We discuss a quantum field detection model comprising two types of detection procedures: maximal detection, where the initial state of the system and detectors undergoes an irreversible evolution, and minimal detection, where the system–detector interaction consists of a small, reversible coupling and posterior maximal detection performed over the detector system. Combined, these detection procedures allow for a time-dependent description of signalling experiments involving yes/no type of questions. A particular minimal detection model, stable in the presence of the vacuum, is presented and studied, successfully reproducing the localization of the state after a detection.

1. Introduction

Quantum mechanics (QM) is an empirically based discipline, and therefore the processes of measurement should play a crucial role in the description of the physical phenomena. In traditional QM, measurements are regarded as a group of actions that, when performed over a system, make them go from having a probabilistic description to having, for at least an instant, one of their possible outcome values completely and classically determined, by making that value known to the observer. When such a measurement is followed by another probabilistic description of the evolution restarting from this newly determined initial condition, this process is often linked with the wave-function collapse hypothesis.

Traditionally, we think about a quantum system in terms of the axiomatic formulation given by von Neumann, where measurable quantities of a system are represented by hermitian operators acting over the
elements—or a mixture of elements—of a Hilbert space, which, in turn, represents the state of the system. Those measurable quantities do not necessarily hold all the physical information we can obtain from a system. Probably the most notable example of this is the work of Glauber regarding photon correlations [1–3], which shows that the important information about a physical system can have nothing to do with observable operators. We believe that in some situations the underlying concept we need to pay attention to is the detection process, which is more related to how we are interacting with the system under observation (SUO). In this paper, we provide a framework to study these processes.

The recently growing field of quantum information in relativistic scenarios and some other applications (e.g. [4,5]) has led to the development of several different detector models [6–8], normally particularly suited to the task at hand. Probably, the most cited example of this is the Unruh–DeWitt detector used in the derivation of the Unruh effect [6]. Similar constructions have been used to test for the generation of entanglement between different systems even in the absence of any mediating excitation of the fields involved [4]. The relative success of such models has raised some issues, most notably regarding their lack of stability in the presence of the vacuum state of the measured field [8]. These problems have even been related to the fundamental questions, what is really a quantum particle? and what does a detector click mean? [9]. Because of these issues, we shall refer to signals and particles interchangeably, that is, the particle concept will be understood only as the physical phenomenon that triggers a particle detector, thereby avoiding a discussion of its real nature.

Detectors are typically regarded as separate quantum systems that can interact with the SUO. In this paper, we discuss a detection model that generalizes two previous models. In the first part of this paper, motivated by the theory of photon correlations, we take the idea that the detection can change the state of the system (SUO and detector) in an irreversible way. This is implemented using semi-unitary (isometric) evolution operators that preserve probability, but that do not possess a well-defined time reversal effect over the initial state. An example of this is given by the detection of a photon via its absorption by a detector. Although the physical processes involved in the absorption of a photon and a massive particle are different, they share the same characteristic of taking away one source signal from the description of the state of system. Indications of this similarity have recently been given in [10] and here we assume that both processes can be described in the same way. Other possibility is to change the state of the detector system in an irreversible way, which is the case when the SUO state triggers some macroscopical amplification in the detector. Both situations will be treated using this particular type of evolution and will be called maximal detection.

The other model that guides our work is the one considered by Mott [11]. He explained why an $\alpha$ particle leaves a straight track in a cloud chamber even when intuitively one would describe the system with a spherically symmetric probability density irradiating from the decaying nucleus. In his time-independent description, the excitation of an atomic detection site conditions the probability of other detector sites being excited, being significant only around a narrow cone streaming from the first excited site and with its axis directed away from the emitting nucleus. In this scenario, one assumes that the interaction between the SUO and the detector is through an interaction Hamiltonian making that part of the evolution unitary and thus reversible. We call such a process minimal detection. The process in which macroscopic clusters of condensation form around the excited detection site is then modelled, in the second step, as a maximal process. We will use this experimental set-up as a way to test our model, showing that our time-dependent treatment follows the expected behaviour.

In §2, we present the two types of detectors mentioned above in terms of what we call the detection architecture. In §3, we set the general conventions and notations used to describe the confined solutions we will use. We then generalize Glauber’s detector model for matter waves to then introduce minimal detectors, giving a model for the interaction with the field to be measured. In §4, the model is applied to the Mott problem. We close in §5 with some final conclusions and remarks.
2. Architecture

Here, we present two general detection ‘architectures’ or descriptions of ways in which detection processes are carried out. We are used to two kinds of measurement processes, as categorized by Pauli ([12] or see [13]). They both depend on using a small quantum system coupled to the SUO to extract the relevant information about the system. Measurements of the first kind—sometimes also referred to as von Neumann measurements—correspond to measurements where the interaction with the measuring instrument changes the state of the SUO as little as possible during the process. Measurements of the second kind, on the other hand, are those that convey the necessary information from the SUO by changing it in a controlled way. In general, a measurement of the first kind will yield the same result upon repetition and we can calculate the post-measurement state using the projection postulate, whereas one of the second kind will give statistically different results on successive runs and the projection postulate will not hold.

In our case, we need to approach this discussion from a different angle. For us, the small quantum system that couples and/or interacts with the SUO, will be called an elementary signal detector (ESD) or just detector, thus the process will be regarded as a detection. A measurement will consist of the interpretation of the information provided by possibly many detectors and macroscopic systems interacting with the SUO and will only have a meaning for a particular observer or set of observers for which the interpretation of said information is the same.

The focus of our discussion will not be on the back-action or the residual changes that the detection process will leave on the SUO but on the kind of process that will yield the detection result. Although in many cases, we will be able to draw a correspondence between our categorization and the usual one, our aim here will be to better understand the detection mechanism in terms of the process taking place during its execution, as described in the following.

(a) Minimal detection plan

The first detection plan is comparable to the usual von Neumann measurements, consisting of a quantum detector system under the control of the experimentalist, coupled to the SUO and evolving alongside it under a unitary evolution. At a certain point in time determined by the observer, the joint evolution is finished, leaving the detector unchanged until it is then inspected or observed by the experimentalist. This inspection of the detector can be viewed as a maximal detection over the minimal ESD (we will discuss this second plan later on). When the coupling interaction can be considered small, the process can be related to indirect measurements; if we impose the coupling term to have some particular set of commutation relations with the free system, we can build a quantum non-demolition measurement, but in general we do not need to make any of these assumptions, the only feature we want from the ESD is that upon inspection it will give us the required information from the SUO. We shall refer to this architecture as the minimal detection plan, as it relegates the actual detection to the second stage, leaving the interface between SUO and observer as a simple unitary evolution.

In a given run of an experiment, each ESD can be accessed only once, over a single interval of time as measured by the observer’s clock. Each such time is called a stage. There is one important difference between the concept of a physical detector and an ESD. The latter is a single theoretical opportunity available to the observer to extract quantum information at a given stage in a process, whereas the former exists in the laboratory and can persist over extended periods of time, generating as many ESDs as the detection architecture requires.

Consider now a simple detector which can only give a no or yes—ground and excited states, respectively—answer to our questions. During the time the detector is coupled to the SUO, there is a unitary regeneration flow back and forth between the ESD ground state and the excited (or signal) state, until a time of the observer’s choosing at which point the physical detector is inspected for a signal, i.e. observed. In this model, there is zero irreversible amplification from the
signal state into other detection channels until the very end of the run, when the observer looks at the physical detector. This form of detection architecture is used in conventional approaches to detection.

Our conventions are as follows. The total Hilbert space $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$, where $\mathcal{H}_S$ is the Hilbert space encoding the quantum degrees of freedom of the SUO and $\mathcal{H}_R$ is the physical detector register space. We use an arbitrary basis $B_S \equiv \{|\alpha\rangle\}$ for the SUO space $\mathcal{H}_S$ with a modified Dirac notation for clarity. A convenient basis for $\mathcal{H}$ is $B \equiv \{|i\rangle \otimes |\alpha\rangle\}$. $\mathcal{H}_R$ is always finite dimensional, whereas $\mathcal{H}_S$ can be infinite dimensional. In the case $\delta_\alpha^\beta$ is the Dirac delta. Then some variant of the Dirac delta.

The one-stage evolution of the joint state can be expressed as

$$U = U_{00}|0\rangle\langle 0| + U_{10}|0\rangle\langle 1| + U_{01}|0\rangle\langle 0| + U_{11}|1\rangle\langle 1|,$$

where $U_{ij}$ are corresponding evolution operators acting over the SUO state $|\psi\rangle$, for each possible detector state.

As the operator $U$ is unitary, in this model of detection interference terms owing to regeneration appear, i.e. there is an amplitude flow from ground state to signal state and then back to ground state. In the maximal detection model discussed in §2b, there is a veto on this subprocess. Although we assume that we can have access to the detector’s degrees of freedom, we know that a coupled quantum system, by itself, constitutes no detection. In other words, we cannot extract information from the SUO by unitary evolution alone. In §2b, we describe how we achieve this by means of ESDs.

(b) Maximal detection plan

In this scenario, the total Hilbert space $\mathcal{H}$ is as before and again we consider the case where there is one actual physical detector in the laboratory. This detector is now such that at each stage, though the ground state $|0\rangle$ of the detector can make a transition to its signal state $|1\rangle$, the signal state $|1\rangle$ cannot make a transition back to the ground state $|0\rangle$. Instead, irreversible amplification processes start to kick in once the detector gets into its signal state. In accordance with the formalism and nomenclature introduced in [14], the detector becomes decommissioned as an active quantum detector capable of transmitting quantum amplitudes further and now acts as a classical source of information. This means in particular that quantum interference/regeneration cannot now occur for that detector once it has reached its signal state $|1\rangle$. With this, a maximal detector can act on its own over an SUO, or it can be the last stage of detection, acting over a minimal detector, that is, maximally detecting the state of a minimal detector.

An example of such detector is given by Glauber’s photon correlation theory. When a photon is caught by a detector there is no going back; no other photon is produced to take its place and the triggering of the detector belongs now to the classical realm, signalling that a particular event has occurred. In this example, we could, this time, draw a comparison between the maximal detection picture and measurements of the second kind, but we note that the motivation is different. For us, it is not important that the SUO is changed (a photon is absorbed) but the fact that the detector has produced a series of effects that makes the absorption of the photon, effectively irreversible. For instance, we could couple the triggering of the maximal detector with a projection operator for a particular SUO state instead to the absorption of a quanta and still generate the same irreversible detection evolution.

Given this detection architecture, we now need a mathematical description that can handle the irreversibility we have described. In [15], such processes are described by a semi-unitary (or isometric) evolution, where the initial Hilbert space $\mathcal{H}_n$ had dimension less or equal to that of the final Hilbert space $\mathcal{H}_{n+1}$. A semi-unitary operator $U$ is such that $U^\dagger U = I_n$, but $UU^\dagger$ is not
necessarily the identity of $\mathcal{H}_{n+1}$. If we consider such evolution at each stage, we can model the amplification processes in the apparatus, as an ever growing dimension of the Hilbert space. The state of the detector will evolve according to the rule $|n\rangle \rightarrow |n+1\rangle$ for $n \geq 1$. That is, a signal state can only evolve to a higher signal state at each stage. This growth of the Hilbert space, and the resulting piecewise evolution, allows us to not include explicitly the complete measurement procedure carried over the detectors or the effects of an environment over the microscopical degrees of freedom.

In this description, the state vector not only encodes the fact that the detector has been triggered but also how long ago (how many stages ago) this happened. At each stage, the dimension of the Hilbert space needs to grow by one to allow for the extra time step elapsed. In this way, we can account for the irreversibility in a physically meaningful way. Detectors which have been triggered and cannot come back will be called decommissioned, because they will no longer affect the evolution of the system.

As illustration of use, we take a system composed by a decaying particle and a detector that can be excited by one of the by-products of the decay. We can take an initial state of the form $|\psi_0\rangle = |\uparrow\rangle \otimes |0\rangle \equiv |\uparrow,0\rangle$, where the $|\downarrow\rangle$ ($|\uparrow\rangle$) represents the (un)decayed state. The evolution of the basis elements during one time step of length $\tau$ can be taken to be

$$U_{\text{max}}(\tau) = (\alpha |\uparrow,0\rangle + \beta |\downarrow,1\rangle)(\uparrow,0) + \sum_{i=1}^{n} |\downarrow,i+1\rangle \langle\downarrow,i| + |\downarrow,0\rangle \langle\downarrow,0| + \sum_{i=1}^{n} |\uparrow,i+1\rangle \langle\uparrow,i|. \quad (2.2)$$

Here, the parameters $\alpha$ and $\beta$ are the amplitudes for the SUO of staying on the excited state and decaying, respectively, and are such that $|\alpha|^2 + |\beta|^2 = 1$. The first two terms provide the desired evolution, the third term gives a reasonable evolution for the uncommon initial condition of having an already decayed particle but no excitation on the detector ($|\downarrow,0\rangle$, while the last one is required to obtain the semi-unitary condition $U_{\text{max}}^{\dagger} U_{\text{max}} = I_0$, but does not represent a physical possibility. Note that in this case $U_{\text{max}}^{\dagger} U_{\text{max}} \neq 1$, so we have strict semi-unitarity.

A related approach has been extensively studied and is commonly known as continuous quantum measurements (see [16,17] for pedagogical introductions to the subject). It basically consists of performing short lived, consecutive weak measurements (measurements with considerable uncertainty about the value of the observable being measured). In the limit when the time intervals of those measurements become infinitesimally short, one obtains a stochastic equation for the state of the measured system, the stochastic part being related to the random nature of the outcome of each measurement.

### 3. Confined system

One long-lasting concern with Mott’s approach, which we address here, is the use of stationary states for the description of an intrinsically time-dependent process. In this approach, we describe source signals and detection apparatus in the same way, using time-dependent quantum fields confined in bounded regions of space. The time-dependent solutions of the field equations for the source signals are allowed to evolve freely over space after their preparation period, but owing to the relativistic nature of their evolution, the solutions remain confined inside the light cone of the initial state. On the other hand, detectors remain confined inside their initial volume during the whole experiment. At any time during the experiment, the observer can question a detector, after which that detector is taken out of the evolution equations, effectively decommissioning that particular subsystem. This gives a fully time-dependent description of the complete process.

Here, we present the general conventions and notations used to describe the confined solutions and their evolutions under two different circumstances: while the boundary conditions are maintained and when those are released.
(a) The set-up

The scenario we want to treat consists of time-dependent sources and detectors with a finite spatio-temporal extension. Although in most applications, one can consider the related scales as large enough to perform the simplifications associated with systems of infinite size, in many modern applications, accounting for the finiteness in time and space of the detection processes becomes a necessity. One example of this is QMs on curved space–times [18] where it may not be possible to define a global particle state [9], so one is forced to define the concept of particle in a localized way. The strategy we adopt to solve these kind of systems consists of three parts. First, we model state preparation, solving the field equations for the sources in a confined region of space. Then, those solutions are left to evolve over unbounded space, leaving their initial confined region. Finally, they are detected in a different, finite region of space–time. Similar localization constructions are commonly used in many applications (e.g. [5,18]), but here we will impose particular requirements over the wave-packets, so it will be useful to describe the conventions we will be using.

We to solve the usual free Lagrangian restricted to a simply connected region of space–time $R$ for a real scalar field $\varphi$,

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi \partial^{\mu} \varphi - m^2 \varphi^2),$$

(3.1)

where we use standard inertial coordinates $\{x^\mu\}$ with metric signature $(1, -1, -1, -1)$ and the convention $\hbar = 1$. The expectation values of the field $\varphi$ are assumed to be zero over the spatial boundaries of $R$ and outside. This accounts for the preparation period, where free signal states are produced. In later stages, these signal states could be subjected to interactions, but they are prepared initially free.

Inside the preparation device, the dynamical field satisfies the Klein–Gordon (KG) equation of motion $(\Box + m^2) \varphi (x) = 0$. For any complex valued solutions $f, g$ to this equation with support over $R$, their scalar product $\langle f, g \rangle_t$ at a given time $t$ is defined by

$$\langle f, g \rangle_t \equiv i \int_{R_t} d^3x (f(t,x) \overrightarrow{\partial_t} g(t,x) - f(t,x) \overrightarrow{\partial_t} g(t,x)),$$

(3.2)

where the integration is over the intersection $R_t$ of $R$ with the hyperplane labelled by $t$ for some particular inertial observer. The scalar product has the same properties as the usual, infinite, case (i) $\langle f, g \rangle_t = -\langle g, f \rangle_t$ (antisymmetry); (ii) $\langle f, g \rangle^*_t = \langle g^*, f^* \rangle_t$ (conjugation); and (iii) $\partial_t \langle f, g \rangle_t = 0$ (stability), assuming that the fields fall off sufficiently rapidly in $R_t$.

We assume that in the region $R$, there is a set of solutions $(f_\alpha)$ to the equation $(\Box + m^2) f = 0$ such that

$$\langle f^*_\alpha, f_\beta \rangle_t = \delta_{\alpha \beta} \quad \text{and} \quad \langle f_\alpha, f_\beta \rangle_t = 0,$$

(3.3)

where the index $\alpha$ may be continuous, discrete or a combination of both. The symbol $\delta_{\alpha \beta}$ represents the Kronecker or (the properly normalized) Dirac delta, depending on the nature of the index. We refer to the $f_\alpha$ as source basis functions.

During preparation, the quantum field $\varphi$ and source basis functions $f_\alpha$ satisfy the wave equation so, inside the support of the basis functions $f_\alpha$, we may take the expansion

$$\varphi(x) = \sum_\alpha \langle f_\alpha(x) \varphi \rangle_t a_\alpha + \langle f^*_\alpha(x) \varphi^* \rangle_t a^*_\alpha,$$

(3.4)

where the coefficients $\{a_\alpha, a^*_\alpha\}$ are invariants. Using the scalar product relations (3.3), we have

$$a_\alpha = \langle f^*_\alpha, \varphi \rangle_t \quad \text{and} \quad a^*_\alpha = -\langle f_\alpha, \varphi^* \rangle_t.$$

(3.5)

The canonical equal-time commutator for the quantum field then gives

$$[a_\alpha, a^*_\beta] = \delta_{\alpha \beta},$$

(3.6)

where $\delta_{\alpha \beta}$ represents the same symbol as in (3.3).

Relative to the chosen inertial frame, Lagrangian (3.1) gives a positive Hamiltonian, so we assume that there is a unique lowest energy eigenstate $|0, t\rangle$ at time $t$, which in the case of
a detector will characterize the no-signal state, and for a source the absence of an excited propagable state.

With expansion (3.4), we find that the operators are annihilated by the ground (no signal) state if for all \(\alpha\). If we also impose that \(\lambda_\alpha > 0\), then we obtain what we will call the signal basis for this model, equivalent to a normal mode expansion. Then the problem reduces to an eigenvalue problem for the \(\lambda_\alpha\), and we shall assume henceforth that this problem can be solved and that a signal basis is being used. Then, during the preparation period, the Hamiltonian takes the form

\[
H = \sum_\alpha \frac{\lambda_\alpha}{2} (a_\alpha^a a_\alpha + a_\alpha a_\alpha^a). \tag{3.7}
\]

The coefficients \(\{\lambda_\alpha\}\) are necessarily non-negative and independent of time \(t\).

(b) The solutions

Besides ortho-normalization conditions (3.3), the solutions for this system will be required to have additional properties. First, we will write them in momentum space at a particular time \(t_0\), via

\[
f_\alpha(t_0, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{f}_\alpha(p) e^{ip \cdot x}, \tag{3.8}
\]

where the on-energy shell condition \(p_0 = \sqrt{p^2 + m^2}\) is implied and will be used in the following. Note that now we have two dual labels, \(\alpha\) and \(p\), which are essentially different but related via \(\tilde{f}_\alpha(p)\), which is the distribution of momentum components contributing to (3.8) for a given \(\alpha\).

This change to momentum space is motivated by the need to relate excited states bounded to different space–time regions to allow them to eventually interact. We do this by considering all those bounded regions to exist inside a bigger one, called the void region \(V\). We then set up a basis set for the solutions in region \(V\), and if we assume this basis to be complete and follow the same rules as the bases existing in the source region \(S\) and in the detection regions \(d_i\), we can write the solutions of the smaller regions in terms of the basis elements of the solutions in the larger one. For practical purposes, we can consider region \(V\) to be of infinite extension, thus allowing us to use the Fourier basis as a translation between the solutions on different regions.

The corresponding orthogonality relation in momentum space is

\[
\int \frac{d^3p}{(2\pi)^3} \frac{(\lambda_\alpha + \lambda_\beta)}{(2p_0)^2} \tilde{f}_\alpha^* (p) \tilde{f}_\beta (p) = \delta_{\alpha\beta}, \tag{3.9}
\]

which is found using (3.3). The completeness relation of the solution set in momentum space is taken to be

\[
\sum_\alpha \frac{\lambda_\alpha}{p_0 + q_0} \tilde{f}_\alpha (p) \tilde{f}_\alpha^* (q) = (2\pi)^3 2p_0 \delta(p - q). \tag{3.10}
\]

The operators \(a_\alpha^a\) and \(a_\alpha^a\) are understood as the annihilation and creation operators for a signal with the quantum numbers given by the set \(\alpha\). Thus, we define them in terms of the underlying operators \(a(p)\) and \(a^*(p)\) of the unbounded free-field theory, i.e.

\[
a_\alpha = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{f}_\alpha (p) a(p). \tag{3.11}
\]

For these solutions, we can also define an \(n\) signal state via the application of \(n\) creation operators

\[
|n(\alpha, \ldots, \nu)\rangle = \frac{1}{\sqrt{n!}} a_\alpha^+ \cdots a_\nu^+ |0\rangle, \tag{3.12}
\]

and the number operator

\[
N = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} a^*(p) a(p), \tag{3.13}
\]

for which is easy to show that \([N, a_\alpha^+] = a_\alpha^+\), so \(|n(\alpha, \ldots, \nu)|N|n(\alpha, \ldots, \nu)\rangle = n\), where \(n\) is the number of creation operators applied in the original state \(|n(\alpha, \ldots, \nu)\rangle\) to the vacuum state \(|0\rangle\). Thus, the
number operator of the second quantized QMs is well defined to operate over \( n \) signals states defined with the confined operators \( a_\alpha^+ \). The confinement of the states arranges for the expectation value of \( N \) to be finite and well behaved.

During the preparation period, before the signal at the source region is released at time \( t_0 \), the evolution can be described via the usual quantum mechanical evolution operator \( U(t) = e^{-iHt} \) where the Hamiltonian \( H \) is the normal-ordered version of the one given in (3.7). This evolution preserves the boundary conditions and allows us to write the one-signal state as

\[
|\psi_\alpha(t < t_0)\rangle = e^{-i\lambda_\alpha t_0} a_\alpha^+ |0\rangle. \tag{3.14}
\]

After time \( t_0 \), the prepared signal is released into the void region (\( V \)) so the free evolution will be given by the Hamiltonian

\[
H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} a^+(p)a(p), \quad t > t_0. \tag{3.15}
\]

This generates a state evolution given by

\[
|\psi_\alpha(t > t_0)\rangle = a_\alpha^+(t) |0\rangle, \tag{3.16}
\]

with

\[
a_\alpha^+(t) = e^{-i\lambda_\alpha t_0} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{f}_\alpha(p) e^{-ip_0(t-t_0)} a^+(p), \tag{3.17}
\]

and where the extra phase, coming from the bounded evolution, can be removed by taking the preparation period to be small or simply taking \( t_0 = 0 \), which we will do in the following. The operators \( \{a(p), a^+(p)\} \) are time independent throughout the whole process, but the operators \( \{a_\alpha, a_\alpha^+\} \) are constant only for times earlier than \( t_0 \). After their release into the void region, they become time-dependent operators, even in the non-interacting case. The difference with interacting quantum fields is given by the fact that we know the precise evolution for the creation and annihilation operators (3.17), which allows us to write the amplitude to detect (by absorption) the created particle at the detector region \( D \) at a given time, as discussed in §3c.

Interactions during the propagation period (after \( t_0 \)) can be taken into account by modifying the Hamiltonian accordingly and treating it perturbatively, or expanding \( a_\alpha \) in an appropriate basis instead of (3.11), as needed.

(c) Maximal detection

Knowing the properties and evolution of the confined states, we are now interested in their detection properties. Here, we will consider a maximal detection process consisting of the absorption of a source signal either at an idealized point of space–time or over a region of space at a determined time as seen by some inertial observer.

It is well known that the solutions of the KG do not produce a positive-definite probability four-current, but by definition we know that a one-signal state is such that, if we have absorbed a quanta at a given point of space, we will get exactly one click. So, for our purposes, it will suffice to define the absorption operator at a point \( x \) as

\[
a(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{ip\cdot x} a(p). \tag{3.18}
\]

This can be incorporated into the maximal detection plan by creating a confined state which is then left to evolve freely for a time \( t \) as in (3.16) to then apply the one-particle detection operator.
defined in (2.2) where we take \( |↑⟩ = a^†(x)|0⟩ \) and \( |↓⟩ = |0⟩ \) and \( β = 1, α = 0 \). Then the amplitude for finding an excited detector acting at a point \( x \) and time \( t \) will be
\[
\Phi_α(t, x) = \sqrt{2\lambda_α} \sum_n (n| \otimes (0| U_{max} U(t) a^†_α |0⟩ \otimes |0⟩) \\
= \sqrt{2\lambda_α} (0| a(x) a^†_α(t)|0⟩ \\
= \sqrt{2\lambda_α} f_α(x),
\]
(3.19)
where \( U_{max} \) is time independent since we are working in the Schrödinger picture and we have made
\[
f_α(t, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{f}_α(p) e^{-i(p_0t - p \cdot x)},
\]
(3.20)
for any time \( t \) after the release of the particle. As \( f_α \) is an element of the signal basis, it fulfils the relation
\[
\int |Φ_α(t, x)|^2 d^3x = 1,
\]
(3.21)
which follows from (3.3) and is thus time independent. This makes \( |Φ_α(x)|^2 \), a worthy candidate for a probability density, that is, the probability of absorbing, or detecting, the particle at a certain point regardless of its momentum, very much like the detectors Glauber used in his theory for photon-correlations [1–3].

This key feature of being able to interpret the modulus square of the signal basis as the probability density of detecting a particle motivates the idea of creating detection regions that can be built to operate in a restricted region of space–time instead of a point detection as in the previous case (3.18). To this end, we now consider the signal state to be \( a^†_α(t)|0⟩ \) (see 3.17) and define a new basis set \( \{g_β\} \), for the detection region \( D \), where the analogue of (3.11) is
\[
b_β = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{g}_β^†(p)a(p).
\]
(3.22)
Although the basis set is changed to reflect the different region of space being occupied by the detector, the operator is still written in terms of the free annihilation operator \( a(p) \), because it is supposed to absorb the same type of particle as the one created at the source region. With this, the amplitude of absorbing in \( D \) the source particle created at \( S \) is
\[
A(t) = ⟨0|b_β a^†_α(t)|0⟩ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \tilde{f}_α(p) \tilde{g}_β^†(p)e^{-i p_0 t}.
\]
(3.23)
We calculate this amplitude numerically for a system with two space dimensions. In figure 1, we see how the modulus square of the amplitude (3.23) builds up after leaving the system to evolve for enough time for the source function to start overlapping with the detection region. For this calculation, the initial state was created taking a two-dimensional box of side \( 1 \) fm centred at the origin, and the KG equation solved for particles of masses from \( m \sim 400 \text{ MeV} \) up to \( m \sim 2000 \text{ MeV} \) imposing the solutions to be zero at the boundaries. In comparison, a deuteron has a RMS charge radius of \( 2.14 \) fm and a mass of \( 1862 \text{ MeV} \), so here we are using particles confined in a smaller region in order to give them a considerable speed once released, that is, when the system is left to evolve with no boundary conditions as in (3.16).

The detector region was a similar box centred \( 3 \) fm away. The first excited state was used for the source and for the detection basis function. The time step was taken to be \( τ \sim 3.3 \times 10^{-24} \) s. The leftmost, narrower peak corresponds to the lighter particle and increasing in mass to the right.

In a simple classical calculation, one finds that taking the initial momentum of the source to be \( p_1 = mvγ \) translates to a speed of \( v \sim 0.3c \) for the heaviest particle, so one could expect a classical particle to arrive at around nine time steps \( (τ) \), which marks the peak of the amplitude as function of time (likewise for the rest of the particles with different masses). After that, the amplitude slowly decays showing that, in this case, the optimal time to perform the detection is at the time given by the classical analogue.
Figure 1. Modulus square of the numerical amplitude (in arbitrary units) for detection according to (3.23) versus time steps. The result is plotted for five different masses; \{400, 800, 1200, 1600, 2000\} MeV c$^{-2}$. See text for further details.

The maximal detection procedure can be easily extended to consider $n$ signals and $m$ detectors ($m \leq n$) by defining the amplitude for a total detection

$$
\Phi_{\text{total}}(t, x_1, \ldots, x_n) = \left(\sqrt{2\lambda_\alpha} \ldots \sqrt{2\lambda_{\nu}}\right)^{-1} \sqrt{n!} \langle 0|a(x_1) \ldots a(x_n)U(t)\hat{a}_\alpha^\dagger \ldots \hat{a}_\nu^\dagger|0\rangle. \tag{3.24}
$$

Then, the probability density for triggering of all $n$ detectors at points $(x_1, \ldots, x_n)$ is

$$
P(t, x_1, \ldots, x_n) = \frac{1}{n_\alpha! \ldots n_\nu!} |\Phi_{\text{total}}(t, x_1, \ldots, x_n)|^2, \tag{3.25}
$$

where the factorials are a normalization factor depending on the multiplicity of the initial signal states; $n_\alpha$ is the number of signals occupying a state $\alpha$ and similarly for the rest of the states taking care to count each state only once. This ensures that the integral of the density $P$ over all the space variables $(x_1, \ldots, x_n)$ is 1.

The probability density of $m$ detectors being triggered at points $(x_1, \ldots, x_m)$ is then

$$
P_m(t, x_1, \ldots, x_m) = \int d^3x_{n-m} P(t, x_1, \ldots, x_n), \tag{3.26}
$$

where the integration is performed over the $n - m$ signals that are not tested by maximal detectors.

(d) Minimal detection

Here, we consider a detection model where a detector is a small quantum mechanical system, weakly coupled to the main physical system via an hermitian interaction Hamiltonian, i.e. via a reversible interaction. The detector is then tested for excitations or changes to its initial state at a given time set by the experimental procedure. In this test, the state of the detector is changed irreversibly via the absorption of a detector excitation. We do this as a simplification of the process of adding an extra maximal detector acting over the minimal detector. We are allowed to do this because, after the test, the minimal detector is taken out of the evolution of the system as a whole, effectively decommissioning that detector. As there is no absorption of the initial signal, it carries on possibly interacting with other detectors along its path until it is finally absorbed in a maximal detection somewhere. This allows for general scenarios of several incoming signal states and a different number of detectors along their way of propagation.
(i) Detector interaction and evolution

The scenario we will use to illustrate the model is given by one source state and $N$ detectors. Initially, the source is set to be on its first excited state inside its preparation region $S$, while the detectors are restricted to their base state on their respective regions $d_i$, $i = 1 \ldots N$. All of these subsystems are embedded inside the larger void region $V$, where interactions other than the system-detector could take place, but we can always make the response of the detectors sensible to the free part of the signal states.

In the same way, the operators $\{a_\alpha, a_\alpha^\dagger\}$ annihilate and create states in the source system, the operators $\{d_i, d_i^\dagger\}$ will play the same role for the $i$th detector (there should be no confusion between the symbol denoting an operator and the one denoting the region in which it operates). In this case, the particular state will be left unspecified and will be assumed to be known for each detector. The detector operators $\{d_i, d_i^\dagger\}$ follow the same commutation relations as the source particle operators and will commute among them and with the external field operators. This choice is motivated by the fact that detectors, being under the experimentalist’s control, can be treated as if the signals corresponded to distinguishable particles.

After time $t_0$, the boundary conditions of the source region no longer condition the state evolves freely through the void region. In principle, the detectors can enter into action at any time, previous or later than $t_0$, but they remain confined through the whole process. The initial Hamiltonian of the whole system will be given by

$$H(t) = H_0^S + \sum_{i=1}^N (H_0^i + H_{sd}^i) \Theta(t_i - t),$$  \hfill (3.27)

where $H_0^{S(i)}$ is the free Hamiltonian for the source ($i$th detector) field and $H_{sd}^i$ is the interaction Hamiltonian between the source and the $i$th detector. The Heaviside step function $\Theta(t_i - t)$ takes out of the evolution the $i$th detector after it has been checked at time $t_i$. Each of these checks corresponds to an attempt of absorption of one excitation of the corresponding detector, thus if we call $|0\rangle$ the lowest energy state for all the subsystems (detectors and source), the state of the complete system at a time $t > t_N$ is given by

$$U_N(t, t_N) d_N U_{N-1}(t_N, t_{N-1}) \ldots d_1 U_0(t_1, t_0) a_\alpha^\dagger |0\rangle,$$ \hfill (3.28)

where the evolution operators are $U_i(t_{i+1}, t_i) = \exp(-i \int_{t_i}^{t_{i+1}} H(t) dt)$. As after time $t_i$, the $i$th detector stops operating, its annihilation operators commute with the Hamiltonian $H_j$ for $j \geq i$, so we can write (3.28) as

$$d_N \ldots d_1 U_N(t, t_N) U_{N-1}(t_N, t_{N-1}) \ldots U_0(t_1, t_0) a_\alpha^\dagger |0\rangle = d_N \ldots d_1 U(t, t_N, \ldots, t_0) a_\alpha^\dagger |0\rangle.$$ \hfill (3.29)

The evolution operator defined in this way is time dependent and in general not unitary, owing to the absorption operators.

Finally, we perform a point-like maximal detection over the source field, yielding the amplitude

$$A_\alpha(t_N, \ldots, t_0, x) = \langle 0|a(t_0, x) d_N \ldots d_1 U(t, t_N, \ldots, t_0) a_\alpha^\dagger |0\rangle,$$ \hfill (3.30)

which is related to the probability of absorbing a source particle at a point $x$ after the detectors 1 to $N$ have been triggered at corresponding times $t_1$ to $t_N$. The SUO annihilation operator is taken at time $t_0$ (in the Schrödinger picture), testing for the absorption of the particle as it was when released, in contrast with the usual S-matrix transition amplitude. The detector is chosen like this because we do not have a relation between the field operators before and after the interaction for finite times, so the logical procedure would be to define the detector’s response in the same way for these two time periods. The fact that the detection is performed at $t_N > t_0$ is then encoded in the fact that the operators sit at the left of the complete evolution. The usual solution to the problem using $\text{in}$ and $\text{out}$ states in the infinitely remote past and future (respectively) does not provide the necessary tools for the computation at hand, so the question we ask avoids this problem by redefining the amplitude of interest. In particular, this means that the S-matrix limit is not
recovered by simply taking the preparation and detection times largely separated, but we would instead need to set the detector to respond to the corresponding out state.

As we can no longer express the required amplitude in terms of the vacuum expectation value of a time-ordered product of field operators—we have only in state operators—we will use the expressions for the evolution of the source creation operator given in (3.17) to go into an interaction picture where the states, instead of the field operators, are evolved freely. With this, we can calculate the lowest order term in the perturbation expansion, which can always be justified by requiring the interaction between the detectors and the source to be sufficiently small compared with the other energy scales involved. This is in fact one of the desired characteristics for these detectors, as we established earlier.

Different intermediate states, consisting on more than one excitation per detector, are possible but would require a higher energy from the source signal. For this reason, we will take the interacting term to be linear on each detector field making (3.30) the leading term in the perturbation expansion up to first non-vanishing order. This choice of taking one excitation of the detector system as the triggering of the device can be contrasted with Zurek’s pointer basis model (see [19] or e.g. [20]), where the state of the device after the measurement has to be found and, in general, does not follow an intuitive rule. We instead define our detectors to have this particular, simple behaviour. On the other hand, it is tempting to associate the discontinuities on the evolution operator with the tracing over of the environment’s degrees of freedom carried on the traditional measurement descriptions. The extent to which our approach could be explained or motivated by any of those is left for future studies.

The jump to the interaction picture is done by letting the state evolve according to the free part of the Hamiltonian \( H_0 = H_0^S + \sum_{i=1}^{N} H_0^I(t_i - t) \), by taking

\[
U(t, t_0) \alpha_\alpha^\dagger(0) = U(t, t_0) e^{iH_0(t-t_0)} e^{-iH_0(t-t_0)} \alpha_\alpha^\dagger(0) = U_I(t, t_0) \alpha_\alpha(0),
\]

where one can show that the evolution operator for this interaction picture is given by

\[
U_I = \mathbb{L} \{ \exp \left[ - i \int_{t_0}^{t} d\tau H_I(\tau) \right] \},
\]

where \( H_I \) is the interacting part of the Hamiltonian (3.27) in the interaction picture i.e. \( e^{-iH_0(t-t_0)} (H - H_0) e^{iH_0(t-t_0)} \), and \( \mathbb{L} \) is the inverse time-ordering operator that arises because we are now evolving the initial state freely, instead of the operators. In other words, as our evolution operator acts over the state already (freely) evolved up to time \( t \), we need to place the Hamiltonians with the larger times arguments to the right. In comparison, in the usual interaction picture evolution operator—represented in terms of regular time ordered exponential—is the solution of \( U = \exp[iH_0(t - t_0)]U_I \), i.e. in reverse order as compared with (3.31).

(ii) Explicit interaction model

To go further in the analysis of the method, it is useful to consider one particular interaction model between the source field and the detectors. Other additional requirements we are going to impose for the interaction term in the Hamiltonian are for it to be Hermitian, linear in the detector field and as we are concerned with massive source fields, we will also require the interaction term to commute with the source-number operator \( N \) (3.13).

The interaction model we use is

\[
H_{id}^I = \int d^3x \phi^+_S(x) \psi_d^I(x) \phi^+_d(x),
\]

where the subscript \( S(d) \) denotes source (detector) fields and the superscripts \(+(-)\) the positive (negative) energy parts of the field, i.e.

\[
\phi^+_S(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{ipx} a^+(p) \quad \text{and} \quad \phi^-_S(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{-ipx} a(p).
\]
The lower contractions needed to calculate reverse-time-ordered products of this interactions can be easily calculated as usual, but noting that the symbol \( \mathcal{L} \) changes the order between interaction terms evaluated at different points, not among the terms comprised in each such terms. Thus,

\[
\langle 0 | \mathcal{L}(\phi_S^+(x)\phi_S^-(y)) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{ip(x-y)} \Theta(x_0 - y_0),
\]

\[
\langle 0 | \mathcal{L}(\phi_S^-(x)\phi_S^+(y)) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{-ip(x-y)} \Theta(y_0 - x_0)
\]

and

\[
\langle 0 | \mathcal{L}(\phi_S^-(x)\phi_S^- (y)) | 0 \rangle = \langle 0 | \mathcal{L}(\phi_S^+(x)\phi_S^-(y)) | 0 \rangle = \langle 0 | \mathcal{L}(\phi_S^+(x)\phi_S^- (x)) | 0 \rangle = 0.
\]

The term where all the fields are contracted is proportional to the integral of \( \Theta(x_0^1 - x_0^2) \ldots \Theta(x_0^n - x_0^n) \) and is then non-zero only when all the temporal variables of the integrations take the same value, thus not contributing to the overall integral. This means at least one of the detectors has to directly interact with the incoming source signal to trigger all the rest of the detectors responses. This would not be the case if, for instance, the interaction term was \( H_{\mathrm{int}} = \int d^3x \phi_S(x) \phi_S^+(x) \phi_S(x) \), making the possibility of separating the source field into positive and negative energy parts a desirable property of the SUO. In cases where this is not possible, e.g. when the detection is performed under an external interaction or in a curved space–time, simple detector models, consisting on powers of the fields, could be triggered by internal excitations of the detecting device itself. The definition of stable detector models will depend on the systems being investigated, and ultimately the need to find such a detector will be given by what is actually being performed via the experimental set-up.

### 4. Mott’s cloud chamber

Making use of (3.30), we can obtain the amplitude of absorbing a particle at a certain point \( x \), given that it has triggered a minimal detector, starting from a particular position in space. The scenario studied here presents a complete dependency on all the time parameters involved: the releasing of the state into the void \( (t_0) \), its detection by a minimal detector \( (t_1) \) and its posterior absorption by a maximal detector \( (t) \) as described in §3c.

The amplitude we are interested in is

\[
\mathcal{A}_\alpha(t, t_1, x) = \langle 0 | \alpha(t_0, x) d_1 \mathcal{U}(t, t_1, t_0) a^\dagger_\alpha | 0 \rangle 
\]

\[
\approx -i \langle 0 | \alpha(t_0, x) d_1 e^{-iH_0 (t-t_1)} \left[ \int_{t_0}^{t_1} d\tau d^3y \psi_d^+(y) \psi_d(y) \phi_S^- \right] a^\dagger_\alpha(t_1) | 0 \rangle,
\]

where the approximation made is to take the first order in the perturbation expansion and the notation \( y \equiv (\tau, y) \) is used. Here, the minimal detection field can be expanded as

\[
\psi_d(x) = \sum_\beta (g_\beta(x) d_\beta + g_\beta^\dagger(x) d^\dagger_\beta)
\]

\[
= \sum_\beta (e^{i\lambda_\beta x_0} g_\beta(t_0, x) d_\beta + e^{-i\lambda_\beta x_0} g_\beta^\dagger(t_0, x) d^\dagger_\beta),
\]

from where we can define the Fourier transform of its base functions \( \tilde{g}_\beta(p) \) following (3.8). We will be concerned with the first excited state of the detector, which we will denote with the subscript 1.

With this, up to first order, amplitude (4.1) can be written in momentum space as

\[
\mathcal{A}_\alpha(t, t_1, x) \approx -2i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \frac{\sin[(p_0 - q_0 + \lambda_1)t_1/2]}{(p_0 - q_0 + \lambda_1)}
\]

\[
\times e^{-i(p_0(t-t_1) - p_x x)} e^{-i(q_0 + \lambda_1)t_1/2} \frac{\sqrt{(q - p)^2 + \mu^2}}{\tilde{g}_\beta^\dagger(q - p) \tilde{f}_\alpha(q)},
\]

(4.3)
Figure 2. Numerical calculation of the amplitude of the Mott process showing its qualitative behaviour. Axes $\hat{x}$ and $\hat{y}$ in femtometre, arbitrary units used in $\hat{z}$.

where $\mu$ is the mass of the minimal detector field. This expression can be used directly to obtain a numerical estimation of the amplitude for this process. We do this in the two-dimensional case for simplicity.

The solution used for $f$ is similar to the one used in §3c but this time the length in the $\hat{x}$-axis is set to be 0.6 fm while leaving the length in the $\hat{y}$-axis at 1 fm. This is to give an initial momentum large enough to make the final momentum (after the collision) still considerably large without reducing the mass of the minimal detector to a scale that could make the numerical calculation unstable. For $g$, both sides are of length 1 fm and centred at $(3 \text{ fm}, 0)$. The mass of the minimal detector field is set to be $\mu \sim 20 \text{ MeV} c^{-2}$. The evolution of $f$ is calculated according to §3b and the times used for the two detections are $t_1 = 9\tau$ for the minimal detection and $t = 9.1\tau$ for the maximal detection, where $\tau$ stands for the time step, as used in figure 1.

The numerical integration is performed using a Monte Carlo method which accounts for the noise in figure 2. In spite of this, we can see the important features of the amplitude. We obtain two clear maxima, one at the detection point and other over the $\hat{x}$-axis, outwards from the source, falling off rapidly in every direction, obtaining a behaviour qualitatively similar to that found by Mott in his time-independent scenario [11].

5. Conclusion and remarks

Different detection models have been used in the literature to account for different phenomena and have proved to be a useful tool to give a functional interpretation of the content of a quantum field even in situations where is not clear how a particle state can be defined. The model presented here consists of two parts: minimal and maximal detectors. It presents a unified way to treat particle detection scenarios provided a suitable interaction between the source and detector fields can be found. In the applications explored here, the maximal detections involved the absorption of a particle, either a source particle or a detector excitation, but there are other possibilities to describe other irreversible evolutions. The non-unitary evolution produced is what one could expect from the extraction of information from a system. Although the model is developed as special relativistic one, we believe that it could be extended to incorporate more general scenarios.

In this framework, the detection of photons and of massive particles is treated alike. We can account for photonic coherences through the maximal detection model and for the recently found coherences [10] in material waves. The form of the absorption operator (3.18) is taken here as
an educated guess, where we know the a grosso modo effect the detection procedure has over the system but the specific details of how that happens and how the absorption operators act over the state are expected to be ultimately defined by the particular experiment being carried out.

Even with these restrictions, the model reproduces the expected qualitative behaviour in the Mott scenario, giving a fully time-dependent, relativistic description of the probability of absorbing the source particle at a determined position of space. The extension of the model to account for more general processes with external interactions, curved space–times and the generation of non-classical correlations between the detection systems is left for future work.

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**References**