} \sigma_3 D_{j3}^{(2)})
\]

(A 10a)

This implies that

\[
\begin{align*}
D_{2×2t}^{(2)} &= \frac{1}{2}(U^T \bar{U}_{xx} + U_{xx} \bar{U} - U_{xt} \bar{U}_x + 6|u|^2 U^T \bar{U})D_{2×2}^{(1)} \\
&+ \left\{ -\frac{1}{4} U^T \bar{U}_t + \frac{1}{2} |u|^2 (u\tilde{u}_x - u_x \tilde{u})\sigma_3 + \frac{1}{4} (u_{xx} \tilde{u}_x - u_x \tilde{u}_{xx})\sigma_3 \right\} \\
D_{33t}^{(2)} &= -i(u\tilde{u}_{xx} + u_{xx} \tilde{u} - u_x \tilde{u}_x + 6|u|^4)D_{33}^{(1)} - \frac{1}{4} (|u|^2) D_{33}^{(0)}.
\end{align*}
\]

(A 10b)

Also, from the x-part of the Lax pair, we have the following equations:

\[
\begin{align*}
D_{2×2x}^{(0)} &= 0 \quad \text{and} \quad D_{33x}^{(0)} = 0. \\
D_{2×2x}^{(1)} &= -\frac{1}{2} U^T \bar{U} D_{2×2}^{(0)} \\
D_{33x}^{(1)} &= i|u|^2 D_{33}^{(0)}
\end{align*}
\]

(A 11a)

\[
\begin{align*}
D_{2×2x}^{(2)} &= -\frac{1}{2} U^T \bar{U} D_{2×2}^{(1)} - \frac{1}{4} U^T \bar{U}_x D_{2×2}^{(0)} \\
D_{33x}^{(2)} &= i|u|^2 D_{33}^{(1)} - \frac{1}{4} (|u|^2) D_{33}^{(0)}.
\end{align*}
\]

(A 11b)

Then from the integral contours γ_j, we can get

\[
D_{2×2}^{(0)} = I_{2×2} \quad \text{and} \quad D_{33}^{(0)} = 1.
\]

(A 12)

**Appendix B. The asymptotic behaviour of c_j(t, k)**

Let

\[
\mu_2(0, t, k) = \begin{pmatrix}
\Phi_{2×2} & \Phi_{j3} \\
\Phi_{3j} & \Phi_{33}
\end{pmatrix}.
\]

The global relation shows that

\[
\Phi_{2×2} \frac{s_{j3}}{s_{33}} e^{-8ik^3 t} + \Phi_{j3} = c_j,
\]

(B 1)

and from equation

\[
\mu_t + [4ik^3 \Lambda, \mu] = V_2 \mu
\]
we get

\[
\begin{pmatrix}
\Phi_{2 \times 2} & \Phi_{3j} \\
\Phi_{3j} & \Phi_{33} \\
\end{pmatrix}
\begin{pmatrix}
\phi_{3j} \\
\phi_{33} \\
\end{pmatrix}
+ 4ik^3 \begin{pmatrix}
0 & 2\phi_{3j} \\
-2\phi_{3j} & 0 \\
\end{pmatrix}
\begin{pmatrix}
\Phi_{2 \times 2} \\
\Phi_{33} \\
\end{pmatrix}
= 4k^2 \begin{pmatrix}
U^T \Phi_{3j} & U^T \Phi_{33} \\
\vec{U} \Phi_{3j} & \vec{U} \Phi_{33} \\
\end{pmatrix}
\]

+ 2ik \begin{pmatrix}
U^T \bar{U} \Phi_{2 \times 2} + U^T \Phi_{3j} & U^T \bar{U} \Phi_{33} + U^T \Phi_{33} \\
\bar{U} \Phi_{2 \times 2} - 2|u|^2 \Phi_{3j} & \bar{U} \Phi_{33} - 2|u|^2 \Phi_{33} \\
\end{pmatrix}
- 4|u|^2 \begin{pmatrix}
U^T \Phi_{3j} & U^T \Phi_{33} \\
\vec{U} \Phi_{2 \times 2} & \vec{U} \Phi_{33} \\
\end{pmatrix}

\]

From the second column of equation (B 2), we get

\[
\Phi_{3j} + 8ik^3 \Phi_{3j} = 4k^2 U^T \Phi_{33} + 2ik(U^T \bar{U} \Phi_{3j} + U^T \Phi_{33})
\]

\[- 4|u|^2 U^T \Phi_{33} - U^T \bar{U} \Phi_{33} + (\bar{u} \bar{u} - u \bar{u}) \sigma_3 \Phi_{3j}
\]

and

\[
\Phi_{33j} = -4k^2 \bar{U} \Phi_{3j} + 2ik(\bar{U} \Phi_{3j} - 2|u|^2 \Phi_{33}) + 4|u|^2 \bar{U} \Phi_{3j} + \bar{U} \Phi_{33}.
\]

Suppose

\[
\begin{pmatrix}
\Phi_{3j} \\
\Phi_{33} \\
\end{pmatrix} = \begin{pmatrix}
\Phi_{3j}^{(1)} \\
\Phi_{3j}^{(2)} \\
\end{pmatrix} = \left( \begin{array}{c} \left( \begin{array}{c}
\vec{U} \Phi_{2 \times 2} + U \Phi_{3j} \\
\bar{U} \Phi_{2 \times 2} - 2|u|^2 \Phi_{3j} \\
\end{array} \right) \\
- \bar{u} \Phi_{3j} + \bar{u} \Phi_{33} \\
\end{array} \right) \right) e^{-8ik^3 t},
\]

where the coefficients \(\alpha_i(t)\) and \(\beta_i(t)\), \(l \geq 0\), are independent of \(k\). To determine these coefficients, we substitute equation (B4) into equation (B3) and use the initial conditions

\[
\alpha_0(0) + \beta_0(0) = (0_{1 \times 2}, 1)^T \quad \text{and} \quad \alpha_1(0) + \beta_1(0) = (0_{1 \times 2}, 0)^T.
\]

Then we get

\[
\begin{pmatrix}
\Phi_{3j} \\
\Phi_{33} \\
\end{pmatrix} = \begin{pmatrix}
0_{1 \times 2} \\
1 \\
\end{pmatrix} + \frac{1}{k} \begin{pmatrix}
\Phi_{3j}^{(1)} \\
\Phi_{3j}^{(2)} \\
\end{pmatrix} + \frac{1}{k^2} \begin{pmatrix}
\Phi_{3j}^{(1)} \\
\Phi_{3j}^{(2)} \\
\end{pmatrix} + \ldots
\]

\[
+ \left[ \frac{1}{k} \begin{pmatrix}
\Phi_{3j}^{(0)} \\
0 \\
\end{pmatrix} + \ldots \right] e^{-8ik^3 t}
\]

From the first column of equation (B 2), we get

\[
\Phi_{2 \times 2j} = 4k^2 U^T \Phi_{3j} + 2ik(U^T \bar{U} \Phi_{2 \times 2} + U^T \Phi_{3j})
\]

\[- 4|u|^2 U^T \Phi_{3j} - U^T \bar{U} \Phi_{2 \times 2} + (\bar{u} \bar{u} - u \bar{u}) \sigma_3 \Phi_{2 \times 2} \]

and

\[
\Phi_{3j}^2 - 8ik^3 \Phi_{3j} = -4k^2 \bar{U} \Phi_{2 \times 2} + 2ik(\bar{U} \Phi_{2 \times 2} - 2|u|^2 \Phi_{3j})
\]

\[
+ 4|u|^2 \bar{U} \Phi_{2 \times 2} + \bar{U} \Phi_{2 \times 2}.
\]

Suppose

\[
\begin{pmatrix}
\Phi_{2 \times 2j} \\
\Phi_{3j} \\
\end{pmatrix} = \begin{pmatrix}
\xi_0(t) + \frac{\xi_1(t)}{k} + \frac{\xi_2(t)}{k^2} + \ldots \\
\frac{v_0(t)}{k} + \frac{v_1(t)}{k^2} + \frac{v_2(t)}{k^3} + \ldots \\
\end{pmatrix} e^{8ik^3 t},
\]

where the coefficients \(\xi_l(t)\) and \(v_l(t)\), \(l \geq 0\), are independent of \(k\). To determine these coefficients, we substitute equation (B7) into equation (B6) and use the initial conditions

\[
\xi_0(0) + v_0(0) = (\bar{u}_{2 \times 2}, 0_{2 \times 1})^T.
\]
Then we get
\[
\left( \begin{array}{c}
\Phi_{2 \times 2} \\
\Phi_{3j}
\end{array} \right) = \left( \begin{array}{c}
1 \\
1
\end{array} \right) + \frac{1}{k} \left( \Phi_{2 \times 2}^{(1)} \right) + \cdots + \left[ \frac{1}{k^2} \left( \Phi_{1\times 1}^{(2)} \right) + \cdots \right] e^{8i\kappa t} \tag{B8}
\]

So, from equation (B 1) and the asymptotic of \( s_{\beta}(k) \) and \( s_{3\beta}(k) \), we get the asymptotic behaviour of \( c_j(t,k) \) as \( k \to \infty \),
\[
c_j(t,k) = \frac{\Phi_{j\beta}^{(1)}}{k} + \frac{\Phi_{j\beta}^{(2)}}{k^2} + \frac{\Phi_{j\beta}^{(3)}}{k^3} + \cdots . \tag{B9}
\]

References