Modification of mean wake flow behind a very slender axisymmetric body of revolution by imposed nonlinear unstable helical modes

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For a sufficiently slender axially symmetric body placed in a uniform stream, only convectively unstable modes are found in previous experiments. This work imposes theoretically and computationally a pair of most unstable helical modes, symmetrically and asymmetrically. The Reynolds stress modification of the developing laminar mean wake flow is modified into an elliptic-like cross section for symmetrical forcing; the consequences of unequal upstream amplitudes are also explored. Energy-transfer mechanisms between the mean flow and the relevant dominant modes and between the modes through ‘triad interactions’ are studied. The results from dynamical considerations provide the physical understanding of the generation of a standing wave mode at twice the azimuthal wavenumber; it is necessary that the wave envelopes of participating modes, including that of the mean flow, overlap in their spatial development, which is a necessary supplement to kinematical conditions for such interactions to take place effectively. Standing wave motions, which are otherwise only found naturally in wakes behind blunt-trailing-edge axisymmetric bodies, can be rendered present through appropriate forcing and nonlinear interactions behind very slender axisymmetric bodies.

1. Introduction

The recent experimental study of wake instability of a series of streamlined axisymmetric bodies in air [1]
points out the importance of aspect ratio of the body, the maximum body diameter $D$ to the body length $L$. Not so slender axisymmetric bodies (their NACA0018 and 0024 sections where $D/L = 18\%$, $24\%$, respectively, at Reynolds number $Re_L = U_0 L / \nu = 10^5$, where $U_0$ is the free stream velocity and \( \nu \) the kinematic viscosity) sustained small reversed flow regions at the trailing edge, reminiscent of absolute instability supported by bluff body wakes, whereas behind the slenderer body section at about the same Reynolds number range, NACA0015 at $D/L = 15\%$, only downstream propagating convective instabilities occurred. Such convective instabilities are experimentally studied for wakes behind very slender axisymmetric bodies in air by Sato & Okada [2] at $D/L = 2\%$ in a Reynolds number range surrounding $Re_L = 10^5$ and by Peterson & Hama [3] in water for $D/L = 6\%$ at $Re_L = 6 \times 10^4$, among others.

This paper focuses on the consequences of forcing convective instabilities in wakes behind very slender axisymmetric bodies with the axis aligned in the streamwise direction. It is prevalently found, both experimentally and theoretically, that the unstable mode is the first helical mode in the linear region [1–5]. The linear analysis of Gold [4] is patterned after the inviscid stability of axisymmetric jet flows of Batchelor & Gill [6].

The plan of the paper is as follows: the basic nonlinear equation (2.1) in its component form is solved with numerical simulation techniques [7] described in the electronic supplementary material, appendix A. The mode numbers used subsequently are ($m,n$), where $m$ denotes the streamwise propagating modes and $n$ denotes the azimuthal modes. A few of the prominent modes that appear in the results include the mean flow (0,0), the imposed pair of first helical modes (1,1), an imposed axisymmetric mode (1,0) and the resulting standing wave mode (0,2) owing to nonlinear interactions and so on. The modifications of the mean flow are duly accounted in the nonlinear simulations, as are the wave–wave interactions. The subsequent interpretations make use of the point of view of nonlinear stability analyses [8–10] in stating conservation equations, obtainable from the full Navier–Stokes equations through Reynolds splitting with appropriate averaging as would be performed in a nonlinear stability analysis. The numerical simulation results are subjected to such interpretations, but no averaging procedures are performed during the course of the simulation.

The linearized (stability) equation (2.5) for the laminar Gaussian wake is first solved as a subsidiary calculation to provide the initial conditions for the nonlinear simulation. To show some consistency, the linear results are compared with experiments in the linear region, but not exhaustively, in terms of the eigenfunction shape, the amplification rate and frequency as a function of the streamwise wavenumber for the first helical mode which sets the stage of initial perturbation conditions.

The temporal disturbance is computed for the wake flow, following numerous previous temporal approximations for spatial problems in the wall-bounded cases [11,12]. The mean flow in the far wake approximation satisfies the parabolic heat equation in the spatial problem, with the Oseen-like approximation to nonlinear convection playing the role of time development equivalent to the Rayleigh transformation of equating time to the streamwise distance scaled by the free stream velocity. The far laminar wake approximation thus recovers the Gaussian profile; in this case, there is no need to insert a force correction into the temporal heat equation for the mean flow as is necessary in the wall-bounded flow problem to continuously mimic the spatially developing Blasius profile [12].

Although a spatial linear stability analysis would be obtainable without difficulty so as to avoid the use of the group velocity to transform between time and the streamwise coordinate, the simplifications in the nonlinear simulation using the temporal analysis are significant and are well illustrated, in addition to Spalart & Yang [12], Kleiser & Zang [11], in other similar nonlinear stability problems [13,14]. In the temporal case, the boundary layer thickness is a function of time and is spatially parallel, which allows the use of streamwise periodic boundary conditions, thus rendering the simulations to be similar to a parabolic time-marching problem. In a spatial simulation, the downstream boundary conditions would be required in an elliptic-like problem [15,16] and are avoided in the temporal simulation with significant simplifications, whereas the major physical nonlinear interaction picture is still obtained.
To perform the simulation of the nonlinear disturbances in the temporal approximation, it is important to use as convection velocity the group velocity of the disturbances, and this is to be found from the characteristics of the linear stability problem. The consideration of the linear problem in this work is thus twofold: while providing the initial disturbance conditions, it also furnishes the group velocity.

In the subsequent nonlinear development, the Gaussian mean flow is modified through the Reynolds shear stress as the disturbances develop nonlinearly. After ascertaining an estimate of the base flow situation, which corresponds to the forced experiments of Sato & Okada [2], further higher amplitude forcing upon the computed flow is obtained. The first helical mode(s) are exploited in some detail to impart, through downstream developing nonlinearities, modification of an otherwise circular cross-sectional mean flow into astoundingly non-circular shape, including asymmetric forcing by a pair of helical modes. This is seen as a preliminary step in the understanding of downstream wake control.

2. Basic equations

The basic equations are stated in forms that will be useful in subsequent applications. For nonlinear numerical computations, the dimensionless incompressible Navier–Stokes equation in rotational form is

\[ \frac{\partial u}{\partial t} + \omega \times u = -\nabla \Pi + \frac{1}{Re_{\delta_0}} \nabla^2 u, \]  

(2.1)

and the continuity is

\[ \nabla \cdot u = 0, \]  

(2.2)

where \( u \) is the total velocity vector (which eventually is split into a mean plus disturbance quantity in interpretation of the results), \( \Pi = p + |u|^2/2 \) is the total pressure, \( p \) is the static pressure, and \( \omega = \nabla \times u \) is the vorticity. The velocities are scaled by the free stream mean streamwise velocity \( U_0 \), the coordinates are scaled by the initial half-wake width \( \delta_0 \), and the time by \( \delta_0/U_0 \). The pressure, which is also expanded into a mean and disturbance part in the interpretation of results, is scaled by \( \rho U_0^2 \), where \( \rho \) is the fluid density. The Reynolds number is \( Re_{\delta_0} = U_0\delta_0/\nu \), where \( \nu \) is the kinematic viscosity.

The vector forms (2.1) and (2.2) can always be expressed in general coordinates by the relevant relations from, for instance, Lagerstrom ([17], §B.5). For cylindrical coordinates, the velocity components are \( u, v, w \) in the respective dimensionless coordinates \( x, r, \theta \), where \( x \) is the streamwise direction from the trailing edge of the body. The Navier–Stokes equations in component form, which is useful for obtaining the linear theory, are available in the literature and are thus not repeated here.

(a) Subsidiary computations: the linear theory. The initial perturbation for nonlinear computations

The initial conditions for the nonlinear computations of (2.1) and (2.2) consist of a mean flow described by the self-similar Gaussian laminar wake solution and imposed disturbances which satisfy the linear inviscid stability equations. The laminar axisymmetric wake mean velocity is written in familiar form [18]

\[ U = 1 - \frac{Q}{X} \exp \left( \frac{U_0 R^2}{4 \nu X} \right), \]  

(2.3)

where \( U \) is the dimensionless mean velocity, \( X, R \) are the dimensional streamwise and radial coordinates, respectively; \( Q \) is determined from the drag force on the body; the laminar centreline velocity defect is proportional to \( X^{-1} \), and the wake half-width is proportional to \( X^{1/2} \). In the
linear stability problem, the flow quantities are rescaled, as for (2.1) and (2.2), by the initial wake half-width $\delta_0$ and the free stream velocity $U_0$. For the linear theory, temporal normal mode decomposition is assumed to be of the form

$$(u, v, w, p) = RL\{F(r), iG(r), H(r), P(r)\} \exp i(\alpha x + n\theta - \omega t),$$

(2.4)

where $RL$ denotes the real part of a complex number, $F, iG, H, P$ are the disturbance eigenfunctions, $\alpha$ is the streamwise wavenumber, $n$ is the azimuthal wavenumber and $\omega = \omega_r + i\omega_i$ is the complex frequency for the temporal problem where $\omega_r$ is the (dimensionless) physical frequency and $\omega_i$ is the (dimensionless) amplification rate. The linear stability equations for the eigenfunctions are obtained by Batchelor & Gill [6], including the effect of viscosity (their (2.4)–(2.7)). The inviscid equations, which suffice for linear instability characteristics in free flows, are obtained for $Re_{\delta_0} \to \infty$, resulting in the much simpler single second-order equation [4] in terms of the pressure eigenfunction

$$p'' + \left(\frac{1}{r} - \frac{2U'}{U - c}\right)p' - \left(\alpha^2 + \frac{n^2}{r^2}\right)p = 0.$$  

(2.5)

The boundary conditions are $p \to 0$, $r \to \infty$; for $n = 0$: $P(0)$ finite; $n = 1, 2$: $P(0) = 0$.

(i) Extracting the group velocity, amplification rates

In order to solve the simpler temporal nonlinear problem, it is necessary to relate the spatial streamwise coordinate to the time via the disturbance group velocity, to be obtained from the linear theory, as in the boundary layer problem in Spalart & Yang [12]. It is well known from observations that the characteristics of the linear theory essentially remain the same as the flow progresses nonlinearly downstream. This is entirely in the spirit of Stuart [19] where the instability characteristics are given by the linear theory, whereas the amplitude functions are solved nonlinearly. This has also been found to be the case in other nonlinear stability problems [20] where the amplification is far from weak.

The details of the linear theory, including applications [2,3], appear in the electronic supplementary material, appendix B. It suffices to summarize that both the phase and group velocities fall in a range of about 0.73–0.89 through the region $X = 20–120$ cm. The closeness of the group, phase and the free stream velocities (unity) is well known for free shear flows. In subsequent nonlinear computations, a unity convection velocity is adopted. It is equivalent to a Rayleigh transformation (or an Oseen approximation for the far wake) but now with good justification. The instabilities found from the linear theory are convective ones only.

(ii) The initial conditions for the nonlinear computations

After understanding consistencies between the linear theory and aspects of the measurements, we now proceed to set down the initial perturbations for the nonlinear computations in terms of the Gaussian mean velocity profile and the linear theory. Nonlinear effects begin to take place in Sato–Okada’s measurements at about $X = 60$ cm. We thus select the characteristics and profile at $X = 40$ cm to commence the numerical integration, where the characteristic half-wake thickness is $\delta_0 = 0.2664$ cm from data, and $U_0 = 10$ m s$^{-1}$ and $Re_{\delta_0} = 1776$. The experimental forcing frequency, 230 Hz, is related to the dimensionless frequency $\omega_r = \alpha c_r$ in the temporal theory by $f = \omega_r U_0/2\pi\delta_0$, thus $\omega_r = 0.3848$ with a streamwise wavenumber of $\alpha = 0.467$. This value is very close to, but slightly to the left of, the wavenumber for maximum growth rate. The initial condition for the total flow at $t = t_0$ (or $x_0$) is the laminar mean Gaussian wake $U_G(r,t_0)$ (which is depicted in (2.3)) plus the chosen eigenfunctions from the linear theory corresponding...
to the initial mean flow: a pair of helical modes of wavenumbers \((m,n) = (1, ±1)\) and one least stable axisymmetric mode \((1,0)\):

\[
\nu(x, \theta, r, t_0) = U_G(r, t)e_x + A_{+1}\nu_{1,+1} \exp i(\alpha x + \theta) + A_{-1}\nu_{1,-1} \exp i(\alpha x - \theta) + A_{0}\nu_{1,0} \exp i(\alpha x) + c.c.,
\]

where \(c.c.\) denotes the complex conjugate, \(e_x\) is the unit vector in the streamwise direction and \(A_{+1}, A_{-1}, A_{0}\) are the maximum amplitudes of the normalized eigenfunctions \(\nu_{1,+1}, \nu_{1,-1}, \nu_{1,0}\), respectively.

The rationale of inserting the \((1,0)\) damped mode in the initial condition is as follows: as the computations progress higher modes are generated through nonlinear triad interactions between wave modes. If the \((1,0)\) mode is initially absent and if only two helical modes are perturbed, then the shape of the resultant Fourier mode spectrum becomes exactly the ‘checkerboard type’; all of the neighbouring diagonal modes of a non-zero mode become zero. In order to fill all the Fourier modes with non-zero values, at least one more wave mode that has odd \(m + n\) needs to be perturbed. Thus, one axisymmetric mode, \((1,0)\), is also perturbed in order to enable all the Fourier modes eventually to have some value as the computation progresses. However, the addition of the \((1,0)\) mode, which initially decays exponentially, otherwise has little dynamical effect on the overall structure of Fourier modes as evidenced by the energy content results in §3. Computations for an initially excluded \((1,0)\) mode are thus not performed.

3. The nonlinear region

Two of the important nonlinear behaviours in mean flow modification is in terms of the enhanced wake spreading rate or wake thickness and the centreline mean velocity acceleration. Theoretically, these are obtained directly from the nonlinear computations. These are compared with experimentally forced wake flow [2], including initial regions of linear instabilities in figure 1a, b, respectively: the open circles are those of the laminar (unforced) wake, the dark circles that of the Sato–Okada forced wake. These are used to estimate the ‘baseline’ perturbation amplitudes.

(a) Estimate of the baseline initial disturbance amplitude

The precise equivalent initial disturbance amplitudes in terms of the experimental acoustic forcing are not known. The measured wake thickness development and the wake centreline velocity acceleration are used to estimate baseline amplitude of the forced experiments of Sato & Okada [2]. Stuart [9] and Meksyn & Stuart [21] advanced the idea that a dominant mode suffices to represent the nonlinear disturbances which enter into the Reynolds stress for mean flow modification. This idea is found, in many instances, to hold for developing flows and in situations far from any weak nonlinear assumptions [20, 22]. For the present, the most amplified helical \(n = ±1\) modes according to the linear theory are assumed to represent the developing nonlinear modes (including consequential generation of other modes through nonlinear interactions), causing the modification of mean flow characteristics.

To this end, several preliminary nonlinear computations of (2.1) and (2.2) with initial conditions (2.6) subjected to different initial disturbance amplitudes are made in order to obtain a wake centreline velocity and wake thickness development that could mimic the experimental values: in figure 1a, b, the dotted lines are for zero perturbation amplitude, the circles are the experimental laminar wake results; the computed solid lines are those that pass through most of the early nonlinear experimental points (dark circles). The latter is designated as case 1: \(A_{±1} = 0.005, A_{0} = 0.0025\), the baseline case of the Sato–Okada forced experiments. Forcing beyond the baseline experiments is computed for enhanced symmetric forcing (case 2: \(A_{±1} = 0.010, A_{0} = 0.005\)) and for asymmetric forcing (case 3: \(A_{+1} = 0.010, A_{-1} = 0.005, A_{0} = 0.005\)). Case 1 and 2 results do not differ qualitatively in terms of details, so that only the case 2 detailed results are shown subsequently.
Figure 1. Wake properties for ascertaining baseline initial amplitude. (a) Centreline velocity as a function of streamwise distance. (b) Half-wake thickness as a function of streamwise distance. Sato & Okada [2] experiments: unforced laminar wake, open circles; baseline forced, dark circles; dotted line: calculated laminar wake; solid line: symmetrical baseline forcing, case 1; dashed line: symmetrical enhanced forcing, case 2; dashed--dotted--dash line: asymmetrical forcing, case 3.

(b) The mean flow equation modified by the Reynolds stress

Although the nonlinear computation of (2.1) and (2.2) uses no averaging procedure, the results can be re-interpreted in terms of averages as in nonlinear stability theory ([9,19], particularly [8]), and as in experiments to aid physical understanding.

In the present temporal computation, nonlinear development in time is relatable to spatial development via the group velocity as already discussed, and is approximated to be that of the free velocity because of their closeness. In this case, the dimensionless time and spatial distance are equal, \( t = x \). This is also equivalent to an Oseen approximation in the far wake in which the nonlinear spatial advection effect is replaced by time and the free stream velocity. With a periodic streamwise condition for the disturbances, the mean flow then becomes indeed parallel and \( U = U(r,t), V = 0 \). The mean flow equation becomes

\[
\frac{\partial U}{\partial t} = \frac{1}{Re_0} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \langle -uv \rangle \right),
\]

where \( \langle \rangle \) denotes the averaging process which produces the mean motion and is here the streamwise and azimuthal averaging, \( \langle -uv \rangle \) is the Reynolds shear stress per unit mass owing to the disturbances, which becomes important as the disturbances become nonlinear [8]. The nonlinear acceleration of the mean centreline velocity and the enhanced spreading of the wake thickness, as depicted in figure 1, are thus interpretable in terms of the Reynolds stress modified...
involving multiple mode interactions in shear flows [23,24].

expected to be present for the energy transfer between (\(i\),\(j\)) mode. The wave mode energy is defined as \(E_{mn}\).

Here, \(E_{00}\) denotes the mean kinetic energy. The mean flow kinetic energy balance is
\[
\frac{\partial E_{00}}{\partial t} = -P - \Phi,
\]
where \(P\) is the energy production of all the disturbances, \(\Phi\) is the rate of viscous dissipation of the mean flow.

The disturbance Reynolds stress in the production mechanism is \((-uv) = \sum_m \sum_n (-u_{mn} v_{mn})\) owing to the orthogonality of the trigonometric base functions. The overall production mechanism is the sum over that for each mode \(P = \sum_m \sum_n P_{mn}\), where energy transfer between the mean flow and each wave mode is \(P_{mn} = \int_0^\infty (-u_{mn} v_{mn}) \frac{\partial U}{\partial r} dr\). The overall disturbance energy balance is thus
\[
\frac{\partial}{\partial t} \sum_m \sum_n E_{mn} = P - \sum_m \sum_n \phi_{mn}.
\]

The double sum of \(E_{mn}\) denotes the overall disturbance kinetic energy and \(\phi_{mn}\) is the disturbance dissipation.

The energy balance for each individual disturbance mode \((m,n)\) is
\[
\frac{\partial E_{mn}}{\partial t} = P_{mn} - \phi_{mn} + WW_{mn}, \quad WW_{mn} = -W_{ij} W_{kl} W_{mn}, \quad W_{ij} W_{kl} W_{mn} = \int_0^\infty \langle -u_{ij} \cdot u_{kl} \cdot \nabla u_{mn} \rangle dr,
\]
which now includes the inter-mode energy transfer mechanism denoted by \(WW_{mn}\) for which the sum of inter-mode energy transfer is zero, \(\sum_m \sum_n WW_{mn} = 0\). The inter-mode energy transfer mechanisms are rather complicated, but they are essentially triad mechanisms in a shear flow [23] and are represented in the present applications following Lee & Liu [24] for interacting modes in a round jet but in compact form. Energy transfer to the \((m,n)\) mode is represented by triple correlation terms on the right-hand side \(WW_{mn} = -W_{ij} W_{kl} W_{mn}\). Equation (3.4) determines the increase or decrease of mode energy \(E_{mn}\) depending on the balance on the right-hand side: the first term is the production of mode energy by the mode-Reynolds stresses working against the rate of shear strain of the mean flow, the second term represents the rate of viscous dissipation of the mode \((m,n)\) energy by viscosity. The last term is the energy exchange between the \((m,n)\) mode and modes \((i,j)\). If positive, then energy is transferred to the \((m,n)\) mode. The third mode \((k,l)\) is expected to be present for the energy transfer between \((m,n)\) and \((i,j)\) modes, as in previous work involving multiple mode interactions in shear flows [23,24].

(d) Anticipating the generation of standing wave mode

The mean flow modification resulting from the initiation of two unstable azimuthal wave modes of the same frequency, but opposite wavenumber, was first noted by Strange [25] in experimental studies of axisymmetric jets, in which a standing wave mode is detected. The generation of the standing wave mode is explained by Cohen & Wygnanski [26] using a normal mode decomposition of the parallel flow inviscid stability equation of motion. The first-order problem is the linearized stability problem (e.g. (2.4)); perturbations to second order yield the kinematic
condition for the temporal problem: for two input helical waves having the same \( \alpha \), opposite azimuthal wavenumbers \( n = \pm 1 \) and the same frequency \( \omega_r \) (and the same amplitude) a standing wave would result with wavenumber 0, frequency 0 and azimuthal wavenumber 2, i.e. \((0,2)\) plus a harmonic axisymmetric wave \((2,0)\). It is clear that there is more to the kinematic condition in a developing flow in that, both experimentally and in the dynamical computations, the nonlinear interactions that produce such results are present as long as the participating modes overlap in their amplitudes in the developing flow. This is demonstrated in the present wake flow problem in the following sections. Further evidence of the mean flow modification in axisymmetric jets is given in Long & Petersen [27].

4. The symmetrical–helical mode forcing (case 2)

Limited case 1 results are shown earlier, which are used to obtain a baseline initial condition to mimic the Sato–Okada experiments. The detailed results for case 2, for enhanced symmetrical mode forcing, are discussed in this section. Although the indications of nonlinearity depicted from centreline wake velocity and wake thickness from figure 1a,b begin about \( X = 80 \) cm, the measured mean velocity profile when properly normalized (not shown here) remains robustly Gaussian up to \( X = 100 \) cm and is confirmed by calculated results within the streamwise distance indicated.

(a) Energy considerations for the symmetric–helical forcing

The energy consideration formalisms discussed in §3c are applied to the computational results for the present symmetrical forcing. The overall development of energy transfer is shown in figure 2 for the mean flow kinetic energy \( E_{00} \). It progressively decreases downstream as its energy goes into the total perturbation kinetic energy \( TPE = \sum_m \sum_n E_{mn} \) (and loss through viscous dissipation, which is not shown). The total perturbation energy initially amplifies exponentially and eventually is limited by nonlinearity as the mean flow energy depletes as well as decreases owing to the rate of viscous dissipation of the perturbations. This is a scenario anticipated by the nonlinear hydrodynamic stability studies of Stuart [9] and Meksyn & Stuart [21].

Shown in figure 3 are the dominant perturbation energies as they develop downstream: initially the forced perturbation mode(s) \((1,1)\) dominates as it goes through exponential amplification but is eventually limited by the inability of the mean flow to maintain its energy supply, and by the transfer of energy to other modes and viscous dissipation. Its harmonic mode \((2,2)\) is generated as well as an axisymmetric harmonic \((2,0)\) but they both play minor roles. Of interest is the generation of a helical–harmonic standing wave mode \((0,2)\), which is absent in the linear stage but, rather, is generated through nonlinear interactions from forcing the pair.

Figure 2. Development of mean flow kinetic energy denoted by \((0,0)\); total perturbation energy \( \sum_m \sum_n E_{mn} \) denoted by \( TPE \) for enhanced symmetric–helical perturbation (case 2).
Figure 3. Development of dominant perturbation energies $E_{mn}$ for enhanced symmetric–helical perturbation (case 2). Prominent is the eventual dominance of a standing wave mode $E_{02}$ denoted by $(0,2)$ downstream in the nonlinear region.

Figure 4. Energy transfer $P_{mn}$ from the mean flow to the wave modes for enhanced symmetric–helical perturbations (case 2).

Figure 5. Inter-mode energy transfer $WW_{02}$ to the standing wave mode $E_{02}$: $(0,2)$ from $(1,1), (1,-1)$. Case 2.

of fundamental streamwise-helical modes as the latter strengthens via energy transfer from the mean motion.

In the wake behind the blunt trailing edge of axisymmetric bodies absolute instability appears in the linear stage [28], and also behind not-so-slender axisymmetric bodies [1]. The present standing wave mode $(0,2)$ is a result of nonlinear interactions and does not satisfy conditions of absolute instability that originate with the linear theory [29–31].

The mean flow to perturbation energy transfer $P_{mn}$ is shown in figure 4 for a few of the dominant modes, the sign in the figure indicates energy transfer from the mean flow. The dominant energy transfer is to the forced fundamental pair of helical modes $(1,1)$ and to the harmonic $(2,2)$ once it is generated through nonlinear self-interaction of the $(1,1)$ mode. All
Figure 6. Development of the cross-sectional distribution of the streamwise-averaged iso-\(u\) contours, symmetrical forcing (case 2): (a) \(X = 40\) cm, (b) \(X = 60\) cm, (c) \(X = 80\) cm, (d) \(X = 100\) cm, (e) \(X = 120\) cm, (f) \(X = 140\) cm. Contour levels in terms of dimensionless iso-\(u\) velocity, higher levels towards the outer wake region. Horizontal axis, \(z\); vertical axis, \(y\). Case 2.
the other modes, generated from the initially excited (1,1) mode, include the helical–harmonic standing wave mode (0,2) and axisymmetric streamwise harmonic mode (2,0). The latter, once generated, becomes a damped mode in that it transfers energy back to the mean flow. The standing wave mode continues to receive energy from the mean flow as other energy transfer mechanisms from the mean flow subside downstream, leading to the persistence of the standing wave mode in figure 3.

The inter-mode energy transfer $W_{mn}$ is discussed for several dominant inter-mode energy exchange mechanisms. The inter-mode energy supply to the standing wave mode (0,2) comes dominantly from the initially perturbed helical mode and is shown in figure 5. Shown in figure C1a in appendix C in the electronic supplementary material is the energy transferred into the (2,0) mode: the dominant source is the initially perturbed fundamental helical mode (1,1); eventually, there is an oscillatory energy return to the (1,1) mode at a weaker pace. Oscillatory energy supply from the harmonic mode (2,2) ensues, but the positive energy supply dominates in a short duration, as does the energy supply from the harmonic–helical, standing wave mode. Although shown, energy exchanges with the initially damped axisymmetric streamwise fundamental mode (1,0) are virtually non-existent. Shown also in appendix C in the electronic supplementary material in figure C1b is the energy lost from the (2,2) mode to the (1,1) mode, whereas energy exchange with the (2,0) mode is oscillatory; the (2,2) mode helps sustain the standing wave mode (0,2) though at a lower pace.

(b) Mean flow (spatial-averaged) modifications for initial symmetric–helical forcing (case 2)

The development of the cross-sectional $(y,z)$ distribution of the streamwise averaged $u$ velocity is shown in figure 6. Initially, the iso-$u$ lines are confined by the outer concentric circles of the axisymmetric laminar wake flow, although the initial forcing of equal amplitude helical mode is visible. The disturbance, as such, interacts with the azimuthally averaged mean flow according to (3.1) through the Reynolds stress $\langle -uv \rangle$. The latter, in turn, shows strong energy conversion to the disturbance with strong Reynolds stress distributions (not shown) precisely in the regions where the streamwise averaged $u$ velocity evolves to two almost distinct wake regions surrounding the initially excited helical modes. The intermediate developing contours exhibit square and elliptic contours and strongly resemble those observed under similar forcing in axisymmetric jet flow experiments [25]. This is somewhat related to Morton’s [32] discussion of the generation of a pair of wakes with equal and opposite fluxes of angular momentum.

5. The asymmetric–helical mode forcing (case 3)

Considering that it would be very difficult to satisfy the symmetry condition exactly in experiments in aligning an axisymmetric body with the axis in the free stream direction [2], it is of interest to study the situation of case 3 in which the pair of helical modes take on different initial amplitudes from case 2: $A_0 = 0.005$, $A_{+1} = 0.010$, $A_{-1} = 0.005$. The effect on mean quantities is also shown in figure 1a,b in this case, as the dot-dashed line for the respective quantities $U_C/U_0$ and $\delta/\delta_0$. They develop more vigorously than the baseline case (case 1, solid line) but less vigorously than the symmetrical forcing with larger amplitude for the $n = -1$ mode (case 2) $A_0 = 0.005$, $A_{\pm 1} = 0.010$ shown in figure 1a,b as the dashed line.

The overall behaviour of $E_{00}$, $\sum_m \sum_n E_{mn}$ for case 3 is, except for details, somewhat similar to that shown in figure 2 for the symmetrical forcing and is thus not shown here. Here, the mode energies for $(1,\pm 1)$ and $(2,\pm 2)$, shown in figure 7, amplify and decay differently owing to the different initial excitation levels for the respective $(1,\pm 1)$ modes. Unlike the symmetric excitation case in figure 3, the standing wave mode is expected to peak further downstream as anticipated in figure C2 in appendix C in the electronic supplementary material. All dominant modes extract energy from the mean flow except the axisymmetric–harmonic mode (2,0). Energy supply to these
Figure 7. Development of perturbation energies for initial asymmetric–helical mode forcing. Case 3.

Figure 8. Inter-wave mode energy transfer to the (0,2) mode, asymmetric forcing. Case 3.

modes from the mean initially dominates but subsequently decays, leaving a net energy supply from the mean to the standing wave mode over large distances downstream. The energy supply to the standing wave mode (0,2) for case 3 is more vigorous than that of case 2 (figure 4). The (0,2) mode persists further downstream in case 3 than in case 2. Shown in figure 8 is the inter-mode energy transfer to the standing wave mode (0,2) from the (1,1) and (1,−1) modes. Because of the difference in the maximum amplitudes of the pair of helical modes in the uneven initial forcing, a net mean drift is anticipated through the generation of a mean azimuthal velocity. Figure 9 shows the modified mean iso-streamwise velocity contours, resulting in the azimuthal tilting owing to this generated mean azimuthal velocity component.

6. Concluding remarks

Both symmetric and asymmetric initial forcing of a pair of most unstable helical modes are qualitatively similar in downstream behaviour, except for the anticipated asymmetrical cross-sectional behaviour in the iso-\( u \) contours (and other flow quantities). The general situation is as follows: the initially forced pair of helical modes grow following the linear theory and generate the three dominant higher modes: wave mode (2,2) from self-interaction and wave modes (0,2) and (2,0) from nonlinear ‘resonant triad’ interactions. Because mode (2,2) comes from self-interaction of mode (1,1), it eventually grows more slowly after the helical modes (1,1) begin to decay. The axisymmetric travelling wave mode (2,0) generated by nonlinear interactions between (1,1) and (1,−1) is also relatively weak because of the net energy transfer back to the mean flow. The standing wave mode (0,2) continues to extract energy from the mean flow as well as from both fundamental helical modes (1,±1). In the fully nonlinear region, the net total energy flow is from the mean flow to the standing wave (0,2). In developing flows, not only is
Figure 9. Development of streamwise-averaged iso-\(u\) contours in the cross-sectional plane, asymmetric forcing (case 3). (a) \(X = 40\) cm, (b) \(X = 60\) cm, (c) \(X = 80\) cm, (d) \(X = 100\) cm, (e) \(X = 120\) cm, (f) \(X = 140\) cm. Contour levels in terms of dimensionless iso-\(u\) velocity, higher levels towards the outer wake region. Horizontal axis, \(z\); vertical axis, \(y\). Case 3.
the kinematic ‘resonance condition’ important to consequential generation of the standing wave mode, but the streamwise duration of overlapping participating modes is also important as to whether streamwise interactions take place, and this can be answered only from the dynamics of the problem. The detection of the standing wave mode in an experimental situation, though available for unstable jet flows, would be most interesting quantitatively for unstable wake flows behind very slender bodies of revolution aligned to the free stream direction.

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References


