Exactification of the Poincaré asymptotic expansion of the Hankel integral: spectacularrly accurate asymptotic expansions and non-asymptotic scales

Eric A. Galapon¹ and Kay Marie L. Martinez¹²

¹Theoretical Physics Group, National Institute of Physics, University of the Philippines, Diliman Quezon City, 1101 The Philippines
²Siliman University, Dumaguete City, 6200 The Philippines

We obtain an exactification of the Poincaré asymptotic expansion (PAE) of the Hankel integral, \( \int_{0}^{\infty} f(x) J_{\nu}(bx) \, dx \) as \( b \to \infty \), using the distributional approach of McClure & Wong. We find that, for half-integer orders of the Bessel function, the exactified asymptotic series terminates, so that it gives an exact finite sum representation of the Hankel integral. For other orders, the asymptotic series does not terminate and is generally divergent, but is amenable to superasymptotic summation, i.e. by optimal truncation. For specific examples, we compare the accuracy of the optimally truncated asymptotic series owing to the McClure–Wong distributional method with owing to the Mellin–Barnes integral method. We find that the former is spectacularly more accurate than the latter, by, in some cases, more than 70 orders of magnitude for the same moderate value of \( b \). Moreover, the exactification can lead to a resummation of the PAE when it is exact, with the resummed Poincaré series exhibiting again the same spectacular accuracy. More importantly, the distributional method may yield meaningful resummations that involve scales that are not asymptotic sequences.

1. Introduction

The Hankel integral, \( \int_{0}^{\infty} f(x) J_{\nu}(bx) \, dx \), arises naturally in many fields of application in physics [1–5]. A recent
example, which has served as the motivation behind this work, arises in the quantum tunnelling problem where the traversal time across a potential barrier appears in the form of a Hankel integral of the zeroth order [5]. On many occasions, it is desirable to obtain an asymptotic estimate of the integral for arbitrarily large \( b \), which is well known. If \( f(x) = \Phi(x) \) is infinitely differentiable at the origin, the Hankel integral has the Poincaré asymptotic expansion (PAE) [6,8–11]

\[
\int_0^\infty \Phi(x) J_\nu (bx) \, dx \sim \sum_{s=0}^\infty \frac{\Phi^{(s)}(0)}{s!} \frac{\Gamma((1/2)(s + 1))}{\Gamma((1/2)(s - 1))} \left( \frac{2}{b} \right)^{s + 1}, \quad b \to \infty.
\]

However, there is a large class of functions in the domain of equation (1.1) for which equation (1.1) itself fails to give any meaningful information on the asymptotic behaviour of the integral for arbitrarily large \( b \). This may happen, for example, when the integral is exponentially small in \( b \), so that it is beyond all orders and equation (1.1) reduces to \( \int_0^\infty \Phi(x) J_\nu (bx) \, dx \sim 0 + 0 + 0 + \cdots \) as \( b \to \infty \). A well-known example of this case is the class of integrals \( I(\omega) = \int_0^\infty e^{-x^2} g(x^2) J_\nu (\omega x) \, dx \) which arises from high-energy nuclear physics [6,12–15]. Also for integer values of \( \nu \), equation (1.1) terminates, giving the impression that the integral reduces to a polynomial in inverse power of \( b \) in the asymptotic limit. However, in general, there are trailing exponentially small terms that are not detected by the PAE (1.1).

Various methods have been developed in obtaining the asymptotic behaviours, and explicit asymptotic expansions of Hankel integrals that are outside the range of the PAE (1.1). For example, Frenzen & Wong [6,14] used the method of shifting the contour of integration in the complex plane (now a standard method in exponential asymptotics [16]) to investigate and obtain the asymptotic behaviour of the class of Hankel integrals \( I(\omega) \). They showed that the asymptotic behaviour of \( I(\omega) \) depends on the analytical properties of the function \( f(z) \) in the complex plane. For \( f(z) \) having algebraic and logarithmic singularities, they developed a method to obtain the explicit asymptotic expansion. When \( f(z) \) is holomorphic in the upper complex plane, they obtained asymptotic expansion with a trailing exponentially small term whose order is only given. But, when \( f(z) \) is an entire function, they obtained only an order estimate, so that no explicit asymptotic expansion was available. Gabutti & Lepora [15] later tackled the same integral and its generalization given by the integral \( \int_0^\infty f(x^2) J_\nu (\omega x) x^{\nu + 1} \, dx \). Their method, which is based on the Laplace transform, gives explicit asymptotic expansions for a limited class of exponentially small Hankel integrals. A powerful method of obtaining explicit asymptotic expansion of a Hankel integral is the Mellin–Barnes method [7], which is a modification of the classical Mellin transform method [6,17,18] based on the asymptotic expansion of the ratio of products of gamma functions [19–23]. This method is capable of obtaining explicit expansions where the method of Frenzen and Wong can give only order estimates. In fact, asymptotic expansions of the examples of this paper can be obtained through the Mellin–Barnes method and will be compared with our results.

However, the above methods are at most prescriptive, providing a means of obtaining the asymptotic expansion of a given specific Hankel integral. What we wish to see is a sufficiently general single expression that includes the dominant and subdominant contributions. In Dingle’s terms [24], we wish to obtain an exact asymptotic expansion for the Hankel integral or to exactify the PAE (1.1) [25,26]. In this paper, we wish to obtain the exactification of (1.1) for the Hankel integral

\[
I(b) = \int_0^\infty \Phi(x) J_\nu (bx) \, dx,
\]

for arbitrary real \( \nu \) and arbitrary function \( \Phi(x) \) in the Schwartz space \( S(\mathbb{R}^+) \) and positive \( b \). We accomplish this by means of the distributional approach in asymptotics developed by McClure & Wong [6,27], which was first used in deriving explicit error terms for the Stieltjes transform [27], and then later applied to the Fourier transform [28], and extended to a large class of integral transforms [9]. Here, we will directly use the original results of McClure and Wong in obtaining
the asymptotic expansion of the Hankel integral $I(b)$. We will obtain the result
\[
\int_0^\infty \Phi(x)f_{\nu}(bx)\,dx = \frac{1}{2} \sum_{s=0}^\infty \Phi^{(s)}(0) \frac{\Gamma((1/2)(v+s+1))}{\Gamma((1/2)(v-s+1))} \left(\frac{2}{b}\right)^{s+1} + \Delta_v(b), \quad b \to \infty, \tag{1.3}
\]
where the first term is the known PAE (1.1) applied to the $\Phi(x)s$; and the second term is the subdominant correction to the standard Poincaré expansion, carrying the exponentially small or beyond-all-order parts of the Hankel integral. The equality in equation (1.3) indicates that no subdominant term is dropped.

The exactified PAE (1.3) will lead us to consider formal series representation of a function $F(z)$ of the form $F(z) = \sum_{n=0}^\infty a_n \phi_n(z)$ as $z \to \infty$, where the $a_n$s are constants independent of $z$ and the $\phi_n(z)$s comprise a sequence of functions of $z$. In this paper, we will refer to the sequence $\{\phi_n(z)\}$ as a scale for the function $f(z)$ or simply a scale. In addition, we will refer to a scale of power-type if $\phi_{n+1}(z)/\phi_n(z) = z^{-\lambda_{n+1}+\lambda_n}$ for some fixed $\lambda_n$ and $j=1,2,\ldots$, otherwise, non-power-type. The scale $\{\phi_n(z)\}$ is referred to as an asymptotic sequence [6] or asymptotic scale [7] if $\phi_{n+1}(z) = o(\phi_n(z))$ as $z \to \infty$. In [29], a scale satisfying the conditions $\phi_{n+1}(z) = O(\phi_n(z))$ and $\phi_{n+2}(z) = o(\phi_n(z))$ as $z \to \infty$ is referred to as a semiasymptotic sequence. Here, equation (1.3) will impose on us to admit scales that are neither asymptotic nor semiasymptotic scale. This clearly muddies the definition of generalized asymptotic series, whose foundation rests on asymptotic scales [7] or its recent suggested further generalization [29], as the formal series $F(z) = \sum_{n=0}^\infty a_n \phi_n(z)$ is no longer necessarily an asymptotic series in the generalized Poincaré sense [6,7]. It is not our objective here to introduce a general theory of asymptotic series that goes beyond the generalized Poincaré series. It is our goal to motivate its further generalization by working out the consequences of the exactified Hankel integral. For our present purposes, we will refer to a formal divergent series as an asymptotic series if it manifests the behaviour of initial convergence, followed by eventual divergence with increasing number of terms in the series for sufficiently large values of the asymptotic parameter.

A given function may be expanded in different scales, potentially in infinitely many ways. Two such expansions are referred to as transformations of each other if one can formally manipulate one to assume the other. For the Hankel integral, we will find that the McClure–Wong method leads to an asymptotic series in a scale other than the asymptotic scale owing to the Mellin–Barnes method. The latter is generally expressed in non-power-type scales, such as the special functions of mathematical physics; the former is expressed in power-type scales. We will demonstrate that the two expansions are formally equivalent by showing that the McClure–Wong expansion can be reduced to the Mellin–Barnes expansion; i.e. the two expansions are transformations of each other. However, their formal equivalence lies only on the surface—beneath is a great gulf separating them in approximating the numerical value of the Hankel integral. We will demonstrate that the asymptotic series owing to the McClure–Wong method possesses a spectacular accuracy in approximating the value of the Hankel integral. Moreover, even in cases where the exactifying term vanishes, the distributional method yields an expansion expressed in mixed scales. One of the terms is in power-type scale of the PAE; the other, in non-power-type scale. We will demonstrate that these two terms can be combined in either one of the scales. Again, we will find that the expansion in non-power-type scale owing to the distributional approach yields a much more accurate approximation of the integral than of the asymptotic expansion in power-type scale of the Poincaré series. More important, we will find that the scale is not an asymptotic or a semiasymptotic scale. These results can be seen as a specific generalization of the known transformations of the PAEs first considered in detail in [30] and further explored in [31–35].

The paper is organized as follows. In §2, we give a brief review of McClure and Wong’s distributional approach; we limit our discussion to what is relevant to the problem at hand. In §3, we apply the distributional method to the Hankel integral to obtain an asymptotic expansion. In §4, we consider different special cases of the application of the exactified PAE and give specific examples. In §5, we demonstrate the spectacular numerical accuracy of the asymptotic expansion owing to the McClure–Wong method compared with that owing to the Mellin–Barnes method.
In §6, we reconsider a known Hankel integral which has a PAE which is exact. We show how the distributional method allows one to resum the series in a specific scale. We will find that the Hankel integral is rewritten in a scale which is not asymptotic but numerically meaningful. In §7, we conclude and discuss briefly the implications of our results to hyperasymptotics and the general theory of asymptotic expansions. A electronic supplementary material is provided to give details to some important derivations.

2. The distributional approach in asymptotics

The basic idea in obtaining the asymptotic expansion of \( I(b) \) for \( b \to \infty \) is to substitute the asymptotic expansion of the Bessel function \( f_s(bx) \) for large \( b \) back into the integral (1.2), and then perform the required integration. However, this leads to the divergent integrals

\[
\int_0^\infty e^{\pm i b x} \phi(x)x^{-k-1/2} \, dx
\]

for positive integer \( k \). One may remedy this by using analytical extension theory, i.e. by restricting \( k \) to values that the integral exists, and then extending the result to the divergent integral by substituting \( k \) with positive integer values. But, naive application of analytical extension theory has been demonstrated to yield incomplete asymptotic expansions—some terms are missing in the expansion [6].

The distributional approach of McClure and Wong gives the proper treatment and interpretation to such divergent integrals. Their idea is to interpret the integrands as distributions and the integral as functionals on them. Only after such identifications that analytical extension can be applied on divergent integrals arising from the method. We will show that applying the distributional approach to our divergent integrals give explicit asymptotic expansions for \( I(b) \). In the following, we summarize the distributional method. We restrict the summary to what is relevant to the Hankel integral. We refer the reader to [6] for a thorough discussion of the distributional method.

Consider a locally integrable function \( f(x) \) on the interval \([0, \infty)\) with polynomial growth at infinity. We now give the distributional meaning to the integral

\[
\int_0^\infty \Phi(x)f(x) \, dx,
\]

where \( \Phi(x) \) is an arbitrary function in the Schwartz space \( S \). For each \( n \geq 1 \), let \( f(x) \) have the asymptotic expansion

\[
f(x) = \sum_{s=0}^{n-1} a_s e^{c x} x^{-\alpha} + f_n(x), \quad x \to \infty,
\]

where \( 0 < \alpha < 1 \) and \( c \) is real. Let \( e_s(x) = e^{c x} x^{-\alpha} \). From [27], the functions \( f(x) \), \( e_s(x) \) and \( f_n(x) \) generate distributions on \( S \) defined as follows:

\[
\langle f, \Phi \rangle = \int_0^\infty f(x) \Phi(x) \, dx, \quad (2.2)
\]

\[
\langle e_s, \Phi \rangle = \frac{1}{s!} \int_0^\infty \Phi^{(s+1)}(x) \int_x^\infty (\tau - x)^s e_s(\tau) \, d\tau \, dx, \quad (2.3)
\]

and

\[
\langle f_n, \Phi \rangle = (-1)^n \int_0^\infty f_n(x) \Phi^{(n)}(x) \, dx = R_n, \quad (2.4)
\]

where \( f_{0,n} = ((-1)^n/(n-1)) \int_x^\infty (\tau - x)^{n-1} f_n(\tau) \, d\tau \) and \( R_n \) is the remainder term.

The usefulness of these distributions rests on their exact relationship. They are related according to

\[
\langle f, \Phi \rangle = \sum_{s=0}^{n-1} a_s \langle e_s, \Phi \rangle - \sum_{s=0}^{n-1} b_s \langle \delta^{(s)}, \Phi \rangle + \langle f_n, \Phi \rangle, \quad (2.5)
\]

where

\[
b_s = \frac{(-1)^{s+1}}{s!} M[f; s + 1] - \sum_{k=0}^{s-1} a_k \frac{\Gamma(s-k-\alpha+1)}{(c/i)^{s-k-\alpha+1}}, \quad (2.6)
\]

and \( \delta^{(s)} \) is the \( s \)th derivative of the Dirac delta function at the origin, and \( M[f; z] \) is the Mellin transform or its analytic extension when the integral diverges.
3. Asymptotic expansion of the Hankel integral

We now apply the distributional approach to the Hankel integral (1.2) for large values of parameter $b$. In order to obtain an expansion in the form of equation (2.1), we write the Bessel function in terms of the Hankel functions using the relationship $2I_\nu(z) = H^{(1)}_\nu(z) + H^{(2)}_\nu(z)$ [36]. The Hankel integral (1.2) then splits into two integrals, $I(b) = I_1(b) + I_2(b)$, with $I_l(b) = \frac{1}{2} \int_0^\infty \Phi(x) H^{(l)}_\nu(bx) \, dx, \ l = 1, 2$. We will apply the distributional approach to the integrals $I_l(b)$ with the identifications $f^{(l)}(x) = H^{(l)}_\nu(bx)$.

Using the known asymptotic expansions of the Hankel functions for large argument [36], we have the following asymptotic expansions for $f^{(l)}(x)$

$$f^{(l)}(x) = H^{(l)}_\nu(bx) = \sum_{s=0}^{n-1} a^{(l)}_s e^{(-1)^{l+1}ibx} x^{-s-1/2} + f^{(l)}_n(x), \ x \to \infty,$$

where the coefficients are given by

$$a^{(l)}_s = \frac{\sqrt{\frac{\pi}{2}}}{2^s s!} \frac{\Gamma(v + k + 1/2)}{\Gamma(v - k + 1/2)} e^{(-1)^{l+1}i\pi/2} \frac{1}{b^{s+1/2}},$$

for $l = 1, 2$.

Now let $e^{(l)}_s(x) = e^{(-1)^{l+1}ibx} x^{-s-1/2}$ and define the distributions according to equations (2.2)–(2.4). We have to consider different distributions corresponding to the two Hankel functions separately. The full expansion for the integral $I(b)$ is obtained by adding these distributions according to

$$I(b) = \frac{1}{2} \left[ \sum_{s=0}^{\infty} a^{(1)}_s \langle e^{(1)}_s, \Phi \rangle + a^{(2)}_s \langle e^{(2)}_s, \Phi \rangle - (-1)^s \Phi^{(s)}(0) B_s \right],$$

where $B_s = b^{(1)}_s + b^{(2)}_s$, and $b^{(l)}_s$, for $l = 1$ and $l = 2$, is obtained from equation (2.6) by substituting the function $f$ with $H^{(l)}_\nu$; we have already extended the summation to infinity in equation (3.3). Equation (2.3) can be further simplified by interchanging the order of integration and performing integration-by-parts to the inner integral. We then have

$$\langle e^{(l)}_s, \Phi \rangle = i^{l+1} \int_0^\infty e^{(-1)^{l+1}it^2} \Phi(t) \, dt,$$

where we have let

$$i^{l+1}_s(b) = \int_0^\infty e^{(-1)^{l+1}ibt} t^{-s-1/2} \Phi(t) \, dt.$$

The integrals $i^{l+1}_s$, which are Mellin transforms, are generally divergent, and they have to be understood as analytic extensions of their convergent versions. For the Mellin transform in equation (2.6), we note that $f^{(1)}(x) + f^{(2)}(x) = 2I_\nu(bx)$. Then, from [6, lemma 4, p. 203], we have

$$M[f^{(1)} + f^{(2)}; s + 1] = \frac{\Gamma((1/2)(v + s + 1))}{\Gamma((1/2)(v - s + 1))} \left( \frac{2}{b} \right)^{s+1}.$$  

Substituting equations (3.4)–(3.6) back into equation (3.3) and performing simplifications, we obtain our main result given by equation (1.3), where the exactifying term is given by

$$\Delta_v(b) = - \sum_{s=0}^{\infty} (-1)^s \frac{\Phi^{(s)}(0)}{s!} \frac{\Gamma((1/2)(v + s + 1))}{\Gamma((1/2)(v - s + 1))} \cos\left( \frac{\pi}{2} (s - v) \right) \sin\left( \frac{\pi}{2} (v + s + 1) \right) \left( \frac{2}{b} \right)^{s+1}$$

$$+ \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{\infty} \frac{\Gamma(v + s + 1/2)}{2^s s! \Gamma(v - s + 1/2) b^{s+1/2}} [e^{i(\pi/2)(s-v-1)b} i^{(1)}_s(b) + e^{-i(\pi/2)(s-v-1)b} i^{(2)}_s(b)].$$
We can further simplify equation (1.3) by expanding the trigonometric factors in the first term of equation (3.7). We find that the Poincaré term gets cancelled out, leaving the expression

\[
I(b) = -\frac{\cos(\pi \nu)}{2} \sum_{s=0}^{\infty} \frac{(-1)^s \Phi^{(s)}(0) \Gamma((1/2)(\nu + s + 1))}{s! \Gamma((1/2)(\nu - s + 1))} \left(\frac{2}{b}\right)^{s+1} \\
+ \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{\infty} \frac{\Gamma(\nu + s + 1/2)}{2^s s! \Gamma(\nu - s + 1/2) b^{s+1/2}} [\cos(\pi/2)(s-\nu-1/2)I_2^{(1)}(b) + \cos(\pi/2)(s-\nu-1/2)I_2^{(2)}(b)].
\] (3.8)

It is this form of the asymptotic expansion that we will use in the sections to follow.

Observe that expansion (1.3) has two distinct terms. The first term is the standard PAE of the Hankel integral, and the second term is the asymptotic component. The process of obtaining the second term involves adding these to the standard Poincaré expansion in the sense of Dingle [24]: an asymptotic series is said to be complete when the subdominant term in the expansion allows us to view equation (3.8) as the completion of the PAE for the Hankel integral.

The above results can be extended to two fronts: first, we can extend the above treatment for \( f(x) = \Phi(x)x^{-\lambda} \), where \( \Phi(x) \) belongs to \( S(\mathbb{R}^+) \), \( 1 > \lambda > 0 \), \( (\nu - \lambda) > -1 \) and \( \Phi(0) \neq 0 \). The only required modification is the identification \( f^{(k)}(x) = x^{-\lambda}H_k^\nu(bx) \). The relevant distributions are \( e^{\pm ibx}x^{-s-\lambda-1/2} \). For \( \lambda \neq 1/2 \), we have the same definitions as in §2; for \( \lambda = 1/2 \), we have a special case that can be treated separately [6]. Second, the application of McClure and Wong’s distributional approach is not limited to the Hankel integral. It may be applied to the general class of integral transforms \( \int_0^\infty \Phi(x)f(bx) \, dx \), where \( f(x) \) has the asymptotic expansion given by equation (2.1), where \( \alpha \) can now be in the interval \( (0, 1] \) and \( c \) can now be zero. The application of the distributional approach for such an integral proceeds as in the Hankel integral: expanding \( f(bx) \) asymptotically as \( b \to \infty \) inside the integral, followed by identifying each integral term that appears after the expansion as a distribution, and finally applying the appropriate relationship among the distributions [6].

4. Special cases and examples

(a) Finite series representation of the Hankel integral

When \( \nu = n + 1/2 \), with \( n = 0, 1, \ldots \), the first term of the expansion (3.8) vanishes, and the second term terminates at the term \( s = n \), yielding

\[
I(b) = \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{n} \frac{1}{2^s s! \Gamma(\nu - s + 1/2) b^{s+1/2}} [\cos(\pi/2)(s-\nu-1/2)I_2^{(1)}(b) + \cos(\pi/2)(s-\nu-1/2)I_2^{(2)}(b)].
\] (4.1)

Because the asymptotic series (3.8) terminates and (3.8) itself is exact, equation (4.1) is then equal to the integral itself. Equation (4.1) provides a finite series representation of the Hankel integral for half-integer orders of the Bessel function.

(i) Example

Let us consider the Hankel integral

\[
h_n(b) = \int_0^\infty e^{-x}J_{n+1/2}(bx) \, dx = \frac{b^n \sqrt{b}}{\sqrt{b^2 + 1}(\sqrt{b^2 + 1} + 1)\sqrt{\sqrt{b^2 + 1} + 1}}, \quad b > 0.
\] (4.2)
which can be obtained by a term-by-term integration of the series expansion of the Bessel function, and then summing the resulting series. The integrals \( I_s^{(1)}(b) = I_s^{(2)*}(b) = \int_0^\infty e^{(ib-1)x}x^{-s-1/2} \, dx \) converge for \( s = 0 \) and diverge for all \( s > 0 \). We assign values to them by means of the analytic extension of the integral

\[
\int_0^\infty e^{(ib-1)x}x^{-\lambda-1/2} \, dx = (1 - ib)^{-1/2 + \lambda} \Gamma \left( \frac{1}{2} - \lambda \right), \quad \text{Re} \lambda < \frac{1}{2}, \quad b \in \mathbb{R},
\]

which is obtained by a simple change of variable in the complex plane.

With the substitution \( \lambda = s \) in the right-hand side of equation (4.3), we obtain the values of \( I_s^{(1)}(b) \) and \( I_s^{(2)}(b) \). Substituting these integrals back into equation (4.1), we arrive at

\[
h_n(b) = \frac{1}{\sqrt{2\pi}} \sum_{s=0}^n \frac{1}{2^s s! ((n-s)!)^2} \left( \frac{\Gamma(1/2 - s)}{b^{s+1/2}} \right) 
\times \left[ e^{i(\pi/2)(s-n-1)}(1 - ib)^{s-1/2} + e^{-i(\pi/2)(s-n-1)}(1 + ib)^{s-1/2} \right].
\]

The equality of equations (4.2) and (4.4) can be established with the help of the identity

\[
\sqrt{1 + ib} = \frac{1}{\sqrt{2}} \left[ \sqrt{\sqrt{1 + ib^2} + 1 + i\sqrt{\sqrt{1 + ib^2} - 1}} \right], \quad b > 0,
\]

with \( \sqrt{1 - ib} \) obtained from equation (4.5) by complex conjugation. This example illustrates how the distributional method, in particular equation (4.1), may yield a finite series representation of the value of a Hankel integral.

(b) Beyond all orders Hankel integrals

When \( \nu = 0 \), odd terms in the PAE vanish by virtue of the factor \( 1/\Gamma((1 - s)/2) \), and only even terms contribute. Hence, when \( \Phi(x) \) is odd in \( x, \Phi(-x) = -\Phi(x) \), the even terms also vanish, and all terms of the PAE entirely vanish. The same is true when the derivative of \( \Phi(x) \) of all orders vanishes at the origin, \( \Phi^{(n)}(0) = 0 \). Under these conditions, the first term of equation (3.8) also vanishes, so that the asymptotic expansion (3.8) reduces to

\[
I(b) = \frac{1}{\sqrt{2\pi}} \sum_{s=0}^\infty \frac{(-1)^s[(1/2)s]^2}{2^s s! b^{s+1/2}} [e^{i(\pi/2)(s-1/2)} I_s^{(1)}(b) + e^{-i(\pi/2)(s-1/2)} I_s^{(2)}(b)].
\]

The vanishing of the PAE implies that the Hankel integral is beyond all orders or exponentially small. The class of integrals \( I(\omega) = \int_0^\infty e^{-ax^2} \sin(x\omega) J_0(bx) \, dx \) belongs to this case.

(i) Example

Let us consider the Hankel integral

\[
I_1(b) = \int_0^\infty e^{-cx^2} \sin(ax)J_0(bx) \, dx
\]

for real \( a, b \) and \( c \). This integral is proportional to the ‘effective index of refraction’ of a Gaussian wavepacket incident upon a barrier potential [5]. To obtain the asymptotic expansion of \( I_1(b) \) for large \( b \), let us instead obtain the asymptotic expansion of the integral \( \tilde{I}_1(b) = \int_0^\infty e^{-x^2 + iax} J_0(bx) \, dx \). Integral (4.7) is obtained by taking the imaginary part of the integral \( \tilde{I}_1(b) \). To evaluate the integrals \( I_s^{(1,2)} \), we use the integral identity [37, p. 365, no. 3.462(1)],

\[
\int_0^\infty x^\lambda-1 e^{-\beta x^2-\gamma x} \, dx = (2\beta)^{-\lambda/2} \Gamma(\lambda) \exp\left( \frac{\gamma^2}{8\beta} \right) D_{-\lambda} \left( \frac{\gamma}{\sqrt{2\beta}} \right)
\]
for $\text{Re}(\lambda) > 0$ and $\text{Re}(\beta) > 0$, where $D_\rho(z)$ is the parabolic cylinder function (PCF). The integrals $I_{s}^{(1,2)}$ are then obtained by analytically extending (4.8) for negative values of $\lambda$. We then have

$$I_{s}^{(1,2)} = \frac{\Gamma(1/2-s)}{(2\pi)^{1/4-s/2}} e^{-(a+b)^2/8c} D_{s-1/2} \left[ -i(a \pm b) \sqrt{2c} \right], \quad (4.9)$$

in which the upper (lower) sign corresponds to $I_1$ ($I_2$).

We next evaluate $\Phi^{(s)}(0)$ for $\Phi(x) = e^{-ax^2+ibx}$. Using the known Rodriguez representation of the Hermite polynomials, $H_{n}(z) = (-1)^n e^{-z^2/2} e^{-z^2}$, we obtain

$$\Phi^{(s)}(0) = \sum_{r=0}^{[s/2]} \binom{s}{2r} e^r H_{2r}(0)(ia)^{s-2r}, \quad (4.10)$$

where $H_{2r}(0) = (-1)^{2r}/r!$ is a Hermite polynomial evaluated at zero, and $[z]$ is the integer part. For $\nu = 0$, we are particularly interested in the even derivatives of the test function $\Phi(x)$. Plugging these results into equation (3.8) gives us a full expansion for the Hankel integral $I_1(b)$. For $I_1(b)$, we are interested in the imaginary part, so we use the functional identity of the PCF in [37, p. 1030, no. 9.248(1)]

$$D_{-p-1}(-iz) = e^{i(p-1)/2} \Gamma(-p) \frac{D_p(z) - e^{i\pi p} D_p(-z)}{\sqrt{2\pi}}, \quad (4.11)$$

Then, because $\Phi^{(2s)}(0)$ is real for all $s$, the imaginary part of $I_1(b)$, which is the integral we seek, is given by

$$I_1(b) = \sum_{s=0}^{\infty} (-1)^s \frac{(1/2)^2(\sqrt{2})^{s-1/2}}{2^{s+1} \sqrt{s!} b^{s+1/2}} \left[ \frac{D_{s+1/2}((b-a)/\sqrt{2c})}{e^{b^2/2c}} - \frac{D_{s-1/2}((b+a)/\sqrt{2c})}{e^{a^2/2c}} \right]. \quad (4.12)$$

Observe that equation (4.12) is expanded in terms of the non-power-type scale given by

$$\phi_s(b) = \frac{1}{b^{s+1/2}} \left[ e^{-(b-a)^2/2c} D_{s-1/2} \left( \frac{b-a}{\sqrt{2c}} \right) - e^{-(b+a)^2/2c} D_{s-1/2} \left( \frac{b+a}{\sqrt{2c}} \right) \right] \quad (4.13)$$

$s = 0, 1, 2, \ldots$ Using the known asymptotic expansion for the PCF, it can be shown that this sequence of functions satisfies $\phi_{s+1}(b)/\phi_s(b) = O(b^{-2})$, as $b \to \infty$ so that $\phi_{s+1}(b) = o(\phi_s(b))$ as $b \to \infty$; i.e. sequence (4.13) is an asymptotic sequence. We can consider the sequence $\phi_s(b)$ as a compound sequence and consider each term separately, which are on their own non-power-type scales. We find that each is an asymptotic sequence.

We now wish to obtain the asymptotic expansion for $I_1(b)$ in power-type scale. We accomplish this by using the asymptotic expansion of $D_v(z)$ given by

$$D_v(z) = z^v e^{-z^2/4} \sum_{k=0}^{\infty} \frac{(-1)^k(-v/2)(1-v/2)k2^k}{k!z^{2k}}, \quad |z| \to \infty \quad (4.14)$$

for $|\arg z| < 3\pi/4$. Note that we have used the equality sign in (4.14) instead of the usual asymptotic symbol $\sim$ to indicate that no subdominant term is dropped in the expansion, i.e. it is complete in the sense of Dingle [24]. By expanding the resulting $(b-a)^{s-2k-1/2}$ and $(b+a)^{-s-2k+1/2}$ terms binomially for large $b$ and collecting terms in powers of $b$, we obtain

$$I_1(b) = \frac{1}{2b} \sum_{m=0}^{\infty} \frac{(1/2)^m}{m!} 3F_1 \left[ \begin{array}{c} -m, -m, m/2 \vspace{0.1cm} \\
1 \\
2 - m \end{array} \right] \left( \frac{a}{b} \right)^m e^{-(b-a)^2/4c} - (-1)^m e^{-(b+a)^2/4c}, \quad (4.15)$$

where $3F_1$ is a specific hypergeometric function. Equation (4.15) is in compound power-type scale. In the provided electronic supplementary material, it is shown that equation (4.15) is equal to the asymptotic expansion owing to the Mellin–Barnes method.
(c) Terminating Poincaré asymptotic expansions

When the order, \( \nu \), of the Bessel function is any positive integer, the PAE terminates, and the distributional method gives a trailing beyond all order correction terms. In particular, for \( \nu = 2n \) and \( \Phi^{(2j+1)}(0) = 0 \) (such as when \( \Phi(x) \) is odd in \( x \)) for \( j = 0, 1, 2, \ldots \), equation (3.8) becomes

\[
I(b) = \frac{1}{2} \sum_{j=0}^{n-1} \frac{\phi^{(2j+1)}(0)}{(2j+1)!} \frac{(n+j)!}{(n+j-1)!} + \frac{1}{2\pi} \sum_{s=0}^{\infty} \frac{\Gamma(2n+s+1/2)}{2^s s! \Gamma(2n-s+1/2) b^{s+1/2}}
\times \left[ e^{i\pi/2(s-2n-1/2)} I_{s}^{(1)}(b) + e^{-i\pi/2(s-2n-1/2)} I_{s}^{(2)}(b) \right].
\] (4.16)

In addition, for \( \nu = 2m+1, m = 0, 1, 2, \ldots \) and \( \Phi^{(2j+1)}(0) = 0, j = 0, 1, 2, \ldots \) (such as when \( \Phi(x) \) is even in \( x \)),

\[
I(b) = \frac{1}{2} \sum_{j=0}^{m} \frac{\phi^{(2j)}(0)}{(2j)!} \frac{(m+j)!}{(m+j-1)!} + \frac{1}{2\pi} \sum_{s=0}^{\infty} \frac{\Gamma(2m+s+3/2)}{2^s s! \Gamma(2m-s+3/2) b^{s+1/2}}
\times \left[ e^{i\pi/2(s-2m-3/2)} I_{s}^{(1)}(b) + e^{-i\pi/2(s-2m-3/2)} I_{s}^{(2)}(b) \right].
\] (4.17)

The first terms of equations (4.16) and (4.17) are the finite sum PAEs, and the second terms are the trailing beyond all order corrections.

(i) Example

Let us consider the integral

\[
I_2(b) = \int_0^\infty e^{-c x^2} \sin(ax) f_{2n}(bx) \, dx,
\] (4.18)

for \( n = 1, 2, \ldots \). We identify \( \Phi(x) = e^{-c x^2} \sin(ax) \), which is odd in \( x \). As the order of the Bessel function is even, equation (4.16) holds for this case. Following the same steps above, we arrive at the expansion

\[
I_2(b) = 2n \sum_{r=0}^{n-1} \frac{(16c)^r \Gamma(n+r+1) U[-r,3/2,-a^2/4c]}{(2r+1)! \Gamma(n-r)b^{2r+2}} + \frac{(-1)^n}{2\sqrt{c}} \sum_{s=0}^{\infty} \frac{(-1)^s (1/2 - 2n)s(1/2 + 2n)s}{s! (b\sqrt{2/c})^{s+1/2}}
\times \left[ e^{-(b-a)^2/8c} D_{s-1/2} \left( b-a \sqrt{2c} \right) - e^{-(b+a)^2/8c} D_{s-1/2} \left( b+a \sqrt{2c} \right) \right],
\] (4.19)

where \( U(a,b,z) \) is the tricomi confluent hypergeometric function. Observe that the correction to the PAE is expressed in the same asymptotic sequence as in the previous example.

For comparison later, we rewrite the exponentially small second term of equation (4.19) in power-type scale of the asymptotic variable \( b \). Again, expanding the PCF using (4.14) following the same procedure above, we arrive at

\[
I_2(b) = 2n \sum_{s=0}^{n-1} \frac{\Gamma(n+s+1)}{\Gamma(n-s)} \frac{(16c)^s U[-s,3/2,-a^2/4c]}{(2s+1)! b^{2s+2}} + \frac{(-1)^n}{2b} \sum_{m=0}^{\infty} \frac{\left( \frac{1}{2} \right)^m}{m!} \left( \frac{a}{b} \right)^m
\times \left[ e^{-(b-a)^2/4c} - (-1)^m e^{-(b+a)^2/4c} \right] \sum_{k=0}^{m/2} \frac{(-c/a^2)^k}{k!(m-2k)!} F_2 \left[ \begin{array}{c} -k, 1/2 - 2n, 1 \frac{1}{2} + 2n \frac{1}{2} - m \frac{1}{2} \end{array} ; 1 \right].
\] (4.20)

Equation (4.20) is also in compound power-type scale. This result can be obtained by the Mellin–Barnes method.

5. Numerical accuracy of the distributional asymptotic expansion

The distributional approach yields an asymptotic expansion that is formally equivalent to the asymptotic expansion in power-type scale owing to the Mellin–Barnes integral method. Here,
we demonstrate that the asymptotic expansion owing to the distributional method gives a much more accurate approximation of the value of the integral than the latter asymptotic expansion.

Let us first look at the general behaviours of the asymptotic expansion (4.12). Figure 1a shows the graph of the magnitude of the terms

\[ \lambda_s = \frac{[(1/2)_s]^2(\sqrt{2c})^{s-1/2}}{2^{s+1}s!^{s+1/2}} \left[ \frac{D_{-s-1/2}(b-a)/\sqrt{2c}}{e^{(b-a)^2/8c}} - \frac{D_{-s-1/2}(b+a)/\sqrt{2c}}{e^{(b+a)^2/8c}} \right] \]  

(5.1)

for the given parameters in figure 1. The \( \lambda_s \)s decrease initially in magnitude and then gradually increase with \( s \), which is the typical behaviour of a divergent asymptotic series. Figure 1b shows the value of the partial sums of equation (4.12) for increasing number of terms in the sum, in particular, the sum \( I_{(b)}^{(N)} = \sum_{s=0}^{N} \lambda_s \). The partial sum oscillates around the exact value, with the amplitude initially decreasing with increasing number of terms and eventually increasing without bound. The closest approach of the partial sum to the exact value occurs near the least term. The approximation to the integral owing to the expansion (4.12) is obtained by truncating it at the least term, i.e. its superasymptotic sum. The Poincaré-type expansion (4.15) is analysed in similar fashion. The asymptotic expansion (4.15) is truncated at the least term to obtain its superasymptotic approximation to the value of the integral. To compare the distributional and the Poincaré type expansions for the integral \( I_2(b) \), we compare only the correction terms to the

**Figure 1.** Behaviour of \( I_1(b) \) with increasing number of terms \( N \) in the expansion. \( a = 1, b = 2, c = 2 \). The exact value of the integral is 0.1453967.
asymptotic series in power-type scale. Comparing the two, the distributional takes much more
optimally truncated distributional expansion is much more accurate than the optimally truncated
obtained by means of arbitrary precision numerical quadrature using Maple 17. Clearly, the
asymptotic series for the integral
\[ \int_0^\infty e^{-x^2} x \sin(x) \sin(bx) dx \]

Table 1. Comparison of the relative error of the optimally truncated power-type asymptotic series and the distributional asymptotic series for the integral \( \int_0^\infty e^{-x^2} J_0(bx) \sin x \, dx \) for different values of the asymptotic parameter \( b \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>power-type</th>
<th>distributional</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{relative error} )</td>
<td>( \text{least term} )</td>
<td>( \text{relative error} )</td>
</tr>
<tr>
<td>1</td>
<td>( 8.1 \times 10^{-2} )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( 7.8 \times 10^{-2} )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( 2.8 \times 10^{-2} )</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>( 8.1 \times 10^{-3} )</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>( 1.0 \times 10^{-3} )</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>( 4.1 \times 10^{-4} )</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>( 1.9 \times 10^{-7} )</td>
<td>28</td>
</tr>
<tr>
<td>8</td>
<td>( 5.8 \times 10^{-10} )</td>
<td>30</td>
</tr>
<tr>
<td>9</td>
<td>( 1.4 \times 10^{-11} )</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>( 1.0 \times 10^{-12} )</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the relative error of the optimally truncated power-type asymptotic series and the distributional asymptotic series for the integral \( \int_0^{\infty} e^{-x^2} \sin(x)J_0(bx) \sin - 2U(0, \frac{3}{4}, -\frac{1}{4})b^{-2} \) for different values of the asymptotic parameter \( b \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>power-type</th>
<th>distributional</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{relative error} )</td>
<td>( \text{least term} )</td>
<td>( \text{relative error} )</td>
</tr>
<tr>
<td>1</td>
<td>( 7.3 \times 10^{-9} )</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>( 2.9 \times 10^{-1} )</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>( 3.3 \times 10^{-2} )</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>( 6.3 \times 10^{-3} )</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>( 1.6 \times 10^{-3} )</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>( 2.0 \times 10^{-5} )</td>
<td>21</td>
</tr>
<tr>
<td>7</td>
<td>( 8.1 \times 10^{-7} )</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>( 2.7 \times 10^{-8} )</td>
<td>44</td>
</tr>
<tr>
<td>9</td>
<td>( 1.9 \times 10^{-10} )</td>
<td>46</td>
</tr>
<tr>
<td>10</td>
<td>( 1.5 \times 10^{-12} )</td>
<td>46</td>
</tr>
</tbody>
</table>

truncated PAE. The behaviour of the asymptotic expansions is compared in the similar fashion.
We find the same behaviour of the partial sum of the expansions. Their approximations to the
value of the integral are just their optimally truncated versions.

Tables 1 and 2 compare the relative error of the optimally truncated asymptotic expansions
owing to the McClure–Wong method and owing to the Mellin–Barnes method for the indicated
parameters and for increasing values of the asymptotic parameter \( b \). The exact values are
obtained by means of arbitrary precision numerical quadrature using Maple 17. Clearly, the
optimally truncated distributional expansion is much more accurate than the optimally truncated
asymptotic series in power-type scale. Comparing the two, the distributional takes much more
6. Resummation and non-asymptotic sequence

We now apply the distributional method to an integral with a known non-trivial PAE and show how a much more accurate expansion is obtained from the PAE. Let us consider the integral

$$F_\nu(x) = \int_0^\infty \frac{f_\nu(xt)}{1+t} \, dt, \quad \nu \neq -1, \quad x > 0. \quad (6.1)$$

The PAE of this integral as $x \to \infty$ was obtained in [7] using Mellin–Barnes method. The asymptotic expansion is given by

$$F_\nu(x) = \sum_{k=0}^\infty (-1)^k \frac{\Gamma((1/2)(\nu+k+1))}{\Gamma((1/2)(\nu-k+1))} \frac{2^k}{\lambda^{k+1}}, \quad x \to \infty, \quad (6.2)$$

which differs in [7] by a shift in the index. In the following we limit ourselves to $\nu$ not equal to an integer.

Now by identification, we have $\Phi(t) = (1+t)^{-1}$ for the integral $F_\nu(x)$. It seems not appropriate to apply the method to this integral, because $\Phi(t)$ does not belong to $S(R^+)$. This is remedied by following Wong [6]. Let $\epsilon > 0$ and introduce the function $\Phi_\epsilon(t) = e^{-\epsilon t}(1+t)^{-1}$. This function belongs to $S(R^+)$. The original integral is recovered in the limit $\epsilon \to 0$. The result is the direct application of equation (3.8) to the function $(1+t)^{-1}$.

To proceed, we compute for the integrals $I_s^{(0)} = \int_0^\infty e^{-(1)^{s+1}ix^t} t^{s-1/2} (1+t)^{-1} \, dt$ for $s = 1, 2, \ldots$ and $l = 1, 2$. Only $s=0$ is convergent; the rest of the integrals diverge and their values will be assigned by analytic continuation of the convergent integral $\int_0^\infty e^{\pm ix^t} t^{-1} (1+t)^{-1} \, dt$, for $x \in \mathbb{R}$ and $0 < \text{Re} \lambda < 1$. An appropriate rotation of the contour of integration along the complex axis allows us to exploit the following integral identity [37] $\int_0^\infty e^{-\mu t} t^{\nu} (1+\beta t)^{-1} \, dt = \beta^{\nu-1} e^{\beta \nu} \Gamma(\nu) \Gamma(1-\nu, \beta \mu)$, for $|\arg \beta| < \pi$, $\text{Re} \mu > 0$, and $\text{Re} \nu > 0$, where $\Gamma(a,z)$ is the incomplete gamma function.

Appropriate identifications of the parameters of the integral identity for the integrals $I_s^{(1/2)}(x)$ yield the desired integrals. Substituting the $I_s^{(1/2)}(x)s$ and $\Phi^{(s)}(0) = (-1)^s s!$ back into equation (3.8) give the expansion

$$F_\nu(x) = -\cos(\pi \nu) \sum_{s=0}^\infty \frac{\Gamma((1/2)(\nu+s+1))}{\Gamma((1/2)(\nu-s+1))} \frac{2^s}{\lambda^{s+1}} + \frac{1}{\sqrt{2\pi}} \sum_{s=0}^\infty \frac{\Gamma(\nu+s+1/2) \Gamma(1/2-s)}{2^s \Gamma(\nu-s+1/2) x^{s+1/2}}$$

$$\times \left[ e^{i(\pi/2)(\nu-s-1/2)} e^{-ix} \Gamma\left(\frac{1}{2} + s, -ix\right) + e^{-i(\pi/2)(s-\nu-1/2)} e^{ix} \Gamma\left(\frac{1}{2} + s, ix\right) \right]. \quad (6.3)$$

On its own, it is not clear how to interpret equation (6.3) or how it is even useful numerically because it is composed of two terms expressed in two different scales.

We appreciate the content of equation (6.3) by rewriting one term in the scale of the other, so that the entire expansion $F_\nu(x)$ is expressed in one scale. Rewriting the second term in the scale of the first term is straightforward by replacing the incomplete gamma function by its (Poincaré) asymptotic expansion in the sector $|\text{Arg} z| < 3\pi/2$. In doing so, we recover the PAE (6.2). The PAE for this case is then exact—there is no neglected subdominant term in the expansion. On the other hand, we reverse the process in rewriting the first term to combine it with the second term. That is we manipulate the first term such that the resulting series will be in terms of the (Poincaré) asymptotic expansion of the incomplete gamma function, and then replace the divergent asymptotic expansion with its analytic sum, which is the incomplete gamma function.
for $s \Gamma_j$ using the theory of distribution has led to the recovery of the PAE of the Hankel integral plus integral. The proper treatment of divergent integrals arising from term-by-term integration itself. The result is

$$F_v(x) = \csc(v\pi) \sum_{s=0}^{\infty} \frac{1}{(2x)^{s+1/2}} \frac{(1/ - v)_s (1/2 + v)_s}{2^s s!(1/2)_s} \times \left[ e^{i(s/2)(s+v-3/2)} e^{-ix} \Gamma^s \left( \frac{1}{2} + s, -ix \right) + e^{-i(s/2)(s+v-3/2)} e^{ix} \Gamma^s \left( \frac{1}{2} + s, ix \right) \right]. \quad (6.4)$$

Details in arriving at equation (6.4) is provided in the electronic supplementary material.

The expansions (6.2) and (6.4) are formally equivalent—one is just the rearrangement of the other. However, from the standpoint of providing a numerical approximation of the value of integral, they are not equivalent. In table 3, we compare the relative error of the optimally truncated expansions (6.2) and (6.4). We find, as in the previous two examples, that the distributional approach gives an asymptotic expansion that yields more accurate approximations than the PAE. The steps leading to equation (6.4) gives us an idea why the resummed version of the PAE is more accurate. In replacing the divergent asymptotic expansion with its corresponding analytic sum (the incomplete gamma function), we have actually included the contribution of the tail of the asymptotic series, which is excluded by superasymptotic expansion of the PAE. The expansion (6.4) then includes contributions from terms that are excluded by the PAE. (See the electronic supplementary material to fully appreciate this statement.)

In the previous two examples, the asymptotic expansions owing to distributional approach are asymptotic sequences. This is not true for the present example. Consider the sequence

$$\varphi_s(x) = \frac{1}{x^{s+1/2}} \left[ e^{i(s/2)(s+v-3/2)} e^{-ix} \Gamma^s \left( \frac{1}{2} + s, -ix \right) + e^{-i(s/2)(s+v-3/2)} e^{ix} \Gamma^s \left( \frac{1}{2} + s, ix \right) \right] \quad (6.5)$$

for $s = 0, 1, 2, \ldots$. From the leading-order asymptotic behaviour of the incomplete gamma function, $\Gamma(a, z) \sim z^{a-1} e^{-z}$ as $z \to \infty$, we have $\varphi_{s+j}(x) \sim 2 \sin(\pi v/2)/x$ as $x \to \infty$ for all $j = 0, 1, 2, \ldots$, so that we have the limit $\lim_{x \to \infty} \varphi_{s+j}(x)/\varphi_s(x) = 1$, for $j = 0, 1, 2, \ldots$, which implies that $\varphi_{s+j}(x) \neq o(\varphi_s(x))$ as $x \to \infty$ for all $j > 0$. Hence, sequence (6.5) is not an asymptotic sequence nor a semiasymptotic one.

### 7. Conclusion

We have applied the distributional method of McClure and Wong in exactifying the Hankel integral. The proper treatment of divergent integrals arising from term-by-term integration using the theory of distribution has led to the recovery of the PAE of the Hankel integral plus

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\varphi_s(x)$</th>
<th>$\varphi_{s+j}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$4.0 \times 10^{-5}$</td>
<td>$1.1 \times 10^{-9}$</td>
</tr>
<tr>
<td>15</td>
<td>$2.3 \times 10^{-6}$</td>
<td>$3.1 \times 10^{-13}$</td>
</tr>
<tr>
<td>20</td>
<td>$3.0 \times 10^{-8}$</td>
<td>$1.3 \times 10^{-17}$</td>
</tr>
<tr>
<td>25</td>
<td>$1.8 \times 10^{-9}$</td>
<td>$1.5 \times 10^{-21}$</td>
</tr>
<tr>
<td>30</td>
<td>$2.5 \times 10^{-11}$</td>
<td>$8.8 \times 10^{-26}$</td>
</tr>
<tr>
<td>35</td>
<td>$1.5 \times 10^{-12}$</td>
<td>$3.1 \times 10^{-30}$</td>
</tr>
<tr>
<td>40</td>
<td>$2.2 \times 10^{-14}$</td>
<td>$7.4 \times 10^{-34}$</td>
</tr>
<tr>
<td>45</td>
<td>$1.3 \times 10^{-15}$</td>
<td>$1.0 \times 10^{-38}$</td>
</tr>
<tr>
<td>50</td>
<td>$1.9 \times 10^{-17}$</td>
<td>$7.7 \times 10^{-42}$</td>
</tr>
</tbody>
</table>

### Table 3. Comparison of the relative error of the optimally truncated Poincaré series and resummed series for $v = 1/3$. 

![Table 3. Comparison of the relative error of the optimally truncated Poincaré series and resummed series for $v = 1/3$.](http://rspa.royalsocietypublishing.org/Downloaded from http://rspa.royalsocietypublishing.org/)
subdominant beyond all order terms. The treatment has yielded an asymptotic expansion that may give a finite series representation to a given Hankel integral, and an asymptotic expansion that provides numerical approximation to the value of the integral with spectacular accuracy.

Our results provide advances in two directions. First, it gives the possibility of direct construction of an exact asymptotic expansion in non-power-type scale which is already a transformation of an asymptotic expansion in power-type scale. In general, it is not difficult to see that the latter expansion (in non-power-type scale) is more accurate than the former expansion (in power-type scale): the fact that somewhere Poincaré-type asymptotic expansions arising in the intermediate steps from the power-type to the non-power-type are being replaced by their closed form analytic versions already indicate that contributions from tails of PAE, which are discarded in supersasymptotic summation, are accounted for in the transformed version of the asymptotic expansion in power-type scale. This is important as transformations into non-power-type scales may provide an alternative to hyperasymptotic summation, which has been developed to account for the contribution of the tails of Poincaré-type asymptotic expansions [38–40]. In hyperasymptotics, one needs repeated re-expansion of the remainder term to obtain more accurate approximation of the sum of the asymptotic expansion. On the other hand, the transformation to a non-power-type expansion may involve only one step, say, by direct application of the McClure and Wong method. As we have seen here, the one-time optimal truncation of the non-power-type expansion is already very accurate; this accuracy may rival the accuracy of hyperasymptotics obtained by multiple optimal truncations and repeated re-expansion of the remainder terms. It is a distinct possibility that the expansion in non-power-type scale can be subjected to hyperasymptotics as well, provided, of course, remainder terms can be explicitly determined, which we lack in the present work and hope to address elsewhere. If it is possible to do hypeasymptotics on the transformed expansion, then it is expected that more accuracy can be wrung from the expansion. Because superasymptotic summation of the expansion is already accurate, adding more accuracy may already yield an ultra-accurate sum of the divergent asymptotic series.

Second, the general theory of Poincaré-type asymptotic expansion is couched in terms of asymptotic scales [6,7]. Despite the caveats and controversies in the general theory [6], generalized asymptotic expansions in asymptotic scales are made meaningful by the fact that their expansions in such scales are unique, and unambiguous order estimates can be made on their remainder terms [7]. This commands keeping the theory in its current form. However, we have seen from our last example that the distributional method has led to transforming the PAE into a series in non-asymptotic scale. The series has behaved similarly with other asymptotic series as it demonstrated the typical behaviour of a traditional asymptotic series: the terms initially decrease in magnitude and eventually increase without bound. Optimal truncation of this series yielded an accurate approximation of the Hankel integral; this shows that asymptotic series may actually be written meaningfully in non-asymptotic scale. This is important as it is well known that generalized asymptotic expansion in asymptotic scale may be useless either analytically or numerically [6]: expansions in non-asymptotic scales may then be even meaningful, at least numerically. Altogether, our results clearly call for a re-examination of the concept of general asymptotic expansion. The numerical usefulness of the expansion that we have obtained involving non-asymptotic scale is a compelling motivation to develop further the general theory of asymptotic expansions.

Funding statement. K.L.M. was supported by DOST-SEI under the ASTHRDP. This work was supported by the University of the Philippines Research Grant Creative and Research Scholarship Fund (2011–2012).

References


