Number and twistedness of strands in weavings on regular convex polyhedra

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This paper deals with two- and threefold weavings on Platonic polyhedral surfaces. Depending on the skewness of the weaving pattern with respect to the edges of the polyhedra, different numbers of closed strands are necessary in a complete weaving. The problem is present in basketry but can be addressed from the aspect of pure geometry (geodesics), graph theory (central circuits of 4-valent graphs) and even structural engineering (fastenings on a closed surface). Numbers of these strands are found to have a periodicity and symmetry, and, in some cases, this number can be predicted directly from the skewness of weaving. In this paper (i) a simple recursive method using symmetry operations is given to find the number of strands of cubic, octahedral and icosahedral weavings for cases where generic symmetry arguments fail; (ii) another simple method is presented to decide whether or not a single closed strand can run along the underlying Platonic without a turn (i.e. the linking number of the two edges of a strand is zero, and so the loop can be stretched to a circle without being twisted); and (iii) the linking number of individual strands in an alternate ‘check’ weaving pattern is determined.

1. Introduction

Woven patterns are common in different cultures and ages of humanity: from ancient (see the German ‘wand’ (‘woven’) for wall) to modern (e.g. the Beijing Olympic Stadium) architecture or from traditional basketry to modern composite manufacturing, even to contemporary arts and crafts (for further detailed references, see [1]). Among all these examples, it was a closed basket [2], i.e. a twofold, two-way woven pattern on a cube, that gave the major inspiration...
Figure 1. Explanation of parameters \((b, c)\) in the case of a square \((a)\) and triangular \((b)\) tiling. \(b\) steps along one direction followed by \(c\) steps in another direction of the underlying tessellation of the plane (at an angle \(\pi/2\) and \(\pi/3\), respectively) leads from one polyhedral vertex to an adjacent one; swapping the parameters affects only the chirality of the pattern. The quadrangulation number \(Q\) or triangulation number \(T\) equals the face area in terms of unit tile areas, as well as the square of edge length in terms of unit tile edge lengths. Part \((c)\) illustrates a twofold, two-way regular alternate weaving on a cube with \((b, c) = (3, 2)\); dark tiles belong to an individual closed strand.

to the following discussion. Some terms used here have already been introduced in Tarnai et al. [1]; the relevant ones are repeated here:

— A wrapping of a polyhedron means a double (or triple) covering of the polyhedral surface by a set of closed strands of constant width but without respect to the order of over- or undercrossings.

— A weaving of a polyhedron means a wrapping with prescribed order of over- or undercrossings.

— A twofold, two-way weaving (e.g. figure 1c) refers to a weaving whose strands run along two (practically orthogonal) directions and produce double covering (in this sense, threefold, three-way weavings will also be referred to).

— In order to characterize the skewness of a weaving pattern according to the Goldberg–Coxeter construction [3,4], the notation \([p, q+b, c]\) is applied with the following meaning: \(p\) stands for the number of edges of a polyhedral face (e.g. \(p = 4\) for the cube wrappings) and \(q\) gives the number of square or triangular tiles (strand overlaps) that meet at tile vertices \((3+;\text{that is, three or more for a cube and }3+, 4+\text{ and }5+\text{ for a tetrahedron, octahedron and icosahedron, respectively})\). Subscripts \(b\) and \(c\) define the slope \(b/c\) or \(c/b\) of a strand to the edges of a face as shown in figure 1a,b. These last two parameters are not well defined for pentagonal faces (that is why the dodecahedron is not mentioned above) but have clear interpretation for either square or regular triangular tilings, giving also the area \(Q(b, c)\) or \(T(b, c)\) of a face in terms of the (unit) strand width. The number of strands in a complete weaving is abbreviated as \(s([p, q+b, c])\), but, where the context makes it possible, only \(s(b, c)\) is written.

Through its relationship with basketry, weaving on convex surfaces implies a structural approach to the problem: tightened closed strands wrapping a polyhedral skeleton can be used as fasteners and modelled by some kind of a truss. Its members are the straight pieces of the strand, while external supports are defined by the edges of the underlying polyhedron. In this context, the number of independent states of self-stress corresponds to the number of strands in a complete wrapping.

The problem of weavings on polyhedra is closely related to some other different fields of science: Deza & Shtogrin [5] studied four-regular graphs (i.e. having all 4-valent vertices) embedded in a sphere. Such graphs that have only triangular and quadrangular faces were
termed octahedrite graphs and it was found that an octahedrite graph has at most six central circuits (closed paths of the graph continuing at the opposite edge at all vertices). The importance of this achievement to weavings is that any crossing in a rectangular twofold, two-way weaving on a convex polyhedron corresponds to a vertex of an octahedrite and vice versa [1]; consequently, any closed weaving pattern \([4, 3+]_{b,c}\) on a cube with \(b, c\) co-primes contains at most six strands (for non-co-prime pairs \(bk, ck\), a bundle of \(k\) strands plays the same role). It is also shown in Deza & Shtogrin [5, theorem 9] that \(s(4, 3+]_{b,c}) = 4\) if \(b\) and \(c\) are odd and the same number of strands can be either 3 or 6 if one of \(b\) or \(c\) is even. It must also be mentioned here that graph theory was found to be an efficient tool in investigating other topological properties (e.g. knots of central circuits) within the framework of the same research [6].

In contrast to the purely topological approach, closed geodesics were investigated on Platonic polyhedra in Fuchs & Fuchs [7], and specifically on the dodecahedron in Fuchs [8]. The parallelism is immediate by considering that a straight strand bent along the edges of a polyhedron follows a geodesic indeed. The results of these papers confirm those obtained for the cube and, in addition, imply the following statements (with \(b, c\) co-primes for all cases):

— any closed geodesic on a tetrahedron is simple, i.e. not self-intersecting, and \(s(3, 3+]_{b,c}) = 3\);
— for the case of the octahedron, \(s(3, 4+]_{b,c}) = 3\) if \(b - c \equiv 0 \mod 3\) and 4 otherwise; and
— for the icosahedron, \(s(3, 5+]_{b,c})\) can assume values 6, 10 or 15.

In any of the results above, it remains unanswered when the number of strands is 3 or 6 for the cube and what the conditions are for different numbers for an icosahedral weaving. Although in Fuchs & Fuchs [7] there is a suggested ‘brute force’ method (to follow the path of the strand by successive matrix multiplications) to get the length of an individual strand (and hence their number), the existence of a simpler method, based only on the algebraic properties of \(b\) and \(c\), is conjectured.

This paper aims to find simple answers to the open questions above as well as to give clear geometrical interpretation of different numbers of strands. Finally, with further use of the tools to be developed here, some other questions on topology (with possible importance for fibre manufacturing) will be answered: when is it possible to wrap a convex regular polyhedron into a (single) closed strand homomorphic to an annulus, i.e. without turns? What can be stated about the ‘twistedness’ (number of complete turns around the midline) of such a closed strand in an alternate weaving?

2. Structural approach and generic number of strands

As regular polyhedra are highly symmetrical objects, it is convenient to use the symmetry-extended stress and mobility count developed in Guest & Fowler [9] for the resultant truss model. Briefly, this method provides a count of representations of self-stresses and mobilities in the symmetry group the object belongs to, and hence may give extra information (compared with the simple Maxwell count) on the existence of self-stresses and mechanisms owing to the symmetry. For details concerning basic group theory, see Bishop [10].

Let the truss model be constructed as follows: joints will correspond to crossings of the midline of strands with the edges of the polyhedron, considering all such crossings to be separate joints even if two strand midlines change direction at the same point. With the notation, for example, of figure 1, a cubic weaving has \(12 \times (b + c)\) nodes because of \(e = 12\) edges of the cube. Truss members (internal constraints, say, of type ‘i’) are put between two joints on a face if they are connected by a strand segment, so their number is \(6 \times 2 \times (b + c)\) owing to \(f = 6\) faces of the cube, each of which is doubly covered by \(b + c\) pieces of strand segments. External constraints can be added to the system by preventing displacements of joints off the edges of the (fixed) polyhedron: this is done by two single d.f. constraints for all joints in two directions (\(r\) bisects the angle of
Figure 2. External and internal constraints of the truss model: each joint (wherever the strand midline changes its direction) is supported in perpendicular directions \( r \) and \( t \) such that only displacements along the edge are allowed. For the sake of simplicity, only one joint with two adjacent edges (strand segments) is shown.

planes of two faces adjacent to the edge, \( t \) is perpendicular to both the edge and \( r \); see figure 2 for details). Obviously, there are \( 12 \times (b + c) \) of both types ‘\( r \)’ and ‘\( t \)’ of constraints.

A simple Maxwell count,

\[
\text{freedoms} - \text{constraints} = \text{mechanisms} - \text{states of self-stress},
\]

would confirm here \((3(12(b + c)) - 12(b + c) - 12(b + c) - 12(b + c) = 0)\) a trivial observation that the numbers of states of self-stress and mechanisms are equal but unknown.

Consider now a natural representation of the symmetry group of the structure (the group \( O \) for the cube). As the whole system is a compound of freedoms (of joint displacements) and constraints, let us represent them as follows: for a single joint \( j \) (spherical, hence unoriented), the trace \( \chi_j(S) \) is 1 under any symmetry operation \( S \) which leaves it unchanged (and 0 otherwise). For all constraint components (all are oriented), traces \( \chi_i(S), \chi_r(S), \chi_t(S) \) are +1 or −1 depending on whether their orientation is kept or reversed by \( S \) (and 0 otherwise). Denoting by \( \Gamma_T \) the representation pertaining to pure translation, the symmetry-adapted Maxwell rule [9] for the present structure reads as follows:

\[
\Gamma_{\text{freedoms}} - \Gamma_{\text{constraints}} = \Gamma_{\text{mechanisms}} - \Gamma_{\text{states of self-stress}}. \tag{2.1}
\]

Representations on the left-hand side can be rewritten (with appropriate abbreviation of subscripts) as \( \Gamma_{\text{fr}} = \Gamma_j \times \Gamma_T \) and \( \Gamma_{\text{cs}} = \Gamma_i + \Gamma_r + \Gamma_t \), yielding

\[
\Gamma_j \times \Gamma_T - \Gamma_i - \Gamma_r - \Gamma_t = \Gamma_m - \Gamma_s. \tag{2.2}
\]

The steps of calculation for the cube are shown in table 1.

Looking at the last row of block (a) of table 1, its reduction into irreducible representations gives

\[
\Gamma_m - \Gamma_s = -A_1 + A_2 + T_1 - T_2, \tag{2.3}
\]

which proves that representation \( \Gamma_s \) contains at least a set of self-stresses whose representation is \( A_1 + T_2 \). Knowing that the dimensions of \( A_1 \) and \( T_2 \) are 1 and 3, respectively, it proves the existence of at least four independent states of self-stress, that is, four strands. Likewise, for block (b) we have

\[
\Gamma_m - \Gamma_s = -A_1 - E + T_1, \tag{2.4}
\]

showing that \( \Gamma_s \supseteq A_1 + E \), representations of one and two dimensions, which indicates the presence of three strands in a ‘generic’ case (as will be shown in the following, these numbers are rather minimal than typical). It is known, however, that 6-tuple strands may also occur, which
Table 1. Representation of stresses and mobilities in a cube wrapping \([4, 3+1]_{bc}\). In block (a), note the character \(\chi_j(C'_j) = 4\) in \(\Gamma_j\), showing that four nodes are left unchanged by a rotation about any \(C'_j\) axis (through opposite edge midpoints). As to the constraints, no truss members lie on \(C'_j\) symmetry axes but four ‘r’ and ‘t’ constraints do; latter ones are reversed. In block (b), only two nodes and constraints of type ‘r’ are left unchanged by a rotation about any \(C'_j\) axis but four truss members remain unchanged (i.e. elongation is mapped to elongation) on each \(C'_j\) symmetry axis.

<table>
<thead>
<tr>
<th></th>
<th>(E)</th>
<th>(8C_3)</th>
<th>(6C'_2)</th>
<th>(6C_4)</th>
<th>(3C'_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_j)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_i)</td>
<td>(3)</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>(\Gamma_h)</td>
<td>(36(b + c))</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_r)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_l)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_t)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_a)</td>
<td>(36(b + c))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_m - \Gamma_i)</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) for exactly one of \(b, c\) odd:

<table>
<thead>
<tr>
<th></th>
<th>(E)</th>
<th>(8C_3)</th>
<th>(6C'_2)</th>
<th>(6C_4)</th>
<th>(3C'_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_j)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_i)</td>
<td>(3)</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>(\Gamma_h)</td>
<td>(36(b + c))</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_r)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(\Gamma_l)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_t)</td>
<td>(12(b + c))</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_a)</td>
<td>(36(b + c))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(\Gamma_m - \Gamma_i)</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

implies three extra states of self-stress pertaining to either \(T_1\) or \(T_2\), because two independent states of self-stress with full symmetry \(A_1\) cannot exist in this model (an orbit of any segment pertains to all possible distinct strands, forcing them to have equal stress in each). In these cases, a pair chosen appropriately out of the six strands will have the same symmetry properties as a single strand has in a generic case.

Although obviously insufficient to answer the question of the number of strands, this method is also applicable for a tetrahedron, octahedron and icosahedron to predict minimal numbers. Without showing all in detail, the representations for all the three cases are as follows. For the tetrahedron (group \(T\)),

\[
\Gamma_m - \Gamma_i = -A - E + T, \tag{2.5}
\]

implying three strands; for the octahedron (group \(O\)),

\[
\Gamma_m - \Gamma_i = -A_1 + A_2 + T_1 - T_2, \tag{2.6}
\]

which completely coincides with the cube with \(b, c\) odd, and thus typically suggests four strands (as it may be six instead of four, it is only possible with two more states of self-stress pertaining to the representation \(E\)). Finally, for the icosahedron (group \(I\)),

\[
\Gamma_m - \Gamma_i = -A + T_1 + T_2 - H \tag{2.7}
\]

gives typically six strands; however, there are cases of 10 (implying four extra states of self-stress pertaining to the representation \(G\)) or 15 (with three extra \(T\) or an extra \(G + H\)).
3. Periodicity and symmetry of numbers of strands

Instead of the structural approach seen before, this section provides an analysis based on pure geometry and symmetry, using the concept of development of the polyhedron along a strand (geodesic), in accordance with Fuchs & Fuchs [7] and Fuchs [8]. These works reached the conclusion that, in some cases, the length (number) of closed geodesics can be obtained by following the symmetry operations defined by the rolling of the polyhedron in different directions. This, on the one hand, requires the knowledge of the order of operations in the development \([p, q]_b,c\) (or briefly, \((b, c)\) henceforth); on the other hand, it needs \(b + c\) successive matrix multiplications. In this section, a simple and new scalar recursive method (in contrast to matrix recursion and path following) will be developed in order to find the number of closed strands for all Platonic except for the dodecahedron.

It was reported in Tarnai [11] that the number of closed strands on a cube exhibits successive symmetry (and, thus, periodicity) if one of \(b, c\) is kept constant. It will be shown here that such symmetries exist for all mentioned polyhedra; furthermore, these symmetries define the table of possible \(b, c\) pairs uniquely. For ‘historical reasons’, the discussion starts with the cube again and is followed by presenting achievements on other Platonic. Note that the dodecahedron is left out of consideration because its development results in no tiling of the plane, while all four other polyhedra develop along a regular square or triangular tiling.

(a) Symmetry of numbers of strands on the cube

Figure 3 shows the number of strands on a cube for some small co-prime pairs \(b, c\). In addition to the already mentioned discovery of symmetries, another one is easily detectable if any series parallel to the diagonals is looked at. One experiments that all horizontal series have points of symmetry at \(0, c\) and \(2c, c\), giving rise to infinitely many other points of symmetry at \(2kc, c\) (\(k\) is arbitrary integer). A similar statement holds in the horizontal direction. If a series along a northeast–southwest diagonal is considered, points of symmetry are found at \(c/2, −c/2\) and \(2c, c\), which generate other points of symmetry at \(c/2 + 3ck/2, −c/2 + 3ck/2\) (and a similar statement holds for the perpendicular direction). These observations imply two statements, as follows.

Proposition 3.1. Numbers for closed strands \((2c + n, c)\) and \((2c − n, c)\) on a cube are equal for arbitrary integer \(n\).

Proof. Consider the midline of a straight strand and the development of the cube along it. Figure 4a shows the domain of a strand midline between two edges crossed equivalently (i.e. at an equal angle and equal distance \(x\) from the next adjacent tiling vertex on the same side); such crossings will be called equivalent crossings. Because only the midline is followed, there is no restriction on the distances of edge crossings from the vertices, it is only assumed that the distance \(x\) in figure 4c is set such that the midline passes through no vertices. It is important to see that the sequence of edge crossings corresponds to a sequence of two rotations by \(\pi/2\), written as \(b\) and \(c\) according to figure 4b. Compare the developments on either side of the dashed \((2c, c)\) line in figure 4c: because of the horizontal symmetry, the sequence of operations \(c^2b\) is interrupted by \(m_i\) occurrences of \(c\) and \(c^{-1}\) on the left- and right-hand sides, respectively, at the same height (say, in the \(i\)th row; see a sequence of operations between two rotations \(b\) in figure 4d). Define two operators expressing the resultant symmetry operations for the ‘flatter’ (left) and ‘steeper’ (right) midline part as follows:

\[
o_f = c^{-m_i}(c^2b)\cdots c^{-m_i}(c^2b)
\]

(3.1)

and

\[
o_s = c^{-m_i}(c^2b)\cdots c^{-m_i}(c^2b).
\]

(3.2)

Note that none of them is necessarily equivalent to the identity (e) but, as in the group \(O\) there is a 24-tuple orbit for each line segment, both operations must correspond to some of the 24 operations in \(O\). Introduce also two possible changes on a given operation \(o\) as explained in figure 5. Let
**Figure 3.** Number of strands on a cube. For the sake of completeness, both positive and negative domains of $b$ and $c$ are shown. Small diamonds indicate points of symmetry (specifically for $c = 5$); the dashed line runs across all points of symmetry ($2c, c$) common to horizontal and northwest–southeast diagonal series.

**Figure 4.** Development of the cube along strand midlines at $(2c + n, c)$ and $(2c - n, c)$; three-dimensional view of a strand midline at $(2 \times 3 - 2, 3)$ (a); fourfold rotation operations $b$ and $c$ (b); development of both midline parts with symmetric horizontal shift $x$ from the dashed $(2c, c)$ line, zigzag lines show the top contour of faces visited (c); sequence of symmetry operations between two adjacent horizontals (in the $i$th row) of the tiling, $\sum_{i=1} m_i = n$ (d).
Figure 5. Changes in operation $o_j$: original operation $o_j$ (a); horizontally reflected operation $o_j^*$ (b); reflected, then inverted operation $o_j^{s-1}$ (c).

$o \to o^*$ mean a reflection of the original operation (figure 5a) about a vertical axis: it results in a change $c \to c^{-1}$ in $o$ (figure 5b), while the inverse operation also reverses the sign of power and order of operations within $o$. Consequently, an operation $o_j o_j^{s-1}$ corresponding to figure 5c can be written as follows:

$$o_j o_j^{s-1} = (c^{s}) (c^{2} b) \cdot (c^{2} b) \cdot (b^{s-1} c) \cdot (c^{m_1} b) \cdot (b^{s-1} c^{m_1}) \cdots .$$

(3.3)

With respect to $c^4 = e$, it follows immediately from equation (3.3) that

$$o_j o_j^{s-1} = e.$$  

(3.4)

Verbally, if a cube is rolled backwards along a horizontally reflected path according to $(2c - n, c)$ until the next equivalent crossing then forwards to $(2c + n, c)$ to the forthcoming equivalent crossing, its position is restored. Mathematically, it means that $o_j = o_j^*$, and if it corresponds to any $C$ rotation (operations in group $O$ are all rotations with the identity $E = C_1$), then $a$ strand domains make up a closed strand and the number of strands in both cases is

$$s(b, c) = \text{order}(O) / 2 \alpha ,$$

(3.5)

where the order of $O$ is 24, and the division is owing to the fact that a full 24-fold orbit of $b + c$ different (oriented) strand segments on each face would result in a fourfold covering instead of two.

**Proposition 3.2.** Numbers for closed strands $(2c + n, c + n)$ and $(2c - n, c - n)$ on a cube are equal for arbitrary integer $n$.

*Proof.* If the strand midlines are shifted by a small $x$ in both the horizontal and vertical sense from the line $(2c - n, c)$ (figure 6a), the resultant operations can be assembled of a series of operator sequences shown in figure 6b as follows:

$$o_s = (cb)^{m_1} (cbc) \cdot \cdots . (cb)^{m_1} (cbc)$$

(3.6)

and

$$o_f = (cb)^{m_1} (cbc) \cdot \cdots . (cb)^{m_1} (cbc).$$

(3.7)

Let us now define a new operator $o \to o^\mu$ where superscript ‘$\mu$’ means a diagonal reflection of the development in line $b = -c$ (technically, it means $b \to c^{-1}$ and $c \to b^{-1}$). Its inverse will then give the original operation sequence in reverse order but swapping $b$ and $c$; the product is therefore

$$o_o o_f^{s-1} = (cb)^{m_1} (cbc) \cdot \cdots . (cb)^{m_1} (cbc) \cdot (bcb)(cbc)^{m_1} \cdots . (bcb)(cbc)^{m_1},$$

(3.8)

and it is reduced again to the identity because $bcbcbc = e$ for the cube. Consequently, a ‘reflected flat rolling’ and a ‘steep rolling’ are equivalent operations and both imply the same number of strands for the reasons given in the proof of proposition 3.1.

**Proposition 3.2.** Numbers for closed strands $(2c + n, c + n)$ and $(2c - n, c - n)$ on a cube are equal for arbitrary integer $n$.

(b) Obtaining the number of strands on the cube

It is left only to see that two kinds of symmetry described above are sufficient to rebuild figure 3 from a single representative $b, c$ of all the three numbers of strands found. It will be shown that, for any co-prime pair $b, c$, the number $s(b, c)$ equals either one of $s(1, 0), s(1, 1), s(2, 1)$. Consider
Figure 6. Development of the cube along strand midlines at $(2c + n, c + n)$ and $(2c - n, c - n)$; strand midlines at $(2 \times 3 + 2, 3 + 2)$ (with operations) and $(2 \times 3 - 2, 3 - 2)$ with symmetric horizontal and vertical shift $\pm x$ (a); sequence of symmetry operations between two adjacent diagonals (in the $i$th slanted region) of the tiling, $\sum_{i=1}^{c} m_i = n$ (b).

Figure 7. Successive reflections 1 and 2 of coordinates $b, c$.

an arbitrary (but positive) co-prime pair (coordinates) $b,c$ such that $b > c$. If $b > 2c$, then reflect $b$ according to proposition 3.1 (called reflection 3.1), otherwise to proposition 3.2 (reflection 3.2); see figure 7 for illustration. Denoting new values of $b$ and $c$ by subscript $n$, the two transformations can be written as follows:

$$(b,c) \rightarrow (b_n, c_n) = (4c-b, c) \quad (3.9)$$

and

$$(b,c) \rightarrow (b_n, c_n) = (3b - 4c, 2b - 3c), \quad (3.10)$$

respectively. Evaluating now the difference of $Q(b,c)$ of the new and old coordinates (this is invariant under swapping of coordinates or change of sign of either one) we have

$$(4c-b)^2 + c^2 - (b^2 + c^2) = 8c(2c-b) \quad (3.11)$$

and

$$(3b - 4c)^2 + (2b - 3c)^2 - (b^2 + c^2) = 12(2c-b)(c-b) \quad (3.12)$$

for the two cases. If $b > 2c$, (3.11) is negative, if $2c > b > c$, (3.12) is negative; that is, the chosen reflection always leads closer to the origin. The process terminates when $b,c$ co-primes do not
Figure 8. Number of strands on an octahedron. Small diamonds indicate points of symmetry (specifically for \( c = 2 \)); the dashed line runs across all points of symmetry (\( c, c \)) common to \( b \)- and \( c \)-directed series (a); 12 symmetry-equivalent coordinate pairs with a constant triangulation number \( T \), unlabelled points are negatives of the opposite ones in both coordinates \( b \) and \( c \) (b).

satisfy either relation above, i.e. \( c = 0 \) (and hence \( b = 1 \), apart from a change of signs and order of coordinates, which are obviously possible owing to the \( C_{2v} \) symmetry of the table \( b, c \) or \( b = c \) (and they both equal 1) or \( b = 2c \) (and \( c = 1 \)). These cases all have known numbers of strands, and the scalar recursive procedure for a given co-prime pair \( b, c \) is then as follows:

(i) \( b := |b|; c := |c| \);
(ii) check whether \( b \geq c \); if not, swap \( b \) and \( c \);
(iii) if \( b > 2c \), \( (b, c) := (|4c - b|, c) \) go to (ii); if \( 2c > b > c \), \( (b, c) := (|3b - 4c|, |2b - 3c|) \) go to (ii);
(iv) if \( b = 1, c = 0, s(b, c) = 3 \); if \( b = 1, c = 1, (b, c) = 4 \); if \( b = 2, c = 1, (b, c) = 6 \).

(Note that non-co-prime pairs can also be considered, and the procedure ends with integer multiples of the three final cases above.) One can see that no extra proof is needed to exclude numbers of strands other than 3, 4 and 6; moreover, expressions (3.11) and (3.12) also show that any reflection changes the value \( b^2 + c^2 \) by an amount divisible by 4. It explains why odd \( b \) and \( c \) behave separately (\( b^2 + c^2 \equiv 2 \) mod 4 holds for them but, unfortunately, the same expression is congruent to 1 modulo 4 for all odd–even pairs). Although the recursive method above is simple and robust (one can easily see that the number of steps cannot exceed \( (b + c)/2 \) as each reflection decreases \( b + c \) by at least 2), it is still not clear, however, whether or not a simple (non-recursive) numeric test to decide the question ‘6 or 3’ exists.

(c) Numbers of strands of octahedral wrappings

The problem of finding these numbers is completely solved [7], but it is shown here that our method also works for triangular tiling. Figure 8a shows the number of strands on an octahedron.
for some small co-prime pairs $b, c$. Because of the triangulation, our table now has $C_{3v}$ symmetry, where symmetry-equivalent coordinate pairs of $b, c$ are generated as follows (see also figure 8b):

$$ (b, c) \rightarrow (c, b) \quad (3.13) $$

and

$$ (b, c) \rightarrow (b + c, -c). \quad (3.14) $$

In figure 9, the two rolling operations (figure 9a,b) and two strand midline domains, slanted and shifted from line $c, c$ (instead of $2c, c$) symmetrically (figure 9c), are demonstrated. Observe that in a triangular tiling only two successive rollings over edges can correspond to a symmetry operation $b$ or $c$; thus, only the positions shown in grey are considered for the polyhedron. As in the case of a cube, the right-hand-side strand development is mirrored horizontally then inverted; the reflection $a_s \rightarrow a_s^* \rightarrow c \rightarrow c^{-1} b \rightarrow c^{-1} b$. Thus,

$$ o_s o_s^{-1} = (c^{m_i})(cb) \cdots (c^{m_i})(cb) \cdot (b^{-1} cc)(c^{-m_i}) \cdots (b^{-1} cc)(c^{-m_i}). \quad (3.15) $$

As $c^2 = e$, we can conclude with similar arguments to those used at the cube that a horizontal reflection

$$ (b, c) \rightarrow (b_n, c_n) = (2c - b, c) \quad (3.16) $$

preserves the number of strands. It is seen immediately that, if $b - c \equiv 0 \mod 3$, $b_n - c_n = c - b$ is also divisible by 3; that is, for all such pairs, $s(b, c) = s(1, 1) = 6$. Without a detailed analysis, just from the simple scheme of symmetries it follows that successive reflections may only terminate at $(1,1)$ or $(1,0)$, apart from the symmetry-equivalent coordinate pairs shown in figure 8b, and $s(1,0) = 4$ ensures that if $b - c \not\equiv 0 \mod 3$ then four closed strands are present.

(d) Numbers of strands of icosahedral wrappings

The results of numerical experiments on the number of strands are presented in figure 10. In addition to the natural symmetry $C_{3v}$, some observations can be made which are strikingly similar to those found at the cube. These symmetry rules are summarized as follows.

**Proposition 3.3.** Numbers for closed strands $(2c + n, c)$ and $(2c - n, c)$ on an icosahedron are equal for arbitrary integer $n$. 
Figure 10. Number of strands on an icosahedron. Small diamonds indicate points of symmetry (specifically for $c = 2$); the dashed line runs across all points of symmetry ($2c, c$) common to $b$- and $c$-directed series.

Proof. Consider now the development of an icosahedron along straight strand segments (figure 11). Based on all arguments used in the proof of proposition 3.1 and also before equation (3.15), the operation

$$o_0 o_s^{-1} = (c^m b)(c^2 b) \cdots (c^m b)(b^{-1} c c^2)(c^{-m_1}) \cdots (b^{-1} c c^2)(c^{-m_1})$$

(3.17)

is equivalent again to the identity because of $c^5 = e$.

Proposition 3.4. Numbers for closed strands $(2c + n, c + n)$ and $(2c - n, c - n)$ on an icosahedron are equal for arbitrary integer $n$.

Proof. With reference to Fuchs & Fuchs [7] again, let the network of figure 6 be re-interpreted as an affine image of triangular tiling (it does not change the order of crossings or proportion of lengths parallel to either $b$ or $c$); the result is displayed in figure 12. Here $o_s^{-1}$ means again a
Figure 11. Development of an icosahedron: fivefold rotation operations according to the two kinds of rollings (a, b); sequence of symmetry operations between two adjacent horizontal (in the i\textsuperscript{th} row) of the tiling (c).

Figure 12. Affine transformation of a development of the icosahedron isomorphic to that of a cube: strand midlines at \((2 \times 3 + 2, 3 + 2)\) (with operations) and \((2 \times 3 - 2, 3 - 2)\) with symmetric horizontal and vertical shift \(\pm x\) (a); sequence of symmetry operations between two adjacent diagonals (in the \(i\)th slanted region) of the tiling (b).

simple reversion of order of operations with a change \(b \leftrightarrow c\) in \(o\), and therefore the product \(o_0 o_j^{\pm 1}\) is written formally as in equation (3.8). As \(c b c b c b = e\) also for the icosahedron, \(o_0 o_j^{\pm 1} = e\).

(e) Obtaining the number of strands on the icosahedron

The arguments in §3b lead formally to the same formulae as equations (3.9) and (3.10) and the recursive method may be visualized by a similar zigzag line obtained by successive reflections along two kinds of directions (parallel to each kind of axis of symmetry). Unlike the cubic or octahedral case, however, no regular pattern modulo 2 or 3 is found. It is explained by the
evaluation of $T(b, c) - T(b, c)$ (instead of the quadrangulation number $Q$, here $T$ is invariant under transformations into 12 symmetry-equivalent coordinates) that

$$
(4c - b)^2 + (4c - b)c + c^2 - (b^2 + bc + c^2) = 10c(2c - b) 
$$

(3.18)

and

$$
(3b - 4c)^2 + (3b - 4c)(2b - 3c) + (2b - 3c)^2 - (b^2 + bc + c^2) = 18(2c - b)(c - b).
$$

(3.19)

These expressions are congruent only modulo 2 but any $T(b, c)$ with co-prime $b, c$ is odd, and no distinction can be made among cases on this basis. In the formal description of the procedure, step (ii) should be slightly modified as follows:

(ii) check whether $b \geq c$; if not, follow (3.13) and (3.14) alternately until it becomes true.

(f) **Symmetry and numbers of strands on different polyhedra**

In the preceding sections, algorithms and formulae were proved to predict the number of closed strands in two- or threefold wrappings but nothing was stated about the symmetry properties of strands. A concise categorization is given below to show the link between the number of strands and the appearance of each strand. Firstly, if $b, c$ are co-prime, then (1) at least one of them must be odd and (2) all possible kinds of edge crossings must be contained in a series of crossings between two equivalent ones. As a consequence, in a wrapping by strands of real width (i.e. leaving no gap or overlap in adjacent parallel segments), each strand must contain some copies of an edge crossing located on a $C_2'$ symmetry axis: let us roll our polyhedron from this point ('edge midpoint') $S$. Secondly, if the polyhedron is a cube and one of $b, c$ is even, each strand contains copies of segments intersected by a $C_4$ axis ('face midpoint'); otherwise (and also in any triangular tiling), each strand contains copies of another edge midpoint (edge crossing at different angle, $\pm \alpha$ is different) $N$. While rolling the polyhedron, sooner or later it arrives at a point $N$ (not to another copy of $S$, because it would imply a closed strand without a point of type $N$ at all; the strand must follow an alternate series of $S$ and $N$). The difference between symmetries is made by the relative positions of these adjacent points of symmetry $S$ and $N$. Figure 13 shows the possibilities for all
Platonics except for the dodecahedron. In general, it is found that $S$ and $N$ imply a new $C'_2$ or $C_2$ axis, and therefore the third symmetry axis $C_i$ perpendicular to both. The order $i = 2, 3, 4$ or 5 depends on the angle between the first two axes, and the closed strand turns out to be an object of the pure rotational symmetry group $D_i$. As $2i$ equals the number of points of symmetry on the strand and the total number of points of symmetry of a wrapping is always $2e$, their quotient gives the number of strands. Note that geometrically it seems possible that adjacent points $S$ and $N$ appear on opposite edges (or even non-adjacent edges of a square face) but they would mean an inexistent number of strands for the respective cases. It is also obvious why the tetrahedron did not require any detailed analysis, because all possible points $N$ have the same relative position to $S$ and there can only appear a triplet of strands.

4. Twistedness of strands

In addition to the problem of the number of strands, another question arises about the number $t$ of complete turns each closed strand makes along its length. Obviously, this number is not unique, insofar as it depends on weaving properties whether or not the strand runs always on the top while rolling. Interestingly, $t$ is not even well defined if the strand runs always on the top: $t$ depends on the starting point of rolling. In this discussion, that starting point $S$ will be assumed to coincide with an edge midpoint. It will be explored what the number $t(b, c)$ depends on if the polyhedron is wrapped in a strand by successive rolling.

Consider two edges of a closed strand as closed loops: their linking number $Lk$ (number of times one is wound around another; for details on knot theory, see [12]) equals exactly the number of (oriented) complete turns of the strand (that is, $Lk = t$). In order to compute this linking number easily, one needs a projection of the link onto a plane called a link diagram. If this is done from an internal point close enough to a vertex of the polyhedron onto the furthest plane tangent to the polyhedron, no new crossings (in addition to those existing on the polyhedral surface) are introduced. Conversely, the above argument makes it possible to count positive and negative crossings (figure 14) directly on the surface of the polyhedron. Note that crossings of loops can be of two kinds in such a diagram: a single crossing of edge loops is found where the strand switches its front and back sides and double crossings occur when the strand midline crosses itself (in [13] it is proved that the numbers of these crossings, called ‘local’ and ‘non-local’, correspond to the double of values of twist ($Tw$) and writhe ($Wr$) of a strand, respectively). The linking number is their sum ($Lk = Tw + Wr$, known as the Călugăreanu–White–Fuller theorem; e.g. [14,15]). Knowing that there is no twist in such a projected link diagram, it follows immediately that $Lk = Wr$; that is, signed crossings of the strand midline should be counted (the term ‘crossing’ will refer to a crossover of strand midline loops and not edge loops henceforth). For all four Platonics in question, one can state the following.

**Proposition 4.1.** The number $t$ of complete turns in a closed strand in a wrapping started at an edge midpoint is zero if the strand visits the opposite edge midpoint and $\pm 1$ otherwise.

*Proof.* As all of our wrappings have at least one $C_2$ axis through edge midpoints, all crossings off this axis are of even number. Consider a planar map of a complete strand started at point $S$. 

![Figure 14. Crossings of loops in a link: positive (a) and negative (b).](http://rspa.royalsocietypublishing.org/Downloaded from http://rspa.royalsocietypublishing.org/ on September 1, 2017)
Figure 15. Self-crossings in an oriented loop of strand midline with a $C_2$ axis: loop crossing both $S$ and $P$ (a); loop crossing $S$ twice, producing there an extra crossing (b). Note that $C_2$ in the link diagram means symmetry in the topological sense only, because the projection distorts the original metrics.

Following from $C_2$ symmetry, all strands either also pass through the opposite edge midpoint of type $S$ or $N$ denoted by $P$ (a) or pass through the same point $S$ with crossing (b); figure 15 shows both cases. Let us reproduce the weaving pattern by following two halves of the strand from $P$ (or from the middle occurrence of $S$ in the case (b)) in both directions such that the right- and left-hand-side parts are put consequently over and under existing segments, then restore the loop orientation by reversing the left-hand-side part only. It is easy to see that each crossing in figure 15a has a symmetry pair now with consistent orientation but opposite up–down position of segments, giving finally zero for $t = Wr = Lk = 0$. In the case (b), the same can be stated about crossings except for that at $S$, and here the answer is therefore $t = W_r = Lk = \pm 1$. ■

It is very important to start wrapping at a $C_2$ axis: choosing another point before the last crossing in this sequence for departure, a single crossing changes its sign and therefore $t$ is incremented by $\pm 2$. Of course, any pattern of over- and undercrossings pertaining to a given strand can be achieved by successive change of sign of crossings, which implies the following.

Corollary 4.2. The number $t$ of complete turns in a closed strand having a $C_2$ axis with over- and undercrossings (i.e. in a given weaving) is even if the strand has no self-crossing at any edge midpoint and odd otherwise.

and

Corollary 4.3. The number $t$ of complete turns in a closed strand with over- and undercrossings conforming to a $C_2$-antisymmetric alternate weaving pattern is zero if the strand has no self-crossing at any edge midpoint and $\pm 1$ otherwise.

In order to see the truth of corollary 4.3, it must be clarified that ‘antisymmetry’ is used for a weaving pattern in the sense that the corresponding symmetry operation swaps under- and overcrossings. As the wrapping in proposition 4.1 is $C_2$-antisymmetric by construction, any other $C_2$-antisymmetric weaving can be obtained by a simultaneous change of signs of both crossings in the $C_2$-symmetric position, thus leaving $Lk$ unchanged.

On this basis, some results will be formulated on cubic weavings in the following section. Note, however, that proposition 4.1 and its corollaries apply to all polyhedra as far as the term ‘edge midpoint’ is reserved for edge midpoints lying on a $C_2$ axis only. Consequently, some of the forthcoming results seem extensible for other polyhedral weavings.

(a) Twistedness of strands in cubic weavings

In a check-weaving pattern on a cube where exactly one of $b, c$ is odd, it is quite obvious that an edge midpoint is on the border between two adjacent tiles (in the ‘top’ and ‘bottom’ positions) of a strand, and thus the complete strand in the weaving is $C_2$-antisymmetric. Corollary 4.3 ensures then that such a strand is homomorphic to an annulus ($t = 0$). If both $b$ and $c$ are odd, edge midpoints are centre points of a tile and the strand in weaving is not $C_2$-antisymmetric but symmetric. Such strands (in alternate weaving), however, belong to the $D_3$ symmetry group and
they have six copies of each crossing except for those that may appear on $C_2'$ axes in triplets. Corollary 4.2 implies that, if there is such a triplet, $t$ is odd. In fact, a stronger condition can be formulated for $D_3$-symmetric strands by observing that a double covering of the cube with $b,c$ odd can be divided into oriented segments such that no following segments have opposite and no adjacent (parallel) ones have identical orientation, as shown in figure 16. Any closed strand on that surface must then follow a given orientation, and alternation of over- and undercrossings has the same rate as that of right and left crossings, making $Lk = t$ to be equal to the total number of crossings (divisible by 6 or by 3 only, as seen above).

In the following discussion, a recursive procedure similar to that in §3b will be presented in order to decide whether the total number of crossings in a strand (with $b,c$ odd) is even or not.

**Proposition 4.4.** Parities of number $t$ for closed strands $(2c + n, c)$ and $(2c - n, c)$ on a cube ($c,n$ are arbitrary odd integers) are opposite.

**Proof.** Recall the construction method of a closed strand shown in figure 13c. The position of eight small dots for possible points $N$ relative to $S$ makes two cases possible: either $N$ is on an
edge adjacent to the edge of $S$ and the strand visits successively the nearest edge midpoint in a common plane and makes no self-crossing in any of these edge midpoints or $N$ is on an edge not adjacent to that of $S$ and the strand visits only three edge midpoints and makes a self-crossing at each. The problem is now reduced to the decision of whether or not two rollings started at $S$ can arrive at both an adjacent and non-adjacent edge midpoint $N$. Figure 17a shows the development of the cube along a strand midline from $S$ to $N$. Similar to figure 4c,d and equations (3.1) and (3.2), the two operation sequences can now be written as

$$o_f = [c^n b]c^{m_i/2}(c^2 b) \cdots c^{m_i}(c^2 b)$$

and

$$o_s = [c^{-u} b]c^{-m_i/2}(c^2 b) \cdots c^{-m_i}(c^2 b),$$

where $u$ stands for the number of vertical tile edges crossed by the ‘steep’ strand in the uppermost row (of half-height) before reaching $N^-$ and $2(u + \sum_{i=1}^{c-1} m_i) + 1 = n$; see also figure 17a. Square brackets indicate here the operations from last $b$ to $N^+$ and $N^-$; the extra $c$ in the first formula is included because point $M$ is closer horizontally to $N^+$ than $N^-$ by one tile. The resultant operation illustrated in figure 17b reads

$$o_f o_s^{-1} = [c^n b](\cdots\cdot c^{m_i}(c^2 b) \cdot (b^{-1} c^2)(c^{-m_i}) \cdots)\cdot [b^{-1} c^{-u}],$$

which simplifies to $c$. Now look at figure 17b again: if starting and final tiles of rolling are related by an operation $c$ then points $N^+$ and $N^-$ lie on opposite edges and thus exactly one of these edges is adjacent to the edge of $S$ wherever $S$ is located.

**Proposition 4.5.** Parities of number $t$ for closed strands $(2c + n, c + n)$ and $(2c - n, c - n)$ on a cube $(c$ is even, $n$ is an arbitrary odd integer) are equal.

**Proof.** With reference to figure 17c, two operation sequences can again be written as follows:

$$o_s = [(bc)^l (bc)](b^2 c)\cdots (bc)^{m_i}(cbc) [(cb)^{n_{i}}(cb)]$$

and

$$o_f = [(bc)^{-l}](b^2 c)\cdots (bc)^{-m_i}(cbc) [(cb)^{n_{i}}(cb)],$$

where $l$ and $r$ count all additional $bc$ or $cb$ operations in the left- and rightmost narrow slanted region of the ‘steep’ strand in figure 17c (the term ‘additional’ is used here to indicate that both sequences $o_s$ and $o_f$ start with one $bc$ operation and $o_s$ terminates with one $cb$ more, meaning three compulsory operations). For that reason, $2(l + r + \sum_{i=1}^{c-1} m_i) + 3 = n$ here. In figure 17d, a scheme of rolling is shown: the double asterisk here means a successive reflection in both $b$ and $c$. The last operation $(c)$ in the sequence is needed to restore the original orientation of $N^+$ in the tiling; for the resultant operation, we have

$$c o_s b^{-1} o_f^{*s-1} = c[(bc)^l (bc)](b^2 c) \cdots (bc)^{m_i}(cbc) [(cb)^{n_{i}}(cb)]$$

$$\times b^{-1}[(bc)(bc)^{-l}](b^2 c) \cdots (bc)^{-m_i}(cbc) [(cb)^{n_{i}}(cb)^{-l}]]].$$

Note that the five middle operations simplify to $cbc$, and because for arbitrary integer $m_i$ (or $r$), $(bc)^{m_i}cbc(b^{-1})^{m_i} = cbc(b^{-1})^{m_i}cbc(b^{-1})^{m_i} = cbc,$ (4.6) can be rewritten as follows:

$$c o_s b^{-1} o_f^{*s-1} = c[(bc)^l (bc)]cbc[(cb)^{n_{i}}(cb)^{-l}]].$$

Regrouping the operations leads now to the form $(cb)^l cbc b c(c b c)^{n_{i}} = e$, with respect to $cbc c b c = e$ for the cube. This fact means that the starting and final tiles are identical to points $N^+$ and $N^{**s-1}$ on the same edge, which is either adjacent to the edge of $S$ or not.

In practice, this result implies a zigzag recursion similar to that described in §3b but the final pair $(b, c) = (1, 1)$ is known; only the number of reflections 3.1 should be checked for parity. Similar arguments may apply for octahedral and icosahedral weavings but detailed analysis is not given here.
5. Conclusion

The problems solved in this paper are as follows: it has been shown that symmetry-adapted structural analysis for prestressed wrappings gives the minimal number of strands only for four examined Platonic polyhedra; recursive algorithms were given to find the number of strands in complete (two- or threefold) weavings for the cube and icosahedron; the method uses only the algebraic properties of numbers \((b, c)\) that determine the skewness of the pattern. A similar method was given for finding the parity of the number of turns in a closed strand on the cube if both \(b\) and \(c\) are odd. Finally, it was proved that every strand with \(C_2\)-antisymmetric alternation makes no turn, as well as every strand on the cube for \(b, c\) odd makes as many turns as the number of crossings in a strand.

Problems that are to be solved, however, still exist. Firstly, it is not known whether or not a closed form instead of a recursive method exists in the cases above (although a negative answer is conjectured). Secondly, it is not straightforward to define a similar alternation for threefold weavings, because adjacent tiles on opposite sides of an edge midpoint in the ‘top’ and ‘bottom’ positions cannot occur, and therefore the complete weaving cannot have \(C_2\) antisymmetry about an axis through the edge midpoint. In addition, a threefold weaving must necessarily have two different kinds of tiling corner points, as neither a \(C_3\) nor a \(C_6\) symmetry around all of these points is possible for alternating strands in a plane tiling. Thirdly, it is a challenging question how either an individual strand or a complete set of strands in a complete weaving can be classified in terms of knot theory.

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