We consider a generalization of the Gelfand problem arising in Frank-Kamenetski theory of thermal explosion. This generalization is a natural extension of the Gelfand problem to two-phase materials, where, in contrast to the classical Gelfand problem which uses a single temperature approach, the state of the system is described by two different temperatures. We show that similar to the classical Gelfand problem the thermal explosion occurs exclusively owing to the absence of stationary temperature distribution. We also show that the presence of interphase heat exchange delays a thermal explosion. Moreover, we prove that in the limit of infinite heat exchange between phases the problem of thermal explosion in two-phase porous media reduces to the classical Gelfand problem with renormalized constants.

1. Introduction

Superlinear parabolic equations and systems of such equations serve as mathematical models of many nonlinear phenomena arising in natural sciences. It is well known that such models may often produce solutions that do not exist globally in time owing to formation singularities. In particular, there are solutions which become infinite either somewhere or everywhere in the spatial domain in a finite time. Formation of such singularities is commonly referred to as blow-up and has attracted considerable attention from scientists and engineers over past decades [1–3]. The classical problem in a theory of blow-up for nonlinear parabolic equations, which is widely known in mathematical literature as a Gelfand problem, reads

\[
\begin{align*}
W_t - \Delta W &= \Lambda g(W) \quad \text{in } (0, T) \times \Omega, \\
W &= 0 \quad \text{on } \partial \Omega \\
W(0, \cdot) &= 0 \quad \text{in } \Omega,
\end{align*}
\]

(1.1)
where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $g : \mathbb{R} \to (0, \infty)$ is a $C^1$ convex non-decreasing function satisfying
\begin{equation}
\int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty \quad \text{for some } x_0 \geq 0,
\end{equation}
and $\Lambda > 0$ is a parameter. This problem was originally introduced in a context of thermo-diffusive combustion as a model of thermal explosion, the spontaneous development of rapid rates of heat release by chemical reactions in combustible mixtures and materials being initially in a non-reactive state [4,5]. Model (1.1) describes an evolution of initially uniform temperature field $W$ which diffuses in space, increases in a bulk owing to the heat release described by a reaction term $\Lambda g$ and is fixed on the boundary (cold boundary). Model (1.1) was derived by Frank-Kamenetskii [6] as a short-time asymptotic of a standard thermo-diffusive model and describes an initial stage of self-ignition of a combustible mixture.

Depending on the parameters of this problem the solutions of (1.1) either blow-up or exist globally. In a context of combustion, the first case corresponds to the successful initiation of the combustion process, whereas the second one corresponds to the ignition failure. Basic physical reasoning discussed in [4,6] and formal (intermediate asymptotics) arguments of Barenblatt presented in [7] suggest that blow-up in model (1.1) occurs exclusively owing to the absence of stationary solutions for this problem. That is, the absence of stationary temperature distribution $w$ that solves, in a weak sense, the following time-independent problem:
\begin{equation}
-\Delta w = \Lambda g(w), \quad w > 0 \text{ in } \Omega \quad \text{and} \quad w = 0 \text{ on } \partial \Omega.
\end{equation}

These formal arguments of Gelfand [7] were made rigorous in Brezis et al. [8]. The following theorem summarizes the main results regarding solutions of problems (1.1) and (1.3) (see [8,9, Theorem 3.4.1] and further references therein).

**Theorem A.** Parabolic problem (1.1) has a classical global solution if and only if stationary problem (1.3) has a weak solution.

There exists $0 < \Lambda^* < \infty$ such that:

(i) for $\Lambda > \Lambda^*$, problem (1.3) has no weak solutions,
(ii) for $0 < \Lambda < \Lambda^*$, problem (1.3) has a minimal classical solution $w_{\Lambda}$,
(iii) $w_{\Lambda}(x)$ is a monotone increasing function of $\Lambda$, and for $\Lambda = \Lambda^*$ problem (1.3) has a weak solution $w^*$ defined by
\begin{equation}
w^*(x) := \lim_{\Lambda \to \Lambda^*} w_{\Lambda}(x).
\end{equation}

The statement of theorem A, from a physical perspective, has a very clear interpretation. Indeed, the parameter $\Lambda$ can be understood as a scaling factor that reflects the size of the domain, which increases as $\Lambda$ increases. Thus, in relatively small domains the cold boundary suppresses intensive chemical reaction in the bulk which leads to a stationary temperature distribution, whereas when the size of the domain exceeds some critical value corresponding to $\Lambda^*$ the cooling on the boundary becomes insufficient to prevent chemical reaction inside the domain $\Omega$, which leads to thermal explosion.

Classical model (1.1) asserts that the process of combustion can be described using a unified single temperature approach. This assumption, which has rather a wide range of validity, however, is not applicable in certain situations. For example, in the combustion of porous materials the difference in temperatures of gaseous and condensed phases can be substantial, which changes a combustion process [5]. As a result, the model describing self-ignition of porous media has to be appropriately modified. Let us note that explosion in two-phase materials has many technological applications ranging from the ignition of metal nanopowders and solid rocket propellants to issues of safe storage of nuclear waste and industrial raw garbage [5,10].

In order to describe explosion in two-phase materials, one may adopt an approach of Frank-Kamenetskii and make a standard reduction of governing equations describing the combustion...
of two-phase porous materials. The conventional system of equations for the dynamics of two-phase material is well known and we refer the reader to Margolis [11] for the details. Partial linearization of these equations incorporating the Frank-Kamenetskii transform [4,6] leads to the following system:

\[
\begin{aligned}
U_t - \Delta U = \lambda g(U) + \nu(V - U), \\
(1.5) \\
\alpha V_t - d \Delta V = \nu(U - V) \quad \text{in} \ (0, T) \times \Omega, \\
U = V = 0 \quad \text{on} \ \partial \Omega
\end{aligned}
\]

and

\[
\begin{aligned}
U(0, \cdot) = V(0, \cdot) = 0 \quad \text{in} \ \Omega;
\end{aligned}
\]

here \(U(t, x)\) and \(V(t, x)\) are appropriately normalized temperatures of condensed (solid) and gaseous phases, respectively, and \(d > 0\) is a ratio of effective gaseous and thermal diffusivity, \(\nu > 0\) is the interphase heat transfer coefficient and \(\alpha > 0\) is a parameter which depends on porosity and ratios of specific heats of the solid and gaseous phases. It is important to note that model (1.5) is formally identical to the one describing the formation of hot spots in transistors. In this case, variables \(U\) and \(V\) can be interpreted as temperatures of electron gas and of the lattice, respectively (e.g. [12]).

As one may expect the behaviour of solutions for problem (1.5) depends crucially on the existence of stationary solutions for the time-independent problem

\[
\begin{aligned}
-\Delta u = \lambda g(u) + \nu(v - u), \\
(1.6) \\
-d \Delta v = \nu(u - v) \quad \text{in} \ \Omega, \\
u, v > 0 \quad \text{in} \ \Omega \\
\lim_{\lambda \to \lambda^*} u = u_\nu \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

The goal of this paper is to study the dynamics of solutions for problem (1.5) and its stationary states described by (1.6). There are two main results of this paper. Our first result states that similar to the classical Gelfand problem blow-up in system (1.5) is fully determined by solutions of problem (1.6). Namely the following holds.

**Theorem 1.1.** If elliptic problem (1.6) has a classical solution, then parabolic problem (1.5) has a global classical solution. If parabolic problem (1.5) has a global classical solution, then the elliptic problem has a weak solution. Moreover, the global classical solution of (1.5) converges in \(L^1\)-norm to a minimal weak solution of (1.6) as \(t \to \infty\).

The precise definition of a minimal classical and weak solution of stationary problem (1.6) is given later in §3. Here, we note only that every classical solution is also a weak solution. On the other hand, weak solutions may have singularities.

In view of this result, detailed information on stationary solutions is needed. This is given by the following theorem.

**Theorem 1.2.** Let \(d > 0\). Then for every \(\nu > 0\) there exists \(0 < \lambda^*_\nu < \infty\) such that:

(i) for \(\lambda > \lambda^*_\nu\), system (1.6) has no classical solutions,

(ii) for \(0 < \lambda < \lambda^*_\nu\), system (1.6) has a minimal classical solution \((u_{\lambda,\nu}, v_{\lambda,\nu})\),

(iii) for \(\nu > 0\), both \(u_{\lambda,\nu}(x)\) and \(v_{\lambda,\nu}(x)\) are monotone increasing functions of \(\lambda\) for every \(x \in \Omega\), and for \(\lambda = \lambda^*_\nu\) system (1.6) has a weak solution \((u^*_\nu, v^*_\nu)\) defined by

\[
(1.7) \quad u^*_\nu(x) := \lim_{\lambda \to \lambda^*_\nu} u_{\lambda,\nu}(x) \quad \text{and} \quad v^*_\nu(x) := \lim_{\lambda \to \lambda^*_\nu} v_{\lambda,\nu}(x) \quad (x \in \Omega),
\]

(iv) \(\lambda^*_\nu \geq \Lambda^*\) and \(\lambda^*_\nu = \lambda^*(\nu)\) is a non-decreasing function of \(\nu > 0\) having the following properties:

\[
\lim_{\nu \to 0} \lambda^*(\nu) = \Lambda^* \quad \text{and} \quad \lim_{\nu \to \infty} \lambda^*(\nu) = \Lambda^*(1 + d),
\]

where \(\Lambda^*\) is the critical value of classical Gelfand problem (1.3).
(v) for $\lambda < \lambda^*_v$, $u_{\lambda, v}(x)$ is a non-increasing function of $v$ for every $x \in \Omega$. For $\lambda < \Lambda^*$ and $v \to 0$ solution $(u_{\lambda, v}, v_{\lambda, v})$ converges uniformly to $(u_0, 0)$, where $u_0$ is the minimal solution of

$$
\begin{cases}
-\Delta u_0 = \lambda g(u_0), & u_0 > 0 \text{ in } \Omega \\
\quad u_0 = 0 \quad \text{ on } \partial \Omega.
\end{cases}
$$

(1.9)

For $\lambda < \Lambda^*(1 + d)$ and $v \to \infty$ solution $(u_{\lambda, v}, v_{\lambda, v})$ converges uniformly to $(u_\infty, u_\infty)$, where $u_\infty$ is the minimal solution of

$$
\begin{cases}
-\Delta u_\infty = \frac{\lambda}{1 + d} g(u_\infty), & u_\infty > 0 \text{ in } \Omega \\
\quad u_\infty = 0 \quad \text{ on } \partial \Omega.
\end{cases}
$$

(1.10)

**Remark 1.3.** The limit weak solution $(u^*_v, v^*_v)$, constructed in part (iii) of theorem 1.2, might be either classical or singular. In the proof of part (i) of theorem 1.2, we show that there exists $\lambda^{**}_v > \lambda^*_v$ such that system (1.6) has no weak solutions for $\lambda > \lambda^{**}_v$. In the case of single equation (1.3), it is known that actually $\lambda^{**} = \lambda^*$; see (i) and (ii) of theorem A. One may expect that a similar result holds for the system considered in this paper. However, the proof of this fact is very delicate even in the case of a single equation (see [8, Theorem 3] and further discussion in [13]). We also want to point out that in the case of single equation (1.3) weak solutions corresponding to $\lambda = \lambda^*$ in most of the cases relevant to applications are in fact classical.

Theorem 1.2 proves that solutions for problem (1.6) behave similarly to solutions of the classical Gelfand problem and that the presence of the heat exchange increases the value of critical parameter $\lambda^*$. These results are quite in line with the physical intuition behind this problem. What we found rather surprising is the limiting behaviour of solutions of (1.6) when $v \to \infty$. Indeed, it is quite remarkable that the substantial heat exchange between the two phases reduces problem (1.6) to the classical Gelfand problem with re-normalized parameters. We also note that this observation in fact also justifies the use of the single temperature model as an effective model for two-phase materials in this asymptotic regime.

The paper is organized as follows: in §2, we give some basic heuristic arguments and present numerical examples which clarify the main results. Sections 3 and 4 are dedicated to the proof of theorems 1.2 and 1.1, respectively.

### 2. Heuristic arguments and numerical examples

In this section, we would like to give some formal arguments and provide results of numerical simulations of problems (1.5) and (1.6) that clarify and illustrate results of theorems 1.1 and 1.2.

Theorem 1.1 basically states that the presence or the absence of global solutions for problem (1.5) is fully determined by the presence or the absence of solutions for system (1.6). Thus, the behaviour of solutions for system (1.5) is essentially similar to the behaviour of solutions for single equation (1.1). It is well known that the dynamics of a system of parabolic equations, in general, is substantially more complex than that of a single equation. However, in our case system (1.1) can, at least formally, be written as a gradient flow

$$
U_t = -\frac{\delta E}{\delta U} \quad \text{and} \quad \alpha V_t = -\frac{\delta E}{\delta V},
$$

(2.1)

with the ‘energy’ functional defined as

$$
E(U, V) = \frac{1}{2} \int_\Omega \left[ |\nabla U|^2 + d|\nabla V|^2 + \nu(U - V)^2 + 2\nu \Phi(U) \right] dx, \quad \Phi(U) = -\int_0^U g(s) ds.
$$

(2.2)

Moreover, system (1.5) is quasi-monotone (quasi-monotone non-decreasing in the terminology of Pao [14]) and thus its classical solutions obey the componentwise parabolic comparison principle [14, Theorem 3.1, p. 393] or [15, Theorem 3.4, p. 130]. Thus, the time evolution for solutions of (1.5) is very much restricted and indeed expected to be similar to that of a single equation. This
situation is somewhat similar to that considered in [16], where self-explosion in confined porous media was considered.

Now let us turn to theorem 1.2. First, we note that the transition from existence to non-existence of solutions to this problem is very similar to that observed in the classical Gelfand problem. This again occurs owing to the presence of a componentwise comparison principle and the fact that system (1.6) is the Euler–Lagrange equation of the functional

$$
\mathcal{E}(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + d|\nabla v|^2 + v(u - v)^2 + 2\gamma \Phi(u)) \, dx,
$$

$$
\Phi(u) = -\int_0^u g(s) \, ds.
$$

(2.3)

In order to understand monotonicity with respect to parameter $v$, it is convenient to rewrite system (1.6) as a non-local equation. Combining the first and the second equations of system (1.6), we have

$$
[-\gamma v^{-1} \Delta + 1](u - v) = \lambda \gamma^{-1} g(u),
$$

(2.4)

where $\gamma = d/(1 + d)$. Thus,

$$
u - v = \lambda \gamma^{-1} [-\gamma v^{-1} \Delta + 1]^{-1} g(u).
$$

(2.5)

Substituting this expression in the first equation of system (1.6) yields

$$
- \Delta u = \frac{\lambda}{1 + d}(1 + d[1 - [-\gamma v^{-1} \Delta + 1]^{-1}]) g(u).
$$

(2.6)

As can be readily seen the operator in curly brackets is an increasing function of $v^{-1}$. This implies that the effective right-hand side of this equation is a decreasing function of $v$ and thus one may expect that $u$ decreases as $v$ increases. Moreover, in the limiting case $v \to \infty$ the right-hand side becomes essentially local. Let us also note that for sufficiently large $v$ the components $u$ and $v$ of system (1.6) away from the boundary are related (at least formally) by a simple formula $v = u - v^{-1} \lambda d g(u)/(1 + d) + o(v^{-1})$ which follows directly from (2.5).

In order to illustrate statements of theorems 1.1 and 1.2, we performed numerical studies of simple one-dimensional versions of problems (1.5) and (1.6) with $\Omega = (-1, 1)$, $d = a = 1$ and $g(u) = e^{\alpha u}$. Physically, these two problems describe stationary temperature distributions and the evolution of temperature fields in a plane-parallel vessel under the assumption of Arrhenius chemical kinetics.

Let us first consider stationary problem (1.6). The solution to this problem was obtained numerically using the conventional shooting method. The numerical study shows that, in full agreement with the statement of theorem 1.2, stationary temperature distribution exists only for values of scaling parameter $\lambda(v)$ which does not exceed some critical value $\lambda^*(v)$. Moreover, this critical value $\lambda^*(v)$ is an increasing function of the heat exchange parameter $v$ and has the following asymptotic properties: $\lambda^*(0) = \Lambda^*$, where $\Lambda^*$ is a critical value of classical Gelfand problem (1.3) (in the considered case $\Lambda^* \approx 0.88$, e.g. [6]) and $\lambda^*(\infty) = (1 + d)\Lambda^* \approx 1.76$. The dependency of the critical value $\lambda^*$ as a function of $v$ is shown in figure 1. In addition, for a fixed value of $\lambda < \lambda^*$ one can see that the $u$ component describing the temperature of the solid phase is decreasing monotonically as $v$ increases, while the temperature of the gas phase (v component) is bounded from above by $u$ and approaches the latter from below as $v$ increases (figure 2).

Finally, let us illustrate the dynamical features of the combustion process given by theorem 1.1. For this reason, let us consider one-dimensional problem (1.5) with all the parameters as above and for a fixed value of $v = 5$ and two values of $\lambda = 1.2$ and $\lambda = 1.5$, one of which is below critical $\lambda^*(5) \approx 1.468$ and the other is above critical. Figure 3 shows the time evolution of temperatures of gas and solid phases in the middle of the vessel $x = 0$, where temperatures of both the solid and the gas have their maximal values as long as the solution exists. As predicted by theorem 1.1 in the case of subcritical $\lambda$ (figure 3a) the solution, after some short transition period, approaches its steady state, whereas for supercritical $\lambda$ (figure 3b) the solution rapidly accelerates and becomes infinite (blows up) in finite time.
Figure 1. The critical value $\lambda^*$ as a function of $\nu$ for the solution of (1.6) with $\Omega = (-1, 1), g(u) = e^u$ and $d = 1$. Dashed line represents $\lambda^*(\infty) \approx 1.76$.

Figure 2. Minimal solution of (1.6) with $\Omega = (-1, 1)$ and $g(u) = e^u, d = 1$ and $\lambda = 1$ with $\nu = 1(a)$ and $\nu = 5 (b). u$, solid line; $v$, dashed line.

Figure 3. Solution of (1.5) with $\Omega = (-1, 1)$ and $g(u) = e^u, d = 1, \alpha = 1, \nu = 5$ and $\lambda = 1.2$ (a) and $\lambda = 1.5$ (b) in the middle of an interval $\Omega$ where the solution has its maximum value as long as it exists. $U(0, t)$, solid line; $V(0, t)$, dashed line.

3. Stationary problem: proof of theorem 1.2

In this section, we study solutions of stationary system (1.6) and discuss their qualitative properties.

First, let us note that if $\lambda = 0$ then system (1.6) becomes linear and has a cooperative structure for all $d, \nu > 0$. Furthermore, as $g(u)$ is a monotone non-decreasing function, (1.6) is a
quasi-monotone non-decreasing nonlinear system in the sense of [14, Theorem 4.1, p. 406] for every \( d, \nu, \lambda > 0 \) and thus can be studied using the comparison type of arguments.

We start with definitions of weak solutions of problem (1.6) as well as weak sub- and supersolutions for this system.

Similar to [8], we say \((u, v)\) is a weak solution of system (1.6) if \( u, v \in L^1(\Omega) \), \( g(u)\delta(x) \in L^1(\Omega) \), where \( \delta(x) := \text{dist}(x, \partial\Omega) \), and

\[
- \int_{\Omega} u\Delta \phi + v \int_{\Omega} u\phi - v \int_{\Omega} v\phi = \lambda \int_{\Omega} g(u)\phi
\]

and

\[
-d \int_{\Omega} v\Delta \psi + v \int_{\Omega} v\psi - v \int_{\Omega} u\psi = 0, \quad \forall \phi, \psi \in C_C^0(\overline{\Omega}).
\]

Note that the assumption \( \phi \in C_C^0(\overline{\Omega}) \) implies \( |\phi| \leq C\delta \) for some constant \( C > 0 \), so the integral on the right-hand side of the first equation is well defined. Note also that zero boundary data are encoded in this definition as we allow test functions \( \phi, \psi \), which have a non-trivial normal derivative, on the boundary.

We say \((u, v)\) is a classical solution of (1.6) if \((u, v)\) is a weak solution of (1.6) and, in addition, \( u, v \in C^2(\Omega) \cap C_0(\overline{\Omega}) \). As usual, \((u, v)\) is a sub- or super-solution of system (1.6) if \( = \) above is replaced by \( \leq \) or \( \geq \), respectively, and in addition only non-negative test functions \( \phi \) and \( \psi \) are considered.

Given two pairs of functions \((u_1, v_1)\) and \((u_2, v_2)\) defined on \( \Omega \), we write \((u_1, v_1) \leq (u_2, v_2)\) provided that \( u_1(x) \leq u_2(x) \) and \( v_1(x) \leq v_2(x) \) for all \( x \in \Omega \). We say that \((u, v)\) is a minimal (super-) solution of (1.6), if \((u, v)\) is a (super-) solution of (1.6) and \((u, v) \leq (\bar{u}, \bar{v})\) for every other supersolution \((\bar{u}, \bar{v})\) of (1.6).

Using these definitions, we can now proceed to a proof of theorem 1.2. The proofs of parts (i–iii) are relatively standard and can be viewed as an extension of similar results of [8,9,13] obtained for solutions for this system.

Proof of part (i) of theorem 1.2. Until the proof of part (iv) of theorem 1.2 we assume that \( \lambda > 0 \) is fixed and, when there is no ambiguity, drop the subscript \( v \) in the notations.

Let \( \mu_1 = \mu_1(-\Delta, \Omega) > 0 \) and \( \phi_1 > 0 \) be the principal eigenvalue and the corresponding principal eigenfunction of \(-\Delta\) in \( H_0^1(\Omega) \) with \( \|\phi\|_1 = 1 \). Recall that, as \( \Omega \) is smooth,

\[
c\delta \leq \phi_1 \leq C\delta,
\]

for some \( C > c > 0 \) (cf. [9, Theorems 3.1.4 and 4.3.1]).

Given \( \lambda > 0 \), let \((u_\lambda, v_\lambda)\) be a weak solution of (1.6). Testing (1.6) against \( \phi_1 \), in the second equation we obtain

\[
\mu_1 d \int_{\Omega} v_\lambda \phi_1 + v \int_{\Omega} v_\lambda \phi_1 - v \int_{\Omega} u_\lambda \phi_1 = 0
\]

or

\[
\int_{\Omega} v_\lambda \phi_1 = \frac{v}{v + \mu_1 d} \int_{\Omega} u_\lambda \phi_1.
\]

Substituting this into the first equation, we then derive

\[
\mu_1(1 + \kappa) \int_{\Omega} u_\lambda \phi_1 = \mu_1 \int_{\Omega} u_\lambda \phi_1 + v \int_{\Omega} u_\lambda \phi_1 - v \int_{\Omega} v_\lambda \phi_1 = \lambda \int_{\Omega} g(u_\lambda)\phi_1,
\]

where \( \kappa = vd/(v + \mu_1 d) \). As \( g(0) > 0 \) and \( g \) is convex, by assumption (1.2) there is a constant \( \eta > 0 \) such that

\[
g(u) \geq \eta u \quad \text{for all } u \geq 0.
\]

Then, from (3.5) we conclude that

\[
\mu_1(1 + \kappa) \int_{\Omega} u_\lambda \phi_1 = \lambda \int_{\Omega} g(u_\lambda)\phi_1 \geq \lambda \eta \int_{\Omega} u_\lambda \phi_1.
\]
This implies that for
\[ \lambda > \frac{\mu_1}{\eta} (1 + \kappa) \quad (3.8) \]
system (1.6) has no weak solutions.

To prove part (ii) of theorem 1.2, we need the following two lemmas.

**Lemma 3.1.** Let \((\phi, \psi)\) be a solution of the following problem:
\[
\begin{aligned}
-\Delta \phi + \nu \phi - \nu \psi &= f & \text{in } \Omega, \\
-d \Delta \psi + \nu \psi - \nu \phi &= 0 & \text{in } \Omega \\
\phi = \psi = 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

Then, for every \(f \in C(\bar{\Omega})\) system (3.9) has a unique classical solution \((\phi, \psi)\). Moreover, \((\phi, \psi) \geq (0, 0)\) provided that \(f \geq 0\). In addition, classical solutions of (3.9) satisfy a strong maximum principle, in the sense that \(f \geq 0\) and \(f \neq 0\) implies that for some \(c, C > 0\) it holds that
\[ c \delta(x) \leq \psi(x) < \phi(x) \leq C \delta(x) \quad \text{for all } x \in \Omega. \quad (3.10) \]

**Proof.** The existence and uniqueness as well as the regularity and positivity properties for systems of type (3.9) follow from well-known results of [17–19].

To prove (3.10), we observe that combining the first and the second equations of (3.9) in a way identical to that discussed in §2 (see equation (2.4)) we obtain
\[ [-\gamma \nu^{-1} \Delta + 1](\phi - \psi) = \gamma f, \quad (3.11) \]
where \(\gamma = d/(1 + d) > 0\). By the strong maximum principle (cf. [9, Theorem 3.1.4]) applied to (3.11) we have \(\phi(x) - \psi(x) > c_1 \delta(x)\) for all \(x \in \Omega\), for some constant \(c_1\). Substituting this into the second equation of system (3.9) and using the strong maximum principle again we obtain the lower bound in (3.10), while from the first equation of (3.9) we derive the upper bound of (3.10) via [9, Theorem 4.3.1].

**Lemma 3.2.** Assume that for some \(\lambda_* > 0\) system (1.6) has a classical supersolution. Then (1.6) has a minimal classical solution \((u_*, v_*)\) for every \(0 < \lambda \leq \lambda_*\).

**Proof.** Let us first observe that if \((\bar{u}, \bar{v})\) is a classical supersolution of (1.6) for some \(\lambda_* > 0\), then \((\bar{u}, \bar{v})\) is also a classical supersolution of (1.6) for every \(0 < \lambda \leq \lambda_*\).

Next, given \(0 < \lambda \leq \lambda_*\), set \((\phi_0, \psi_0) = (0, 0)\). For \(k \in \mathbb{N}\), recursively define \((\phi_k, \psi_k)\) as the unique positive solution of the linear system
\[
\begin{aligned}
-\Delta \phi_k + \nu \phi_k - \nu \psi_k &= \lambda g(\phi_{k-1}) & \text{in } \Omega, \\
-d \Delta \psi_k + \nu \psi_k - \nu \phi_k &= 0 & \text{in } \Omega \\
\phi_k = \psi_k = 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(3.12)  

By lemma 3.1, it is clear that
\[ 0 \leq \phi_1(x) \leq \bar{u}(x) \quad \text{and} \quad 0 \leq \psi_1(x) \leq \bar{v}(x) \quad (x \in \Omega). \quad (3.13) \]
Assume that for some \(k \in \mathbb{N}\) it holds that
\[ 0 \leq \phi_{k-1}(x) \leq \phi_k(x) \leq \bar{u}(x) \quad \text{and} \quad 0 \leq \psi_{k-1}(x) \leq \psi_k(x) \leq \bar{v}(x) \quad (x \in \Omega). \quad (3.14) \]

Then, taking into account the monotonicity of \(g\), we obtain
\[
\begin{aligned}
-\Delta (\phi_{k+1} - \phi_k) + \nu(\phi_{k+1} - \phi_k) - \nu(\psi_{k+1} - \psi_k) &= \lambda (g(\phi_k) - g(\phi_{k-1})) \geq 0 & \text{in } \Omega, \\
-d \Delta (\psi_{k+1} - \psi_k) + \nu(\psi_{k+1} - \psi_k) - \nu(\phi_{k+1} - \phi_k) &= 0 & \text{in } \Omega \\
\phi_{k+1} - \phi_k = \psi_{k+1} - \psi_k &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(3.15)  

By lemma 3.1 and the principle of mathematical induction, we conclude that the sequence \((\phi_k, \psi_k)\) is monotone non-decreasing. Similarly, we deduce that \((\phi_k, \psi_k)\) is uniformly bounded by \((\bar{u}, \bar{v})\), so
that for all \( k \in \mathbb{N} \) it holds that
\[
0 \leq \phi_k(x) \leq \phi_{k+1}(x) \leq \tilde{u}(x) \quad \text{and} \quad 0 \leq \psi_k(x) \leq \psi_{k+1}(x) \leq \tilde{v}(x) \quad (x \in \Omega).
\] (3.16)
Therefore, the sequence \((\phi_k, \psi_k)\) converges pointwisely in \( \Omega \), and we denote
\[
u_k(x) := \lim_{k \to \infty} \phi_k(x) \leq \tilde{u}(x) \quad \text{and} \quad v_k(x) := \lim_{k \to \infty} \psi_k(x) \leq \tilde{v}(x) \quad (x \in \Omega).
\] (3.17)
By the standard elliptic regularity (cf. [9, Proof of Theorem 3.3.3, Step 3]), \((\nu_k, v_k)\) is a classical solution of nonlinear system (1.6). Moreover, because the construction of \((\nu_k, v_k)\) does not depend on the specific choice of a supersolution \((\tilde{u}, \tilde{v})\), we conclude that \((\nu_k, v_k)\) is a minimal solution of (1.6).

Now we turn to the proof of part (ii) of theorem 1.2.

**Proof of part (ii) of theorem 1.2.** Let \( \Lambda^* \) be the critical value of the classical Gelfand problem (1.3). For \( 0 < \lambda < \Lambda^* \), let \( u_0 := w_\lambda \) be the minimal classical solution of (1.3). As \( g \) is positive and monotone non-decreasing, it is clear that \((u_0, u_0)\) is a classical supersolution of nonlinear system (1.6) for every \( 0 < \lambda \leq \Lambda \). It follows from Lemma 3.2 and upper bound (3.8) in the proof of part (i) of theorem 1.2 that the set of \( \lambda > 0 \) where (1.6) has a minimal classical solution is a bounded, non-empty interval. Thus, we defined
\[
\lambda^* := \sup \{ \lambda > 0 : (1.6) \text{ has a minimal classical solution} \}.
\] (3.18)
This completes the proof of parts (i) and (ii) of theorem 1.2.

As a next step, we continue to the proof of part (iii) of the main theorem.

**Proof of claim (iii) of theorem 1.2.** Given \( 0 < \lambda < \lambda^* \) and \( 0 < \varepsilon < \lambda \), we observe that \((u_\lambda, v_\lambda)\) is a supersolution of (1.6) with \( \lambda \) replaced by \( \lambda - \varepsilon \). Therefore, \((u_{\lambda-\varepsilon}, v_{\lambda-\varepsilon}) \leq (u_\lambda, v_\lambda)\). Let \( \phi := u_\lambda - u_{\lambda-\varepsilon}, \psi := v_\lambda - v_{\lambda-\varepsilon} \). Then, taking into account the monotonicity of \( g \) we see that
\[
\begin{align*}
-\Delta \phi - v(\psi - \phi) & = \lambda g(u_\lambda) - (\lambda - \varepsilon)g(u_{\lambda-\varepsilon}) > 0 \quad \text{in} \quad \Omega, \\
-\Delta \psi - v(\phi - \psi) & = 0 \quad \text{in} \quad \Omega \\
\phi = \psi = 0 \quad & \text{on} \quad \partial \Omega.
\end{align*}
\] (3.19)
By the strong maximum principle of [9, Theorem 3.1.4] we conclude that for some \( c_\varepsilon > 0 \) it holds that
\[
u_\lambda(x) \geq u_{\lambda-\varepsilon}(x) + c_\varepsilon \delta(x) \quad \text{and} \quad v_\lambda(x) \geq v_{\lambda-\varepsilon}(x) + c_\varepsilon \delta(x) \quad (x \in \Omega).
\] (3.20)
In particular, \( u_\lambda(x) \) and \( v_\lambda(x) \) are strictly monotone increasing functions of \( \lambda \), for every \( x \in \Omega \).

Furthermore, as \( g(0) > 0 \) and \( g \) is convex, by assumption (1.2) there is a constant \( m_\varepsilon > 0 \) such that, for all \( s \geq 0 \),
\[
\frac{\lambda^*}{2} g(s) \geq \mu_1(1 + \kappa)s - m_\varepsilon.
\] (3.21)
Now, testing (1.6) against \( \phi_1 \) and using (3.4) and (3.5) we obtain
\[
\lambda \int_\Omega g(u_\lambda) \phi_1 = \mu_1(1 + \kappa) \int_\Omega u_\lambda \phi_1 \leq \frac{\lambda^*}{2} \int_\Omega g(u_\lambda) \phi_1 + m_\varepsilon \int_\Omega \phi_1.
\] (3.22)
We conclude that
\[
\lim_{\lambda \to \lambda^*} \int_\Omega g(u_\lambda) \phi_1 \leq 2 \frac{m^*}{\lambda^*} < \infty.
\] (3.23)
Let \( \zeta \) be the unique positive solution of the following problem:
\[
\begin{align*}
-\Delta \zeta & = 1 \quad \text{in} \quad \Omega \\
\zeta & = 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\] (3.24)
Similar to (3.2), by [9, Theorems 3.1.4 and 4.3.1] we conclude that
\[
c_\delta \leq \zeta \leq C_\delta.
\] (3.25)
Testing (1.6) with $\phi = \psi = \zeta$, we obtain
\[
\int_{\Omega} u_\lambda + v \left( u_\lambda - v \lambda \right) \zeta = \lambda \int_{\Omega} g(u_\lambda) \zeta \quad \text{and} \quad d \int_{\Omega} v_\lambda + v \left( v_\lambda - u_\lambda \right) \zeta = 0. 
\] (3.26)

Adding these equations together and taking into account (3.2), (3.25) and (3.23), we have
\[
\int_{\Omega} (u_\lambda + d v_\lambda) = \lambda \int_{\Omega} g(u_\lambda) \zeta \leq c_1 \lambda \int_{\Omega} g(u_\lambda) \phi_1 < \infty. 
\] (3.27)

In the view of the positivity of $u_\lambda$ and $v_\lambda$, we conclude that $u_\lambda$ and $v_\lambda$ are bounded in $L^1(\Omega)$.

As both $u_\lambda(x)$ and $v_\lambda(x)$ are increasing in $\lambda$, we conclude that $(u_\lambda, v_\lambda)$ converges to $(u^*, v^*)$ in $L^1(\Omega)$, and $g(u_\lambda)$ converges to $g(u_\lambda)$ in $L^1(\Omega, \delta(x) \, dx)$. Similar to [8, Lemma 5], it follows that $(u^*, v^*)$ is a weak solution of (1.6) with $\lambda = \lambda^*$.

We now proceed to the proof of the final two parts of theorem 1.2. First, we will study monotonicity properties of the minimal solutions $u_{\lambda,v}$ and $v_{\lambda,v}$ with respect to $v$, then we consider the limiting behaviour of the solutions as $v \to 0$, and finally the limiting behaviour of the solutions as $v \to \infty$.

The following lemma establishes the monotonicity of the $u$ component with respect to parameter $v$.

**Lemma 3.3.** $\lambda_\nu^* = \lambda_\nu^*(\nu)$ is a non-decreasing function of $\nu > 0$. Moreover, for $0 < \nu < \tilde{\nu}$ and $\lambda < \lambda_\nu^*$, let $u_{\lambda,v}$ and $u_{\lambda,\tilde{\nu}}$ be the first components of the minimal solutions of problem (1.6). Then $u_\nu \geq u_{\tilde{\nu}}$ in $\Omega$.

**Proof.** Recall that the minimal solution $(u_{\lambda,v}, v_{\lambda,v})$ was constructed by iterations given by system (3.12). Let $(\phi_k, \psi_k)$ and $(\tilde{\phi}_k, \tilde{\psi}_k)$ be solutions of (3.12) corresponding to $(\lambda, v)$ and $(\lambda, \tilde{v})$, respectively. We are going to show that $v < \tilde{v}$ implies that $\phi_k > \tilde{\phi}_k$ in $\Omega$ for each $k$. As a result, sequences converging pointwise to solutions $u_{\lambda,v}$ and $u_{\lambda,\tilde{v}}$ are ordered. Hence, the limits are ordered.

Subtracting the first and the second equations of system (3.12), we observe that
\[
- \Delta(\phi_k - \psi_k) + v \left( \frac{1 + d}{d} \right) (\phi_k - \psi_k) = \lambda g(\phi_k - 1) \quad \text{in} \quad \Omega, 
\] (3.28)

which implies that
\[
\phi_k \geq \psi_k \quad \text{in} \quad \Omega, 
\] (3.29)

for each $k$. By adding the first and the second equations of system (3.12), we also have that
\[
- \Delta(\phi_k + d \psi_k) = \lambda g(\phi_k - 1) \quad \text{in} \quad \Omega, 
\] (3.30)

for each $k$ and $v$. Needless to say that identical equations hold for $\tilde{\phi}_k$ and $\tilde{\psi}_k$.

Now let us show that $\phi_1 \geq \tilde{\phi}_1$. Indeed as $\phi_0 = \tilde{\phi}_0 = 0$ in $\Omega$, we have from (3.30)
\[
- \Delta(\phi_1 + d \psi_1) = - \Delta(\tilde{\phi}_1 + d \tilde{\psi}_1) = \lambda g(0) \quad \text{in} \quad \Omega, 
\] (3.31)

and therefore
\[
\phi_1 + d \psi_1 = \tilde{\phi}_1 + d \tilde{\psi}_1 \quad \text{in} \quad \Omega. 
\] (3.32)

Taking the difference of the first equations of system (3.12) for $\nu$ and $\tilde{\nu}$ after some algebra we obtain
\[
- \Delta(\phi_1 - \tilde{\phi}_1) + v(\phi_1 - \tilde{\phi}_1) - v(\psi_1 - \tilde{\psi}_1) = (\tilde{\nu} - \nu)(\tilde{\phi}_1 - \tilde{\psi}_1) \quad \text{in} \quad \Omega. 
\] (3.33)

Using (3.32) and (3.29), we obtain from (3.33)
\[
- \Delta(\phi_1 - \tilde{\phi}_1) + v \left( \frac{1 + d}{d} \right) (\phi_1 - \tilde{\phi}_1) = (\tilde{\nu} - \nu)(\tilde{\phi}_1 - \tilde{\psi}_1) > 0 \quad \text{in} \quad \Omega, 
\] (3.34)

and thus
\[
\phi_1 > \tilde{\phi}_1 \quad \text{in} \quad \Omega. 
\] (3.35)
Let us show now that \( \phi_k > \tilde{\phi}_k \) provided \( \phi_{k-1} > \tilde{\phi}_{k-1} \). Indeed, assume that
\[
\phi_{k-1} > \tilde{\phi}_{k-1} \quad \text{in } \Omega,
\] (3.36)
then by (3.30) we have
\[
- \Delta (\phi_k + d \psi_k) = \lambda g(\phi_{k-1}) \quad \text{and} \quad - \Delta (\tilde{\phi}_k + d \tilde{\psi}_k) = \lambda g(\tilde{\phi}_{k-1}) \quad \text{in } \Omega,
\] (3.37)
and using the fact that \( g \) is increasing and assumption (3.36), we have
\[
g(\phi_{k-1}) > g(\tilde{\phi}_{k-1}),
\] (3.38)
which together with (3.37) gives
\[
\phi_k + d \psi_k > \tilde{\phi}_k + d \tilde{\psi}_k \quad \text{in } \Omega.
\] (3.39)
Combining the first equations of system (3.12) for \( u \) and \( \tilde{v} \) we have
\[
- \Delta (\phi_k - \tilde{\phi}_k) + v(\phi_k - \tilde{\phi}_k) - \nu(\psi_k - \tilde{\psi}_k) = \lambda (g(\phi_{k-1}) - g(\tilde{\phi}_{k-1})) + (\tilde{v} - v)(\tilde{\phi}_k - \tilde{\psi}_k) > 0 \quad \text{in } \Omega.
\] (3.40)
Note that the positivity of the right-hand side of equation (3.40) follows from (3.38) and (3.29).

Using (3.39) from (3.40), we have
\[
- \Delta (\phi_k - \tilde{\phi}_k) + v \left( 1 + \frac{d}{\nu} \right) (\phi_k - \tilde{\phi}_k) > 0 \quad \text{in } \Omega,
\] (3.41)
which yields
\[
\phi_k > \tilde{\phi}_k \quad \text{in } \Omega.
\] (3.42)
In view of (3.35), inequality (3.42) holds for each \( k \geq 1 \). As \( \phi_k \to u \) and \( \tilde{\phi}_k \to \tilde{u} \) we conclude that
\[
u_{\lambda,v} \geq \tilde{u}_{\lambda,\tilde{v}} \quad \text{in } \Omega.
\] (3.43)
By construction, it follows that \( \lambda_{v}^{*} = \lambda^{*}(v) \) is a non-decreasing function of \( v > 0 \), which completes the proof. \( \blacksquare \)

Let us now consider the limiting behaviour of system (1.6) as \( v \to 0 \).

**Proposition 3.4.** For \( \lambda < \Lambda^{*} \) and \( v \to 0 \), the minimal solution \( (u_{\lambda,v}, v_{\lambda,v}) \) converges uniformly to \((u_0,0)\), where \( u_0 \) is the minimal solution of (1.9).

**Proof.** Let \( \lambda < \Lambda^{*} \) and let \( u_0 \) be the minimal classical solution of Gelfand problem (1.9). Then \((u_0, u_0)\) is a supersolution of (1.6) and in particular for all \( v > 0 \) it holds that
\[
(u_{\lambda,v}, v_{\lambda,v}) \leq (u_0, u_0).
\] (3.44)
The minimal solution \( v_{\lambda,v} \) of the second equation of (1.6) can be represented as
\[
v_{\lambda,v} = v[-\Delta + v]^{-1} u_{\lambda,v}.
\] (3.45)
As \( u_{\lambda,v} \) is uniformly bounded in \( L^{\infty}(\Omega) \) and \([-\Delta + v]^{-1} \) is a bounded operator from \( L^{\infty}(\Omega) \) into \( L^{\infty}(\Omega) \), we have
\[
0 < v_{\lambda,v} \leq vC \quad \text{in } \Omega,
\] (3.46)
for some \( C > 0 \) independent of \( v \), that is,
\[
\| v_{\lambda,v} \|_{L^{\infty}} \to 0 \quad \text{as } v \to 0.
\] (3.47)
Next, as \( g \) is of class \( C^{1} \), combining the first equations of (1.6) and (1.9) and setting \( w_{v} := u_0 - u_{\lambda,v} \) we obtain
\[
- \Delta w_v = \lambda(g(u_0) - g(u_{\lambda,v})) + v(u_{\lambda,v} - v_{\lambda,v}) = g'(\xi)w_v + v(u_{\lambda,v} - v_{\lambda,v}),
\] (3.48)
where \( \xi_\nu \in L^\infty(\Omega) \) satisfy
\[
 u_{\lambda,\nu} \leq \xi_\nu \leq u_0 \quad \text{in } \Omega.
\] (3.49)

As \( u_0 \) is a minimal solution of corresponding Gelfand problem (1.9), it is stable [8] in the sense that the operator \(-\Delta - \lambda g'\gamma(\xi_\nu)\) is invertible in \( L^2(\Omega) \). In the view of (3.49), the operator \(-\Delta - \lambda g'\gamma(\xi)\) is also invertible in \( L^2(\Omega) \). This allows us to rewrite (3.48) as follows:
\[
w_\nu = v[-\Delta - \lambda g'\gamma(\xi_\nu)]^{-1}(u_{\lambda,\nu} - v_{\lambda,\nu}).
\] (3.50)

As \([ -\Delta - \lambda g'\gamma(\xi_\nu) ]^{-1}\) is bounded from \( L^\infty(\Omega) \) into \( L^\infty(\Omega) \), while \( u_{\lambda,\nu} \) and \( v_{\lambda,\nu} \) are uniformly bounded in \( L^\infty(\Omega) \), we conclude that
\[
0 < w_\nu \leq C_\nu \quad \text{in } \Omega.
\] (3.51)

Therefore, \( \|w_\nu\|_{L^\infty} \to 0 \) as \( \nu \to 0 \) and the assertion follows.

We now give several lemmas needed to study the behaviour of the solution for problem (1.6) in the limit of \( \nu \to \infty \). To shorten the notation, we denote
\[
K_\nu := d[1 - v[-\gamma \Delta + v]^{-1}],
\] (3.52)
where \( \gamma = d/(1 + d) \). Clearly, \( K_\nu \) is a bounded linear operator in \( C(\Omega) \) and in \( L^p(\Omega) \), for any \( 1 \leq p \leq \infty \). Similar to the derivation of (2.6), we see that if \((u, \nu)\) is a classical solution of (1.6) then \( u \) is a classical solution of the non-local equation
\[
-\Delta u = \frac{\lambda}{1 + d}(1 + K_\nu)g(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\] (3.53)
and
\[
v = u - \lambda \gamma [-\gamma \Delta + v]^{-1}g(u).
\] (3.54)

We present two standard results about the properties of the operator \( K_\nu \).

**Lemma 3.5.** For all \( \nu > 0 \), \( K_\nu \) is a positive operator in \( L^\infty(\Omega) \), i.e. for every \( f \in L^\infty(\Omega) \), \( f \geq 0 \) implies \( K_\nu f \geq 0 \). Moreover, \( \|K_\nu\|_{L^2 \to L^2} = 1 \) and \( K_\nu \) strongly converges to zero as \( \nu \to \infty \), i.e. for every \( f \in L^2(\Omega) \), \( \lim_{\nu \to \infty} \|K_\nu f\|_{L^2} = 0 \).

**Proof.** Observe that the resolvent operator \([-\gamma \Delta + v]^{-1}\) is well defined in \( L^2(\Omega) \) for all \( \nu > 0 \), and by spectral theorem,
\[
\|v[-\gamma \Delta + v]^{-1}\|_{L^2 \to L^2} = v(\gamma \mu_1 + v)^{-1} < 1,
\] (3.55)
that is, \( v[-\gamma \Delta + v]^{-1} \) is a contraction in \( L^2(\Omega) \). Then by [20, Proposition 1.3], the family \([-\gamma \Delta + v]^{-1}\) is a strongly continuous contraction resolvent, that is,
\[
\|f - v[-\gamma \Delta + v]^{-1}f\|_{L^2} \to 0 \quad \text{for every } f \in L^2(\Omega).
\] (3.56)
Moreover, by [20, Definition 4.1 and Chapter 2.1], \([-\gamma \Delta + v]^{-1}\) is sub-Markovian, that is, for all \( f \in L^2(\Omega) \),
\[
0 \leq f \leq 1 \quad \text{implies } 0 \leq v[-\gamma \Delta + v]^{-1}f \leq 1.
\] (3.57)
As \( K_\nu = I - v[-\gamma \Delta + v]^{-1} \), we conclude that \( K_\nu \) is a positive operator in \( L^\infty(\Omega) \) for all \( \nu > 0 \), and that \( \|K_\nu f\|_{L^2} \to 0 \) for every \( f \in L^2(\Omega) \).

**Lemma 3.6.** Let \( \lambda < \lambda_{v_0}^1 \) for some \( v_0 > 0 \). Then \( \mu_1(-\Delta - (\lambda/(1 + d))g'(u_0,\nu)) > 0 \) for all \( \nu > v_0 \).
Proof. Let \((u_{\lambda,v}, v_{\lambda,v})\) be the minimal positive solution of (1.6). Consider the eigenvalue problem for the linearized system

\[
\begin{align*}
-\Delta \phi + (v - g'(u_{\lambda,v}))\phi - v \psi &= \mu \phi & \text{in } \Omega, \\
-d \Delta \psi + v \psi - v \phi &= \mu \psi & \text{in } \Omega, \\
\phi &= \psi = 0 & \text{on } \partial \Omega.
\end{align*}
\]

(3.58)

and

It is known that system (3.58) admits the principal eigenvalue \(\bar{\mu}_{\lambda,v,1}\), the corresponding eigenfunction \((\phi, \psi)\) can be chosen positive and \(\phi, \psi \in C^2(\Omega) \cap C_0(\bar{\Omega})\) (see [18, Theorem 1.1]).

Rearranging system (3.58) as in §2, we see that \(\bar{\mu}_{\lambda,v,1} > 0\) satisfy the non-local equation

\[
-\Delta \phi - \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v} - \phi) = \bar{\mu}_{\lambda,v,1} \phi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega.
\]

(3.59)

Similar to [9, Proposition 3.4.4], assume that \(\bar{\mu}_{\lambda,v,1} < 0\). Given \(\varepsilon > 0\), we compute

\[
-\Delta(u_{\lambda,v} - \varepsilon \phi) - \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v} - \varepsilon \phi)
\]

\[
= \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v}) - \varepsilon \left( \frac{\lambda}{1 + d}(1 + K_v)g'(u_{\lambda,v})\phi - \bar{\mu}_{\lambda,v,1} \varepsilon \phi - \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v} - \varepsilon \phi) \right)
\]

\[
= -\frac{\lambda}{1 + d}(1 + K_v)(g(u_{\lambda,v} - \varepsilon \phi) - g(u_{\lambda,v}) + \varepsilon g'(u_{\lambda,v})\phi) - \bar{\mu}_{\lambda,v,1} \varepsilon \phi,
\]

(3.60)

where

\[
(1 + K_v)(g(u_{\lambda,v} - \varepsilon \phi) - g(u_{\lambda,v}) + \varepsilon g'(u_{\lambda,v})\phi) = o(\varepsilon),
\]

(3.61)

as \(g \in C^1, u_{\lambda,v}, \phi \in L^\infty(\Omega)\) and \((1 + K_v)\) is bounded in \(L^\infty(\Omega)\). As \(\bar{\mu}_{\lambda,v,1} < 0\), we deduce that

\[
-\Delta(u_{\lambda,v} - \varepsilon \phi) - \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v} - \varepsilon \phi) \geq 0,
\]

(3.62)

for all sufficiently small \(\varepsilon > 0\).

Using the definition of \(K_v\), we then conclude that

\[
-\Delta(u_{\lambda,v} - \varepsilon \phi) + v(u_{\lambda,v} - \varepsilon \phi - v_{\lambda,v}) - \lambda g(u_{\lambda,v} - \varepsilon \phi)
\]

\[
= -\Delta(u_{\lambda,v} - \varepsilon \phi) - \frac{\lambda}{1 + d}(1 + K_v)g(u_{\lambda,v} - \varepsilon \phi) \geq 0 \quad \text{in } \Omega,
\]

(3.63)

and further we note that

\[
-\Delta v_{\lambda,v} + v(v_{\lambda,v} - u_{\lambda,v} + \varepsilon \phi) = \varepsilon \phi \geq 0,
\]

(3.64)

for all sufficiently small \(\varepsilon > 0\). This means that \((u_{\lambda,v} - \varepsilon \phi, v_{\lambda,v})\) is a supersolution of system (1.6). By lemma 3.2, we conclude that system (1.6) admits a solution \((\bar{u}_{\lambda,v}, \bar{v}_{\lambda,v})\) with \(0 < \bar{u}_{\lambda,v} < u_{\lambda,v}\). But this contradicts the minimality of \((u_{\lambda,v}, v_{\lambda,v})\). Hence \(\bar{\mu}_{\lambda,v,1} \geq 0\).

Now observe that

\[
\int_\Omega K_v g'(u_{\lambda,v}) \phi^2 \geq g'(0) \sigma_1(v) \int_\Omega \phi^2,
\]

(3.65)

where \(\sigma_1(v) = \lambda \frac{d}{d(1 + \varepsilon)} \mu_1(-\Delta + 1)^{-1} > 0\) is the smallest eigenvalue of \(K_v\) in \(L^2(\Omega)\). This implies that \(\mu_1(-\Delta + g'(u_{\lambda,v})) > \bar{\mu}_{\lambda,v,1} \geq 0\). \(\blacksquare\)

Using results presented above we can now describe the limiting behaviour of the solution for problem (1.6).

**Proposition 3.7.** For \(\lambda < \Lambda^*(1 + d)\) and \(v \to \infty\), the minimal solution \((u_{\lambda,v}, v_{\lambda,v})\) converges uniformly to \((u_\infty, u_\infty)\), where \(u_\infty\) is the minimal solution of (1.10).
Proof. Using representation (3.53), we see that
\[ -\Delta u_\lambda \nu = \frac{\lambda}{d + 1} g(u_\lambda \nu) + \frac{\lambda}{d + 1} \mathbf{K}_\nu g(u_\lambda \nu) \geq \frac{\lambda}{d + 1} g(u_\lambda \nu). \] (3.66)
In the view of the positivity of \( \mathbf{K}_\nu \) (lemma 3.5), we conclude that \( u_\lambda \nu \) is a supersolution of (1.10). As \( u_\infty \) is the minimal solution of (1.10), we see that
\[ u_\lambda \nu \geq u_\infty. \] (3.67)
By lemma 3.3, \( u_\nu \) is monotone decreasing as \( \nu \to \infty \), so for a \( \tilde{\nu} > 0 \) and all \( \nu \in [\tilde{\nu}, \infty) \) we have
\[ u_\nu \geq u_\lambda \nu \geq u_\infty. \] (3.68)
In particular, \( \{u_\lambda \nu \}_{\nu \geq \tilde{\nu}} \) and \( \{g(u_\lambda \nu)\}_{\nu \geq \tilde{\nu}} \) are bounded in \( L^2(\Omega) \), for every \( \nu \in [1, \infty) \). As \( \mathbf{K}_\nu \) is bounded and \( (-\Delta)^{-1} \) is compact in \( L^p(\Omega) \) for every \( p \in (1, \infty) \), and (3.66) can be rewritten as
\[ u_\lambda \nu = (-\Delta)^{-1} \left( \frac{\lambda}{d + 1} g(u_\lambda \nu) + \frac{\lambda}{d + 1} \mathbf{K}_\nu g(u_\lambda \nu) \right), \] (3.69)
we conclude that, for a sequence \( \nu_n \to \infty \), \( u_{\lambda \nu_n} \) converges to a limit \( \tilde{u}_\infty \). Moreover, as \( u_{\lambda \nu_n}(x) \) is monotone decreasing in \( x \) (lemma 3.3), we conclude that \( \lim_{\nu \to \infty} u_{\lambda \nu} = \tilde{u}_\infty \) and \( \lim_{\nu \to \infty} g(u_{\lambda \nu}) = g(\tilde{u}_\infty) \). Using strong convergence of \( \mathbf{K}_\nu \) to zero (lemma 3.5), we conclude that
\[ \|\mathbf{K}_\nu g(u_{\lambda \nu})\|_{L^2} \leq \|\mathbf{K}_\nu g(\tilde{u}_\infty)\|_{L^2} + \|\mathbf{K}_\nu\|_{L^2 \to L^2} \|g(u_{\lambda \nu}) - g(\tilde{u}_\infty)\|_{L^2} \to 0, \] (3.70)
as \( \nu \to \infty \).
Combining (3.53) and (1.10) and setting \( w_\nu := u_\lambda \nu - u_\infty \), we obtain
\[ -\Delta w_\nu = \frac{\lambda}{d + 1} (g(u_\lambda \nu) - g(u_\infty)) + \frac{\lambda}{d + 1} \mathbf{K}_\nu g(u_\lambda \nu) \]
\[ = \frac{\lambda}{d + 1} g'((\xi_\nu)w_\nu + \frac{\lambda}{d + 1} \mathbf{K}_\nu g(u_\lambda \nu), \] (3.71)
where \( \xi_\nu \in L^\infty(\Omega) \) and
\[ u_\infty \leq \xi_\nu \leq u_\lambda \nu. \] (3.72)
By lemma 3.6, the operator \( -\Delta - (\lambda/(1 + d))g'(u_\lambda \nu) \) is invertible in \( L^2(\Omega) \). Then, in view of (3.72), the operator \( -\Delta - (\lambda/(1 + d))g'(\xi_\nu) \) is also invertible in \( L^2(\Omega) \). This allows us to rewrite (3.71) as follows:
\[ w_\nu = \frac{\lambda}{d + 1} \left[ -\Delta - \frac{\lambda}{1 + d} g'(\xi_\nu) \right]^{-1} \mathbf{K}_\nu g(u_\lambda \nu). \] (3.73)
In view of (3.70) and as the operator \( [-\Delta - (\lambda/(1 + d))g'(\xi_\nu)]^{-1} \) is bounded in \( L^2(\Omega) \), we conclude that \( \|w_\nu\|_{L^2} \to 0 \) as \( \nu \to \infty \), and thus \( \|u_{\lambda \nu} - u_\infty\|_{L^2} \to 0 \) as \( \nu \to \infty \). In particular, this implies that \( \tilde{u}_\infty = u_\infty \).
Furthermore, using the standard bootstrap argument we improve the convergence to conclude that \( \|w_\nu\|_{L^\infty} \to 0 \) as \( \nu \to \infty \), and thus \( \|u_{\lambda \nu} - u_\infty\|_{L^\infty} \to 0 \) as \( \nu \to \infty \).
Finally, by (3.54),
\[ u_{\lambda \nu} = u_{\lambda \nu} - \lambda \gamma [-\gamma \Delta + v]^{-1} g(u_{\lambda \nu}) \] (3.74)
As \( [-\gamma \Delta + v]^{-1} \) is bounded as the operator from \( L^\infty(\Omega) \) into \( L^\infty(\Omega) \), we also conclude that \( \|u_{\lambda \nu} - u_{\lambda \nu}\|_{L^\infty} \to 0 \) as \( \nu \to \infty \).

We are now in a position to complete the proof of theorem 1.2.

Proof of parts (iv) and (v) of theorem 1.2. The claim follows immediately from lemma 3.3 and propositions 3.4 and 3.7.
4. Parabolic problem: proof of theorem 1.1

In this section, we present a proof of theorem 1.1. This theorem can be viewed as an extension of the result obtained in Gordon [16] for a system which describes thermal ignition in one-phase confined materials, which in turn is an extension of the result for the classical Gelfand problem given in Brezis et al. [8].

In order to proceed, we will need the following lemma.

Lemma 4.1. Let $U, V$ be a global classical solution of (1.5). Then, $(U_t, V_t) \geq 0$ in $\Omega$.

Proof. Differentiating system (1.5) with respect to time and setting $\xi = U_t, \eta = V_t$ we have

\[
\begin{align*}
\xi_t - \Delta \xi &= \lambda g'(U)\xi + \nu(\eta - \xi), \\
\alpha\eta_t - d\Delta \eta &= \nu(\xi - \eta) \quad \text{in } (0,T) \times \Omega, \\
\xi &= \eta = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(4.1)

and

\[
\xi(0,\cdot), \eta(0,\cdot) \geq 0 \quad \text{in } \Omega.
\]

Linear system (4.1) is quasi-monotone and thus the componentwise comparison principle holds [14]. As $\xi = \eta = 0$ is a subsolution, we have $\xi, \eta \geq 0$ for all $t \geq 0$ in $\Omega$. \[\blacksquare\]

Now we turn to the proof of theorem 1.1.

Proof of theorem 1.1. First we claim that if (1.6) has a classical solution, then (1.5) has a global solution. This follows directly from the fact that system (1.5) is quasi-monotone, and thus a comparison principle holds componentwise [14].

Now, let us show that the existence of a global solution for problem (1.5) implies the existence of a weak solution for (1.6).

Let us first note that by lemma 4.1 $U, V, U_t, V_t \geq 0$ for all $x \in \Omega$ and $t \geq 0$, so that solutions of problem (1.5) are non-negative and non-decreasing.

Next, observe that for each $\phi, \psi \in C^2(\Omega)$ with $\phi = \psi = 0$ on $\partial \Omega$ we have

\[
\frac{d}{dt} \int_\Omega U\phi + \int_\Omega U(-\Delta \phi) = \lambda \int_\Omega g(U)\phi + \nu \int_\Omega (V - U)\phi
\]

and

\[
\alpha \frac{d}{dt} \int_\Omega V\psi + d \int_\Omega V(-\Delta \psi) = \nu \int_\Omega (U - V)\psi.
\]

(4.2)

As before, let $\mu_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ in $\Omega$ and $\phi_1 > 0$ be the corresponding eigenfunction, with $\|\phi_1\| = 1$. Setting $\phi = \psi = \phi_1$ in (4.2) we have

\[
\frac{d}{dt} \int_\Omega U\phi_1 + \mu_1 \int_\Omega U\phi_1 = \lambda \int_\Omega g(U)\phi_1 + \nu \int_\Omega (V - U)\phi_1
\]

and

\[
\alpha \frac{d}{dt} \int_\Omega V\phi_1 + \mu_1 d \int_\Omega V\phi_1 = \nu \int_\Omega (U - V)\phi_1.
\]

(4.3)

We first claim that $\int_\Omega U\phi_1$ and $\int_\Omega V\phi_1$ are uniformly bounded in time.

From the first equation of (4.3), non-negativity of $V, \phi_1$ and Jensen’s inequality we have

\[
\frac{d}{dt} \int_\Omega U\phi_1 + (\mu_1 + \nu) \int_\Omega U\phi_1 = \lambda \int_\Omega g(U)\phi_1 + \nu \int_\Omega V\phi_1 \geq \lambda \int_\Omega g(U)\phi_1 \geq \frac{\lambda}{2} g\left( \int_\Omega U\phi_1 \right).
\]

(4.4)

By assumption (1.2), we have $g'(s) \to \infty$ as $s \to \infty$. Therefore, there is a constant $M_1 > 0$ such that

\[
\lambda g(s) - (\mu_1 + \nu) s \geq \frac{\lambda}{2} g(s), \quad \text{for } s \geq M_1.
\]

(4.5)

Now assume that $\int_\Omega U\phi_1 = M_1$ at $t = t_0$, then for $t \geq t_0$ we have

\[
\frac{d}{dt} \int_\Omega U\phi_1 \geq \frac{\lambda}{2} g\left( \int_\Omega U\phi_1 \right),
\]

(4.6)
which contradicts (1.2) and thus
\[ \int_{\Omega} U\phi_1 \leq M_1. \quad (4.7) \]
In view of (4.7), we have from the second equation of (4.3) that
\[ \alpha \frac{d}{dt} \int_{\Omega} V\phi_1 + (\mu_1 d + v) \int_{\Omega} V\phi_1 = v \int_{\Omega} U\phi_1 \leq vM_1, \quad (4.8) \]
which immediately implies that
\[ \int_{\Omega} V\phi_1 \leq M_2 \quad (4.9) \]
for some constant \( M_2 > 0 \). Finally integrating (4.3) on \((t, t + 1)\) and taking into account that \( g(U)\) and \( U \) are non-decreasing, we have
\[ \lambda \int_{\Omega} g(U(t))\phi_1 \leq \int_{t}^{t+1} \int_{\Omega} g(U)\phi_1 \leq \int_{t}^{t+1} \int_{\Omega} U(t+1)\phi_1 + (\mu_1 + v) \int_{t}^{t+1} \int_{\Omega} U\phi_1 \leq (1 + \mu_1 + v)M_1, \quad (4.10) \]
and thus
\[ \sup_{t > 0} \int_{\Omega} g(U)\phi_1 \leq \frac{1 + \mu_1 + v}{\lambda} M_1. \quad (4.11) \]
Now let us show that both \( U \) and \( V \) are bounded in \( L_1 \) uniformly in time. Let \( \zeta \) be the solution of (3.24). Observe that estimates (4.7), (4.9) and (4.11) imply that
\[ \int_{\Omega} U\zeta, \int_{\Omega} V\zeta, \sup_{t > 0} \int_{\Omega} g(U)\zeta \leq M_3 \quad (4.12) \]
for some constant \( M_3 \) independent of time. Next setting in (4.2) \( \phi = \psi = \zeta \) and integrating the result on \((t, t + 1)\) we have
\[ \begin{align*}
\int_{\Omega} U &\leq \int_{t}^{t+1} \int_{\Omega} U(t, \cdot)\zeta - \int_{\Omega} U(t+1, \cdot)\zeta + \lambda \int_{t}^{t+1} \int_{\Omega} g(U)\zeta + v \int_{t}^{t+1} (V - U)\zeta \leq M_4, \\
d \int_{\Omega} V &\leq d \int_{t}^{t+1} \int_{\Omega} V = \alpha \left( \int_{t}^{t+1} \int_{\Omega} V(t, \cdot)\zeta - \int_{\Omega} V(t+1, \cdot)\zeta + v \int_{t}^{t+1} (V - U)\zeta \leq M_5. \right)
\end{align*} \quad (4.13) \]
The first and the last inequalities in both equations of (4.13) hold because \( U \) and \( V \) are non-decreasing functions of time (lemma 4.1) and by (4.12), respectively. Thus,
\[ \sup_{t > 0} \|U(t)\|_{L^1(\Omega)} \leq M_4 \quad \text{and} \quad \sup_{t > 0} \|V(t)\|_{L^1(\Omega)} \leq M_5. \quad (4.14) \]
From (4.14) and monotone convergence theorem, we deduce that \( U \) and \( V \) have a limit \( u, v \) in \( L^1(\Omega) \). Moreover by (4.10), we have that \( g(U) \) converges to \( g(u) \) in \( L^1(\Omega, \delta(x)\,dx) \) as \( t \to \infty \). Integrating (4.2) on \((t, t + 1)\) we have
\[ \begin{align*}
\int_{\Omega} U\phi^{t+1}_1 - \int_{\Omega} U\phi_t + \int_{\Omega} (-\Delta \phi) = \lambda \int_{\Omega} g(U)\phi + v \int_{\Omega} (V - U)\phi \\
\end{align*} \quad (4.15) \]
and
\[ \begin{align*}
\alpha \int_{\Omega} V\psi^{t+1}_1 - d \int_{\Omega} V(-\Delta \psi) = v \int_{\Omega} (U - V)\psi.
\end{align*} \]
Finally, letting \( t \to \infty \) we have
\[ \begin{align*}
\int_{\Omega} u(-\Delta \phi) = \lambda \int_{\Omega} g(u)\phi + v \int_{\Omega} (v - u)\phi \\
\end{align*} \quad (4.16) \]
and
\[ \begin{align*}
d \int_{\Omega} v(-\Delta \psi) = v \int_{\Omega} (u - v)\psi.
\end{align*} \]
Therefore, the \( L^1 \)-limit of \( U \) and \( V \) as \( t \to \infty \) is a weak solution of (1.6), as defined in (3.1). Let us also note that \( U \) and \( V \) are non-decreasing in time. In view of the parabolic comparison principle
for (1.5), we conclude that the limit of $U$ and $V$ is a minimal weak solution of elliptic problem (1.6).

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