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This paper presents the results of both experimental and theoretical investigations for the dynamics of a steel disc spinning on a horizontal rough surface. With a pair of high-speed cameras, a stereoscopic vision method is adopted to perform omnidirectional measurements for the temporal evolution of the disc’s motion. The experiment data allow us to detail the dynamics of the disc, and consequently to quantify its energy. From our experimental observations, it is confirmed that rolling friction is a primary factor responsible for the dissipation of the energy. Furthermore, a mathematical model, in which the rolling friction is characterized by a resistance torque proportional to the square of precession rate, is also proposed. By employing the model, we perform qualitative analysis and numerical simulations. Both of them provide results that precisely agree with our experimental findings.

1. Introduction

A rigid body spinning on a rough surface is commonly observed in nature. Studies on this topic started a long time ago when, for example, Euler first analysed the motion of a rigid body around a moving axis [1]. Since then, results on the dynamics of a rigid body have been reported in a tremendous body of literature [2–5]. Owing to Moffatt’s recent work [6], this classical problem has recently revived interest for studying the energy dissipation in the motion of a ‘Euler’s disc’, namely a circular (homogeneous) metal disc spinning on a flat surface [7–14]. In this system, while its mechanical energy is dissipating, paradoxically, the speed of the disc’s rolling increases rapidly.
By introducing viscous dissipation caused by the sheared air between disc and table, Moffatt [6] predicted that the rate of dissipation of energy would take a power-law character. As dissipation of energy follows this power-law character, the precession goes to infinity before the disc ceases its motion in a finite-time interval. He attributed a finite-time singularity to this kind of paradoxical phenomenon.

Moffatt established his calculation by assuming that the disc would approximately roll in a quasi-steady state around its centre of mass without any slippage at the contact point. Under such a motion state, the angular momentum theorem makes \( \psi^2 \sin \theta \) a nearly constant value, and the energy \( E \) involved in the disc motion is estimated with \( E = 3mgrr \sin \theta / 2 \), where \( \psi \) is the precession rate, \( r \) the radius of the disc and \( \theta \) the nutation angle. Based on the two important relationships, together with the air viscous mechanism, Moffatt successfully deduced an approximate solution for the precession rate, \( \psi(t) \propto (t_f - t)^{-1/n} \), where \( n = 6 \) is the dissipation exponent and \( t_f \) is the instant at which the disc comes to rest. This solution not only provides a specific exponent convenient for experimental validation, but also exposes a singularity in association with the anomalous phenomenon of the disc motion in reality. Furthermore, this air-drag model brings a prediction that the contact between the disc and table separates when the time goes closely enough to \( t_f \).

Although the power-law dependence of the viscous mechanism presents an appealing feature in describing the disc motion, much controversy exists on the question of what is the dominant mechanism of energy dissipation for this system. By comparing the motion of a coin both in vacuum and air, Engh et al. [15] discovered that the presence of air has little effect on the motion of the coin, so they suggested that sliding friction at the point of contact may play the dominant role of dissipating energy. However, Moffatt [16] in his response refuted this statement because he believed that the steady rolling state would be an available assumption in the final stage of the disc motion. Moffatt’s assumption was further supported by the subsequent experiments conducted by Petrie et al. in [13], where no slip state was observed during the motion of a Euler’s disc.

In order to quantify the energy dissipation, many researchers have implemented experiments to measure the motion of the disc by different methods. McDonald & McDonald [7] adopted a flashlight and a phototransistor to measure the precession rate, which was observed to follow an exponent \( n = 4 \). Easwar et al. [9] reported an exponent \( n = 3 \) through experiments using a single high-speed camera to measure the precession rate. Caps [8] performed a series of experiments to separately measure the inclination angle, precession rate and angular velocity around the symmetric axis of the disc, and presented the exponent with a value varying between \( n = 3 \) and 4. Leine [10] adopted a high-speed camera to synchronically measure the inclination angle and the precession rate, and suggested that the exponent is either \( n = 4 \) or 3. Despite the diversity in the measured exponents, none of the results in these experiments agree with Moffatt’s prediction.

By physical intuition, one may be inclined to believe rolling friction rather than air viscosity responsible for the dominant mechanism in the energy dissipation. This opinion is supported by the fact that the disc behaves differently on different surfaces [8,9]. For instance, some authors [6,13] found that the motion of a commercial ‘Euler disc’ could last for about 80–100 s, whereas other experiments [7,10], including the one that will be presented in this paper, revealed that the motion of a similar disc only lasted about 7–8 s. The difference between them is that the commercial ‘Euler disc’ has a carefully bevelled edge, whereas the tested disc in our experiments has a rather sharp edge. Intuitively, the sharp edge implies a greater pressure on the substrate so that it results in an increase in rolling friction to decrease the rolling motion rapidly. With respect to this fact, Moffatt [16] admitted that ‘rolling friction is an equally plausible candidate’, although he chose air viscosity for convenience in his analysis.

Although a general opinion is tending to believe that rolling friction dominates the energy dissipation, the absence of its fundamental description still blocks its entrance into the modelling of the disc motion. Easwar et al. [9] predicted that rolling friction dissipates energy with a rate being proportional to the precession rate. However, no clear explanation was given in the modelling of rolling friction. Leine [10] suggested that rolling friction may satisfy a dry contour
friction law or a viscous contour friction law [11]. In particular, the dry contour friction law follows the conventional definition of rolling friction [17], in which a torque proportional to the normal contact force resists the disc motion. However, when this model is incorporated into simulations, numerical results given by Kessler et al. [18] revealed that energy would decrease asymptotically, namely the disc does not stop in finite time. This numerical phenomenon is not in accord with a variety of experimental observations.

This paper purposes to describe an experimental investigation for the dynamics of a disc with a sharp edge spinning on a flat surface. Firstly, we build a set of apparatuses to make a pendulum-based ball collide against the disc standing on the flat surface to start its motion. This design, same as the one in our prior work for investigating the impact dynamics in a disc–ball system [19], allows us to precisely repeat experiments with one specific initial conditions. Secondly, a pair of high-speed cameras were synchronized to sample the images of the disc in motion. Following Zhong et al. [20] in experimentally investigating the motion of a freely falling thin disc in water, we develop an image processing method to identify the kinematical quantities related to its six degrees of freedom. Knowledge of the time history of the position and orientation gives important information to detail the disc dynamics, and consequently to quantify the evolution of energy with regard to time.

Based on the experiment observations, we expect to answer the questions as follows: (1) whether and when the disc can enter into a pure rolling motion; (2) how the energy dissipates along the evolution of the dynamics; (3) which assumptions adopted in Moffatt’s analysis are available; (4) what is the model of the rolling friction; and (5) what happens when the disc is very close to the table. Correct answers for the above questions not only can help distinguish the mechanism of energy dissipation in the disc motion, but also may bring out the physics underlying rolling friction.

This paper is organized as follows. In §2, we introduce the equation of Euler’s disc rolling on the plane. Experiment set-up and measurement methodology are presented in §3. We report in §4 the experimental and theoretical results together with comparisons between them. Energy dissipation mechanism is discussed and numerical investigations are done in §5. Finally, we conclude and summarize the paper in §6.

2. Description of a disc in rolling motion

In this section, we will give a detailed description for the kinematics of the disc, then present the equations governing the disc’s rolling motion. The conditions for the simplified model adopted by Moffatt [6] will also be discussed.

(a) Kinematics

Consider a disc with radius \( r \) and thickness \( 2h \), rolling on a plane (figure 1). Describing its kinematics requires six degrees of freedom and two coordinate system frames. An inertial coordinate frame \( I = (O, i, j, k) \) is attached to the table. Denote by \( O \) the disc’s centre of mass, positioned with coordinates \( (x, y, z) \) in frame \( I \). At point \( O \), we introduce the frame \( R_O = (O, e_1, e_2, e_3) \), which is obtained by subsequently rotating the frame \( I \) over an Euler’s precession angle \( \psi \) and then over an Euler’s nutation angle \( \theta \). Unit vector \( e_3 \) is the axis of revolution of disc, and \( e_1 \) remains parallel to the table. Denote by \( \phi \) the revolution angle of the disc around axis \( e_3 \). The configuration of the disc can be fully described by six independent variables, \( (x, y, z, \psi, \theta, \phi) \).

Suppose that the disc moves with an absolute angular velocity \( \mathbf{\Omega} \), whose components, \( (\omega_1, \omega_2, \omega_3) \) in frame \( R_O \), are related with three Euler’s parameters by the equations as follows:

\[
\begin{align*}
\omega_1 &= \dot{\theta}, \\
\omega_2 &= \dot{\psi} \sin \theta \\
\omega_3 &= \dot{\psi} \cos \theta + \phi
\end{align*}
\] (2.1)

and
Let the disc contact the table at point $A$. The geometric centre of the bottom surface of the disc is denoted with $B$. Note that frame $R_O$ is not body-fixed, but moves along with disc’s motion such that $e_2$ is always parallel to $r_{AB}$, a vector from point $A$ to $B$. Suppose that $v_A$ represents the temporal velocity of contact point $A$ in disc. According to the kinematical relationship for a point in a rigid body, we have

$$v_A = v_O + \Omega \times (r_{OB} + r_{BA}),$$

(2.2)

where $v_O$ is the temporal velocity of the disc’s centre of mass at point $O$, and $r_{OB}$ is the position vector along the direction from point $O$ to $B$.

We introduce a coordinate frame $R_A = (A, m^0, n^0, k)$ at contact point $A$ for convenience in describing the forces and torques applied in the disc by the table. Both vectors $m^0$ and $n^0$ are in the horizontal plane of the table, and $n^0$ is a unit vector always parallel to $e_1$, while $m^0$ is perpendicular to $n^0$. If the disc purely rolls on the table, velocity at point $A$, $v_A$ will be equal to zero. This corresponds to a set of non-holonomic constraint equations defined at velocity level [4]. According to equation (2.2), together with the transition matrices between different frames, the nonholonomic constraint equation $v_A = 0$ in frame $R_A$ can be expressed as

$$
\begin{align*}
    v^m_A &= \dot{x}_\psi + \dot{y}_\psi - h\dot{\psi}s_\theta + r\dot{s}_\theta = 0, \\
    v^n_A &= -\dot{x}_\psi + \dot{y}_\psi + h\dot{\theta}c_\theta + r\dot{s}_\theta = 0, \\
    v^k_A &= \dot{z} + h\dot{\theta}s_\theta - r\dot{s}_\theta = 0,
\end{align*}
$$

(2.3)

where $s_{(\cdot)}$ and $c_{(\cdot)}$ are, respectively, the abbreviations for sine and cosine functions, for example, $s_\psi$ means sin($\psi$). The dynamics of the disc subject to non-holonomic constraints has been widely studied in history, and many mature results can be found in existing literature [2,21,22].

(b) Governing equations of the disc in rolling motion

The disc has a mass $m$ and an inertial tensor $\Theta_C$ about its centre of mass $O$. The three principal moments of the inertial tensor are $J_1 = J_2 = 1/4mr^2 + 1/3mh^2$ and $J_3 = 1/2mr^2$. Note that the axis of moving frame $R_O$ always coincides with the principal axis of the inertial tensor. Denote by $G_O$ the angular momentum of the disc motion about its centre of mass $O$. Vector $G_O$ in frame $R_O$ can be expressed as

$$G_O = J_1\omega_1e_1 + J_2\omega_2e_2 + J_3\omega_3e_3.$$

(2.4)
In spite of the potential complications in the mechanics of the contact, we can always adopt a resultant force $F_A$ and a net moment $M_A$ to represent the force and torque of the table acting on the disc. In frame $R_A$, the two force vectors take a form as follows:

$$
F_A = F_n n^0 + F_m m^0 + F_z k
$$

and

$$
M_A = M_n n^0 + M_m m^0 + M_z k.
$$

(2.5)

Considering that the disc moves under gravity, we express the governing equations of the motion of the disc’s centre of mass in frame $R_A$ as follows:

$$
\begin{align*}
\ddot{x}c_\psi + \dot{y}s_\psi &= \frac{1}{m} F_n, \\
\ddot{y}s_\psi + \dot{x}c_\psi &= \frac{1} {m} F_m
\end{align*}
$$

and

$$
\ddot{z} = \frac{1}{m} F_z - g.
$$

(2.6)

Balance of angular momentum of the disc about its centre of mass $O$ leads to

$$
\frac{dG_O}{dt} = \frac{dG_O}{dt} + \mathbf{\Omega} \times G_O = M_A + r_{OA} \times F_A,
$$

(2.7)

where $\frac{dG_O}{dt}$ is the relative derivative of the components of $G_O$ in frame $R_O$, and $r_{OA}$ is the position vector from $O$ to $A$.

From equation (2.7), we obtain the equations governing the motion of the disc’s attitudes

$$
\begin{align*}
J_1 \ddot{\theta} - J_1 s_\theta c_\theta \dot{\psi}^2 + J_3 s_\theta \omega_3 \dot{\psi} &= -(rc_\theta - hs_\theta)F_z + (rs_\theta + hc_\theta)F_m + M_n, \\
J_1 \frac{d}{dt}(\dot{\psi}s_\theta) + J_1 \dot{\theta} \dot{\psi} c_\theta - J_3 \dot{\theta} \omega_3 &= -hF_n + c_\theta M_m + s_\theta M_z
\end{align*}
$$

(2.8)

and

$$
J_3 \dot{\omega}_3 = rF_n - s_\theta M_m + c_\theta M_z.
$$

Noting that equations in (2.6) and (2.8) are unclosed provided no conditions are supplemented for determining the contact force $F_A$ and net moment $M_A$.

Suppose that the disc purely rolls on the table. Furthermore, assume the net moment $M_A$ is small enough to be ignorable. By differentiating (2.3) with regard to time, together with (2.6), we can eliminate the unknown components $(F_n, F_m, F_z)$ of the contact force $F_A$ in (2.8). This leads to the equations of governing the disc’s dynamics in purely rolling motion

$$
\begin{align*}
I_1 \ddot{\psi} c_\theta s_\theta + (I_3 s_\theta + m rhc_\theta) \dot{\psi} \dot{\omega}_3 - m rh \dot{\psi}^2 s_\theta^2 + mg(rc_\theta - hs_\theta) &= 0, \\
I_2 \frac{d}{dt}(\dot{\psi} s_\theta) + J_1 \dot{\theta} \dot{\psi} c_\theta - J_3 \dot{\theta} \omega_3 - m rh \dot{\omega}_3 + m h \dot{\psi} \dot{\theta} (rs_\theta + hc_\theta) &= 0
\end{align*}
$$

(2.9)

and

$$
I_3 \dot{\omega}_3 - m rh \frac{d}{dt}(\dot{\psi} s_\theta) = m r \dot{\psi} \dot{\theta} (rs_\theta + hc_\theta),
$$

where $I_1 = I_1 + m(r^2 + h^2)$, $I_2 = I_1 + m h^2$ and $I_3 = I_3 + m r^2$.

Though strong couplings exist in the kinematical quantities, if the action from net moment $M_A$ is ignorable, the motion of the disc can be predicted using equations (2.1), (2.3), (2.6), (2.8) and (2.9). Furthermore, no energy is dissipated in the rolling motion of the disc. Namely, the total energy

$$
E = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} (J_1 \omega_1^2 + J_1 \omega_2^2 + J_3 \omega_3^2) + mgz
$$

(2.10)

will remain unchanged.

(c) Rolling motion in a regular precession state

Let us analyse a particular type of rolling motion, termed as a ‘regular or steady precession’ state, in which nutation angle is constant. In this case, it is not difficult to deduce that both the precession
rate $\dot{\psi}$ and revolution rate $\dot{\phi}$ also remain unchanged. Denote the constant angle of nutation with a value of $\theta = \theta_0$, the constant precession rate with a value of $\dot{\psi} = p$, the third component of the angular velocity with a constant value of $\omega_3 = q$.

According to the first equation in (2.9), the constant values of $p$ and $q$ are not independent but satisfy the relationship as follows:

$$
(I_3 s_{\theta_0}^2 + mrc_{\theta_0}s_{\theta_0}h)pq - (J_1 s_{\theta_0}c_{\theta_0} + mrh^2_{\theta_0})p^2 + mg(rc_{\theta_0} - hs_{\theta_0}) = 0. 
$$

As the disc is purely rolling in this special type state, from the velocity constraint (2.3) of the contact point, we easily get the trajectory of the centre of mass of the disc

$$
\begin{align*}
  x &= x_0 + (\sin(pt + \psi_0) - \sin \psi_0)R_O, \\
  y &= y_0 + (\cos(pt + \psi_0) - \cos \psi_0)R_O \\
  z &= r \sin \theta_0,
\end{align*}
$$

where $R_O = (r - hps_{\theta_0})/p$ and $x_0, y_0, \psi_0$ are constants depending on the initial conditions of the special motion of the disc.

Clearly, equation (2.12) shows a circular trajectory with radius $R_O$ on a plane above the table with a distance $z = r \sin \theta_0$. The contact point also goes along a circle with a radius of $R_A$, whose value is equal to $R_A = R_O + r \cos \theta_0 - h \sin \theta_0$.

(d) Assumptions in Moffatt’s analysis

For convenience of reading this paper, let us present a short review for the Moffatt’s analysis. In [6], he assumes that the disc is thin so that its thickness is negligible. The points $A$ and $B$ are instantaneously at rest, such that the disc is instantaneously rotating about line $AB$ with an angular velocity $\Omega$ (Figure 2). Under such assumption, the balance of angular momentum leads to a formula

$$
p^2 \sin \theta_0 = \frac{4g}{r},
$$

which is termed as a ‘adiabatic’ condition [6].

In fact, this condition can also be deduced using (2.11). Since $q \ll p$ and $h \ll r$, by ignoring the small quantities with higher orders, equation (2.11) also results in the same condition as shown in (2.13).
As the thickness of the disc is not regarded, the ‘adiabatic’ condition in (2.13) makes the total energy of the disc be expressed as

\[ E = \frac{3}{2}mgr\sin\theta_0. \]  

(2.14)

Noting that the equations in (2.13) and (2.14) hold true only if the disc’s motion is in a steady precession state. If the state is perturbed by periodic perturbations, both the equalities just represent the average solutions for the disc motion neighbouring to a stable steady precession state. Stability analysis around the equilibrium of the steady precession state can be found in existing literature [2,7,10,21]. In order to avoid unnecessary digression, here we directly present the result given by Mcdonald et al. [7]. Suppose that nutation angle \( \theta_0 \) is perturbed to oscillate with a circular frequency \( \varpi \), there is a relationship between \( \varpi \) and \( p \) is given by

\[ \varpi p = \sqrt{\frac{3}{5}} \approx 0.77. \]  

(2.15)

We will check the relationships shown in (2.13)–(2.15), according to the measurements in our experiments.

3. Experiment set-up and measurement methodology

(a) Experiment apparatus

Experiments were conducted in a disc–ball experimental apparatus, which consists of a pedestal, a pendulum-based steel ball and a steel disc with sharp edges. A schematic of the experimental set-up is shown in figure 3. An electromagnetic device is used to hold and release the pendulum ball to hit the disc at a planned point. The disc stationarily stands with \( \theta_0 = \pi/2 \) on the pedestal. The impacted surface of the disc is positioned with a distance of the ball’s radius from the equilibrium point of the pendulum ball so that the ball hits the disc with a horizontal velocity perpendicular to the surface of the disc. We refer the details of the experiment set-up to [19].

Different test cases were performed with the same steel disc with radius \( r = 37.5 \) mm, thickness \( 2h = 6 \) mm and mass \( m = 209.69 \) g by setting the pendulum-based ball with the same impacting velocity but hitting the disc surface at different points. The position of the impacted site \( D \) is distinguished by lateral displacement (designated by \( a \)) and vertical displacement (designated...
by $b$) measured from the centre of the impacted surface of the disc. The pedestal is a plate made of steel with 10 mm thickness or covered with 3 mm glass in order to change the surface roughness. Table 1 shows the test cases performed in our experiments. For each test case, at least three trials were performed in order to check the repeatability error.

### (b) Measurement methodology

The positions and attitudes of the disc were measured using a stereoscopic vision method [20]. A pair of high-speed cameras (Lavision HighSpeedStar 4G) were fixed to photograph the disc. The orientations of the cameras were selected carefully so that the images of the disc were formed in a good visual angle. The cameras were synchronized by a video distributor and the images were sampled by a frame grabber at a frame rate of 1000 f.p.s. with a resolution of $512 \times 512$ pixels with 256 grey scales.

Six dotted markers (figure 3) were painted on the impacted surface to facilitate image recognition. Once the disc was impacted by the pendulum-based ball, its motion could be recorded by the two cameras synchronically. In the cases studied in our experiments, the impact surface of the disc faces the cameras, so the markers are always observable in the recorded images.

A pinhole camera model was used to describe the relation between a spatial point and an image pixel. We obtained the calibration information for the two CCD cameras by the method in [23], in which a standard chessboard was used. Using an in-house software package based on well-established image processing algorithms, we analysed the pictures of the chessboard over different orientations to determine the camera’s matrices. The matrices contain the intrinsic parameters related to the focal length and distortion parameters of the lenses, and the external parameters, such as the entries of the transition matrix between the pixel coordinate system fixed on the image and the inertial coordinate system fixed on the table.

After calibration, we digitized the current disc’s configuration using pairs of images captured at the same instant. The motions of the disc in all six degrees of freedom were then calculated based on image processing and post-estimation methods, which gave a measurement accuracy of 0.005 mm in position and $0.05^\circ$ in rotation. Appendix A gives the detailed illustration for data processing. A differential method was adopted to obtain the velocities for the six degrees of freedom.

### 4. Experiment results

Based on the detailed measurements, we carry out phenomenological analysis to answer the first three questions mentioned in the introduction: (1) can the disc enter into a purely rolling state? (2) does the energy dissipate with a power-law character? and (3) are the assumptions inserted in Moffatt’s analysis suitable?

#### (a) Phenomenological analysis

In order to make phenomenological analysis reliable, we check the repeatability error for the experiments over the same initial conditions. Figure 4 presents the time histories of the nutation...
angle $\theta(t)$ for three trials of the experiments in Case I. Owing to uncertainty of the experiments, a small discrepancy exists in the terminal instants of the three trials. Nevertheless, the repeatability error among the three curves is small enough so that we believe the experimental results reliable. These curves also show that the nutation angle $\theta(t)$ frequently fluctuates up and down, while gradually decreasing from $90^\circ$ to $0^\circ$. This implies that the disc motion has an oscillatory nature [12], while energy involved in the disc’s motion always decreases.

In order to avoid the uncertainty of the roughness of the contact surface, we glued a piece of paper with thickness 0.1 mm on the geometric centre of the bottom surface of disc, such that the disc will lie on the table with a small inclined angle when its motion stops. This small inclined angle can also be identified from our experiments. The inset in figure 4 reveals that the final value of $\theta(t)$ is not exactly equal to zero, but of a value $0.5^\circ$. This in turn supports that our measurement method can provide sufficiently accurate data.

Unless explicitly illustrated, in the following we will just focus on one typical trial in Case I as an example to demonstrate the experimental phenomenon involved in the disc motion. Figure 5 shows the trajectory of the disc’s centre of mass projected onto the horizontal plane. The trajectory...
Figure 6. Trajectories of the centre of mass (red line) and the contact point (dotted blue line) from $t = 0.606$ s to 1.239 s. Markers P1, P2, P3, P4 signify the instants when nutation angle minimizes. (Online version in colour.)

Initially takes a bowknot-like shape, then quickly enters into a petaloid shape, and eventually converges to a nearly fixed point.

The bowknot-like path in the trajectory corresponds to the disc motion in a slip state at the contact point $A$, during which the nutation angle at the beginning of the motion quickly decreases in a short time interval (about 0.3 s observed from figure 4). After that, oscillation occurs in nutation angle, and the precession motion also changes, accompanying the movement of the contact point on the horizontal plane. The combination of the motion in nutation and precession makes the trajectory of the disc’s centre of mass a petaloid shape. To picture the scenario, figure 6 shows the trajectory of the moving contact point from $t = 0.606$ to 1.239 s, superimposed by one of the disc’s centre of mass in a full petaloid shape, where four points P1, P2, P3 and P4 serve as the markers to distinguish the instants when the nutation angle reaches a minimal value. Clearly, as a full petaloid shape is formed, there are five circles in the trace of the moving contact point. This means that the oscillation of the nutation angle finishes four periods, while the precession angle evolves five circles.

Essentially, the oscillatory behaviour exhibited in the disc motion originates from the gyroscopic effects inherent in three-dimensional rigid dynamics. When the tangential velocity at the contact point vanishes, the disc will purely roll on the table, and feature a quasi-steady rolling state accompanying a nutation oscillation. Recall equation (2.11) that gives a specific condition responsible for an ideal regular precession state. At the instant when the disc enters into a purely rolling state, an oscillation must almost occur since equation (2.11) can hardly be exactly satisfied. If there is no energy dissipation, the oscillation will take a constant amplitude and a constant frequency, depending upon the incident conditions of the disc in a purely rolling state. With the dissipation of energy, however, the amplitude of the oscillation decreases while its frequency changes along with precession rate. In order to illustrate this point, we count the number of the precession circle and the times of the fluctuation of the nutation angle from the beginning to the end of the disc motion. As shown in figure 7, there is an approximately linear relationship with a slope of a value of 0.79. The value is close to the one given in equation (2.15), in which $\omega/p \approx 0.77$ for a thin disc moving in a disturbed steady rolling state.

Observing the velocity at the contact point offers a direct insight into its contact state in association with friction. Figure 8 shows the evolution of the amplitude of the velocity at the contact point versus time. By figure 8, together with the consideration of the inevitable uncertainty in experiments, we can generally affirm that the velocity of the contact point vanishes after $t > 3$ s. Namely, after that time, the disc purely rolls on the table without slippage at the contact point.

To exhibit the motion of the disc in its final stage, figure 9 plots positions of contact point during the last one second before the disc stops spinning. These contact-point positions of many
precession periods reveal a nearly perfect circular path with a radius of 37.2858 mm, very close to the disc's radius, 37.5 mm. This agrees with figure 5, where the trajectory of the disc's centre of mass converges to a nearly fixed point. The experimental observation further confirms that the disc motion in its final stage is approximately in a steady procession state.

To exhibit the evolution of angular velocity in the disc motion, figure 10 shows the evolution of the procession rate. During most fraction of the whole process of the disc motion, the precession rate fluctuates along with the oscillatory nutation motion, and changes its average value slowly. However, during the last one second it increases rapidly, and revokes its large amplitude in an extremely small time interval to terminate the disc motion. We attribute the physical mechanism of suddenly braking the disc motion to the frictional contact between the table and the paper glued on the bottom surface of the disc. The inset in figure 10 details the evolution of the procession rate near at its final stage, where \( \dot{\psi} \) decreases rapidly within 20 ms. The angular velocity component \( \omega_3 \) is quite small in comparison with the precession rate while \( \omega_2 \) is also small in amplitude but decreases slowly, as shown in figure 11.

Recall that Moffatt's analysis is based on two assumptions that value of \( \dot{\psi}^2 \sin \theta \) remains unchanged and the energy involved in the disc motion is equal to \( E = 3mgr \sin \theta / 2 \). The omnidirectional observation in our experiments allows us to check these assumptions in detail. Figure 12 plots the temporal values of \( \dot{\psi}^2 \sin \theta \) during the whole process of the disc motion.

**Figure 7.** Number of precession circle versus the one of the nutation oscillation from beginning to the end of the disc's motion. (Online version in colour.)

**Figure 8.** Amplitude of the velocity of the contact point, superimposed with red dotted line, corresponding to a threshold value of \( \nu_A \) for distinguishing the velocity vanishing. (Online version in colour.)
Figure 9. Trajectory of the contact point in a time interval from 6.899 to 7.499 s, superimposed with a fitted circle. (Online version in colour.)

Figure 10. Temporal evolution of precession rate $\dot{\psi}$. Inset shows the variation of $\dot{\psi}$ in the final stage of the disc’s motion. (Online version in colour.)

Figure 11. The angular-velocity components of $\omega_3$ and $\omega_2$ along directions of $e_3$ and $z'$, respectively.
Figure 12. Experiment verification of the adiabatic condition: the values of $\dot{\psi}^2 \sin \theta$ versus time. (Online version in colour.)

Figure 13. Time history of the mechanical energy in the disc motion, superimposed with a curve standing for $3mgr \sin \theta / 2$. (Online version in colour.)

Though its simultaneous value fluctuates with the oscillation of the nutation angle, its average basically is constant near to $4g/r$. However, the asymmetric shape of the curve with respect to the line of $4g/r$ means that it lies not at the exact average of the adjacent peak and valley values of $\dot{\psi}^2 \sin \theta$.

With the experimental results, we use (2.10) to obtain the energy involved in the disc motion. Figure 13 shows the comparison of the measured energy and the value of $3mgr \sin \theta / 2$. Good agreements between them imply that it is an available assumption of scaling the energy by $E = 3mgr \sin \theta / 2$, as stated in [6].

In order to clearly exhibit the energy dissipation of the disc’s motion in a quasi-steady rolling state, we focus on the final stage of the disc motion in the last 3 s. The time to collapse, $t_f$, is determined based on the image process with a measurement accuracy of 0.2 ms.

Figure 14 presents the evolution of the energy as a function of time on a ln–ln plot for the three experimental cases. For all the experiments of a steel disc moving on a steel surface, the energy evolution clearly reveals a good power-law behaviour that spans a little over four decades of time. This means that the major mechanism of energy dissipation remains the same for the disc motion over different initial conditions.

Figure 15 presents the ln–ln plot of the nutation angle $\theta$ as a function of time. The evolution of precession rate versus time is shown in figure 16. Although both the curves reveal a relatively
good power-law behaviour, large fluctuation exists due to the oscillatory nature around a quasi-steady procession motion. This may be one of the reasons why existing experiments exhibited different values for the exponent of the power law in describing the evolution of precession rate and the nutation angles.

Denote by $n_E$, $n_\theta$, $n_\dot{\psi}$ the exponents in the evolutions of energy $E$, nutation angle $\theta$ and precession rate $\dot{\psi}$ versus time interval $(t_f - t)$, respectively. By setting $(t_f - t) = 0.5\,s$, we report the values of these exponents in table 2. It is clear that $n_E$ and $n_\theta$ fit well in a same value of $n_\theta \approx n_E \approx 2/3$. This result is not a surprise because figure 13 has confirmed that the energy approximately satisfies a relationship $E = 3mg r \sin \theta/2$ during most time of the disc motion. The value of the exponent in the evolution of the precession rate approximately takes a value of $n_\dot{\psi} \approx -1/3$. This also agrees with the findings in figure 12, in which the average of $\dot{\psi}^2 \theta$ nearly takes a constant as time evolves so that it is easy to infer that $n_\dot{\psi}$ should be approximately $-1/3$ if $n_\theta \approx 2/3$.

We also performed experiments by setting the disc moving on a glass surface. The values of $t_f$, $n_E$, $n_\theta$ and $n_\dot{\psi}$ are reported in table 3, in which the results are obtained by setting $t_f - t = 0.3\,s$. In comparison with the steel surface, the disc under the same initial conditions will preserve its motion in a relatively long time. Nevertheless, the power-law behaviours seem to remain unchanged although the friction on a glass–steel surface is much different from the one on a steel–steel surface.
Figure 16. Evolution of the precession rate on a ln–ln plot for three cases. (Online version in colour.)

Table 2. Dissipation coefficient of disc rolling on steel surface and the time duration.

<table>
<thead>
<tr>
<th>case no.</th>
<th>exp no.</th>
<th>$n_E$</th>
<th>$n_θ$</th>
<th>$n_ψ$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1</td>
<td>0.6833</td>
<td>0.6819</td>
<td>−0.3209</td>
<td>7.510</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.6397</td>
<td>0.6280</td>
<td>−0.2880</td>
<td>7.520</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.6705</td>
<td>0.6814</td>
<td>−0.3052</td>
<td>7.410</td>
</tr>
<tr>
<td>Case II</td>
<td>1</td>
<td>0.6630</td>
<td>0.6730</td>
<td>−0.2690</td>
<td>5.909</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.6875</td>
<td>0.6866</td>
<td>−0.3391</td>
<td>5.187</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.7078</td>
<td>0.6834</td>
<td>−0.2684</td>
<td>5.681</td>
</tr>
<tr>
<td>Case III</td>
<td>1</td>
<td>0.6800</td>
<td>0.6894</td>
<td>−0.3367</td>
<td>7.757</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.6727</td>
<td>0.6601</td>
<td>−0.2541</td>
<td>7.035</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.6075</td>
<td>0.6191</td>
<td>−0.2990</td>
<td>7.184</td>
</tr>
<tr>
<td>average</td>
<td></td>
<td>0.6872</td>
<td>0.6670</td>
<td>−0.2978</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 3. Dissipation coefficient of disc rolling on glass surface and the time duration.

<table>
<thead>
<tr>
<th>case no</th>
<th>exp no.</th>
<th>$n_E$</th>
<th>$n_θ$</th>
<th>$n_ψ$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case II</td>
<td>1</td>
<td>0.6220</td>
<td>0.6112</td>
<td>−0.2917</td>
<td>8.263</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.6389</td>
<td>0.6214</td>
<td>−0.2894</td>
<td>8.053</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.6065</td>
<td>0.6067</td>
<td>−0.2900</td>
<td>8.165</td>
</tr>
<tr>
<td>Case III</td>
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<td>0.6052</td>
<td>0.5957</td>
<td>−0.2902</td>
<td>12.852</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.6548</td>
<td>0.6127</td>
<td>−0.2970</td>
<td>13.477</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.6125</td>
<td>0.6131</td>
<td>−0.2957</td>
<td>12.962</td>
</tr>
<tr>
<td>Case IV</td>
<td>1</td>
<td>0.5746</td>
<td>0.5769</td>
<td>−0.2776</td>
<td>11.397</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.7031</td>
<td>0.7125</td>
<td>−0.3368</td>
<td>10.948</td>
</tr>
<tr>
<td>average</td>
<td></td>
<td>0.6272</td>
<td>0.6188</td>
<td>−0.2961</td>
<td>—</td>
</tr>
</tbody>
</table>
5. Energy dissipation mechanism: rolling friction model

With the experimental results of six degrees of freedom obtained by omnidirectional measurement, the disc’s purely rolling state in its final stage has been certified. In addition, the analysis for the experimental results also reveals that the pure rolling motion can be generally represented by a steady precession state. Namely, the energy involved in the disc motion is approximately scaled by \( E \propto \sin \theta \), and the value of \( \dot{\psi}^2 \sin \theta \) is constant. More importantly, the evolution of energy dissipation versus time exhibits a good power-law character. In terms of the phenomenological analysis, we expect to establish a model for rolling friction.

(a) Modelling rolling friction

As illustrated in §1, the resistance torque induced by rolling friction is basically represented by

\[
M_A = M_m m^0 + M_n n^0 + M_z z^0, \tag{5.1}
\]

where \( z^0 \) is the same as \( k \) introduced previously. At the contact point, we may easily understand that there exists a small region with distributed contact stress and deformation responsible for the resistance torque [17]. Under the physical picture, we attribute the resisting moments \( M_m \) and \( M_n \) to the effects from the inhomogeneous distribution of the normal stress on the contact region, while the resisting moment \( M_z \) to the tangential stress distribution.

Usually, the component \( M_z \) is termed as pivoting friction against the twist motion [10,11]. According to the results presented in [24,25], the pivoting frictional moment can be approximately expressed as

\[
M_z = -k_z F_z \text{Sign}(\omega_z), \tag{5.2}
\]

where \( k_z \) is a coefficient depending on material properties. Symbol function \( \text{Sign}(\omega_z) \) is adopted to represent the direction of \( M_z \) reverse to the one of the \( \omega_z \), which is the component of the angular velocity \( \Omega \) along the direction of \( z \)-axis.

The inhomogeneous distribution of the normal stress on the contact region is due to the movement of the contact point. Usually, these effects are scaled by a classical rolling friction law, in which a frictional moment takes a function just proportional to the normal contact force. The classical rolling friction law may be a good approximation for the contact between bodies with a slow velocity in rolling motion [17]. In the scenario of a steel disc rolling on a rigid surface, we conjecture that viscosity comes from the complicated evolution of the normal contact stress between the disc and the table [26]. We prefer to select a viscous friction model to scale the resisting moment \( M_n \) against the nutation movement.

\[
M_n = -k_n F_z \dot{\theta}^2 \text{Sign}(\dot{\theta}). \tag{5.3}
\]

Similarly, the resisting moment \( M_m \) against the precession movement reads as

\[
M_m = -k_m F_z \dot{\psi}^2 \text{Sign}(\dot{\psi}). \tag{5.4}
\]

The coefficients \( k_n \) and \( k_m \) in (5.3) and (5.4) are constants related to the materials of contacting bodies.

(b) Discussion for the energy dissipation and the termination of the disc motion

As the model of rolling friction is inserted into the dynamics of the disc motion, the rate of dissipation of energy can be calculated by

\[
\frac{dE}{dt} = M_A \cdot \Omega. \tag{5.5}
\]
In frame $R_A$, the absolute angular velocity $\boldsymbol{\Omega}$ is expressed as

$$\boldsymbol{\Omega} = \dot{\theta} \mathbf{n}^0 - \dot{\psi} s_\theta \mathbf{m}^0 + \omega_z \mathbf{z}^0.$$  \hspace{1cm} (5.6)

Therefore, we have

$$\frac{dE}{dt} = -(k_z \omega_z \mathrm{Sign}(\omega_z) + k_n \dot{\psi}^3 \mathrm{Sign}(\dot{\psi}) - k_m \dot{\psi}^2 \ddot{\psi} s_\theta \mathrm{Sign}(\ddot{\psi})) F_z. \hspace{1cm} (5.7)$$

In terms of the experimental observations shown in figure 11, the values of $\omega_z$ and $\omega_3$ are generally small and decrease slowly. As $k_z$ takes a small value, the contribution from $M_z$ to energy dissipation is small enough to be ignorable. Also, rolling friction represents the accumulative effects of energy dissipation in the long-term behaviours of the dynamics. Therefore, coefficients $k_n$ and $k_m$ should be in a neighbouring order of magnitude since they reflect a similar mechanism. In addition, in comparison with $\psi$, the value of $\dot{\theta}$ is small enough. This means that $M_n$ is much smaller in amplitude than $M_m$, so that $M_m$ is the dominant rolling friction in accounting for the dissipation of energy. By considering that the contact region at the contact point takes a prolate shape with an inhomogeneous distribution of the normal stress, this scenario agrees with the physical intuition that the resistance torque is mainly applied along $\mathbf{m}^0$ direction.

As $\omega_3$ goes to zero, corresponding to a scenario that the disc seems to be still while the contact point moves rapidly, we know from equation (2.1) that $\dot{\psi} \approx -\dot{\theta} \cos \theta$. In this case, by neglecting the small terms, equation (5.7) is approximately expressed as

$$\frac{dE}{dt} \approx -k_m \dot{\psi}^3 c_\theta s_\theta \mathrm{Sign}(\ddot{\psi}) F_z. \hspace{1cm} (5.8)$$

Note that $c_\theta \approx 1$ and $s_\theta \approx \theta$ as $\theta$ is small. In addition, $F_z$ is of the magnitude near to $mg$ and can be seen as a constant in this analysis. Together with the condition of $\dot{\psi}^2 \theta \approx 4g/r$, this makes equation (5.8) be further simplified as

$$\frac{dE}{dt} \approx -mgk_m \left(\frac{4g}{r}\right)^{3/2} \theta^{-1/2}(t). \hspace{1cm} (5.9)$$

Noting that $dE/dt \approx 3mgr\dot{\theta}(t)/2$, so we have

$$\frac{3}{2} \dot{\theta}(t) \approx -k_m \left(\frac{4g}{r}\right)^{3/2} \theta^{-1/2}(t). \hspace{1cm} (5.10)$$

Suppose that $t_f$ is the terminal time related to $\theta(t_f) = 0$. Integration of equation (5.10) gives an approximate solution to $\theta(t)$

$$\theta(t) \approx k_m^{2/3} \left(\frac{4g}{r^{5/3}}(t_f - t)^{2/3}\right). \hspace{1cm} (5.11)$$

Clearly, the approximate solution makes the exponents defined in §4 take values of $n_E = n_\theta = 2/3$, and $n_\psi = -1/3$. These values agree well with the experimental results shown in tables 2 and 3.

Suppose that $\ddot{\theta}_0$ corresponds to the average value of the nutation angle at an instant ($t = 0$). Based on $\ddot{\theta}_0$, we can use equation (5.11) to estimate the value of terminal time $t_f$. It is obvious that $t_f$ increases along the values of $\ddot{\theta}_0^{3/2}$ and $t_f^{5/2}$, but linearly decreases with $k_m$. Note that the value of $k_m$ should strictly depend on the physical property of the contact interface. This is the reason why the disc behaves differently on different surfaces.

When the precession motion follows $\psi \propto (t_f - t)^{-1/3}$, it is certainly singular as $t \rightarrow t_f$. To give a resolution for the singularity, Moffatt conjectured that the disc may rise off the surface with the increase in the vertical acceleration. The explanation was also questioned by many researchers [3,27]. Intuitively, one may tend to believe that the disc would terminate its motion by the collapse without leaving from the table. In our recent work for a planar disc falling down to a surface [19], theoretical analysis and experiments also indicated that it is impossible for the disc to leave the surface.

Recall that the approximate solution in equation (5.11) is valid only when normal contact force $F_z$ is approximately in a constant value. In terms of $F_z = mr\ddot{\theta} + mg$, this simplification is
basically safe because \( \dot{\theta}(t) \) changes slowly during most time of the disc motion. Nevertheless, the statement that \( \dot{\theta}(t) \) goes to infinity as \( t \to t_f \) is not true because the value of \( F_z \) is also responsible for energy dissipation. Let us replace \( F_z \) in equation (5.8) by \( F_z = mr\ddot{\theta} + mg \). Together with \( E \propto \theta \), equation (5.8) is rewritten as
\[
\dot{\theta} \approx -\dot{\theta}^{-1/2}(r\ddot{\theta} + g). \tag{5.12}
\]

If the magnitude of \( \ddot{\theta} \) is increased to a value \( |\ddot{\theta}| = g/r \), equation (5.12) indicates that the value of \( \dot{\theta} \) should converge to zero. Note that the increase in the absolute value \( |\dot{\theta}| \) only permits the magnitude of \( \dot{\theta} \) to increase monotonically. This means that the threshold value, \( |\dot{\theta}| = g/r \), responsible for the occurrence of the disc detaching from the table, is inaccessible. Thus, there must be some new mechanism involved, such as impact or dissipation caused by machining errors, to terminate the disc’s motion. In the tested disc of our experiments, the paper glued on the bottom surface of the disc plays a role of magnifying the roughness of the contact interface to arrest the motion of the disc.

(c) Comparison between numerical and experimental results

We simulate the dynamics of the disc by employing the rolling friction model described above and then compare the numerical simulation with experimental results. Numerical simulations are performed based on the nonholonomic constraint equations in (2.3), the governing equations of the motion of the disc’s centre of mass in (2.6), and the equations governing the motion of the disc’s attitudes in (2.8), together with the model of rolling friction given by (5.2)–(5.4). The values coefficients of rolling friction used in simulation are set as follows: \( k_c \) and \( k_n \) are set zero because they typically contribute little to the energy dissipation; \( k_m = 0.381 \times 10^{-7} \) when the disc is rolling on the steel surface and \( k_m = 0.28 \times 10^{-7} \) for the glass surface.

In simulations, we focus on the last 1.5 s before the disc stops its motion. The initial conditions to advance the simulation are obtained from experimental data with a slight adjustment of

![Figure 17. Comparison between simulations and experiments of a disc rolling on steel surface. (Black solid line represents for simulation data, while red dotted line is for data in experiment.) (Online version in colour.)](image)
making contact point satisfy the constraint of its velocity vanishing. Considering that the paper glued on the bottom surface of the disc makes $\theta$ unable to reach zero, the simulations are terminated when nutation angle reaches a threshold value $\theta \leq 0.5^\circ$.

Figure 17 presents the evolutions of the nutation angle and the precession rate of the disc moving on a steel surface (Cases I and III). It can be seen that the numerical simulations based on the rolling friction model proposed above can precisely reproduce the experimental results. For the cases of the disc moving on a glass surface (Cases II and IV), good agreement between our numerical and experimental results is also found, as shown in figure 18.

Before ending this section, let us check the approximation solution given in equation (5.11). By specifying $t_f - t = 1.5\,s$ to equation (5.11), we find that the average values of the nutation angle at time $t$ are equal to $\bar{\theta}_0 = 7.93^\circ$ and $6.46^\circ$, for the disc on the steel and glass surfaces, respectively. These values basically lie at the starts of the curves in figures 17 and 18. This means that equation (5.11) can approximately capture the average motion of the disc as it moves in a purely rolling state and the approximation of $\sin \theta \sim \theta$ can be established.

6. Conclusion

The spinning motion of a disc, which is activated by a pendulum-based ball, on a steel or a glass surface was investigated experimentally. A stereoscopic vision technique was used to perform omnidirectional observations for the six degrees of freedom of describing the motion of disc in three dimensions. Knowledge of the time histories of position and orientation details the disc dynamics, thus quantifies the energy dissipation induced by the complicated interplay at a moving contact point.
According to our experimental observations, the disc under different conditions can quickly enter into a quasi-steady procession state that satisfies the assumptions of angular momentum and energy as given by Moffatt [6]. Nevertheless, the evolution of energy versus time follows a power law with an exponent equal to $2/3$, which disagrees with the value given in Moffatt’s theory. The evidence from our experiments supports the idea that the viscosity is truly responsible for the dissipation of energy, but originates from rolling friction rather than the sheared air between the disc and the table.

Based on a physical picture of the interaction at the contact point, we scale the rolling friction in association with angular velocity to reflect the viscous effect. Qualitative analysis reveals that the model of rolling friction admits the evolution of energy to behave in a power law with an exponent agreeing with our experiment measurements. Furthermore, good agreement between numerical and experimental results supports that the model of rolling friction can enter into the dynamics of the spinning disc.

By the stereoscopic vision method, together with a qualitative analysis under the mechanism of rolling friction, we show that the disc always contacts the table during its motion. Nevertheless, since our experiments were performed by testing a disc with a piece of thin paper that cannot effectively reveal its dynamical behaviour in the cases of its nutation angle really decreasing to zero, other experimental technique, such as electrical measurements, may need to be developed in future for checking the contact state of the disc in its motion with an extremely smaller nutation angle.

Basically rolling friction represents a mechanism of scaling the energy dissipation in a long-term behaviour of contact dynamics and is a manifestation of the small scales in size and time involved in a moving contact point. Although a clear explanation for the viscous mechanism in rolling friction still remains open, the spinning disc is truly a typical example of exhibiting its influence on the macroscopic motion. Good agreement between our experiments and simulations seems to shed light on the mechanism of the rolling friction, which provides a moment proportional to the square of precession rate to resist the disc motion.

Acknowledgements. The authors thank Dr Hongjie Zhong for his kind help in developing stereoscopic vision technique in our experiments.

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Appendix A. Data processing

A stereoscopic vision method was used to determine the disc six degrees of freedom. After calibration for the two CCD cameras to obtain camera matrices, a pinhole camera model was used to describe the relationship between a spatial point and an image pixel.

$$
\begin{bmatrix}
u \\
v \\
1
\end{bmatrix} = C 
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix},
$$

where $(u, v)$ are the pixel coordinates, $C$ is the camera matrices and $(X, Y, Z)$ are the physical coordinates of point $P$ in a laboratory frame.

As shown in figure 3, six circle spots are marked on the disc. Suppose a dotted marker $i$ on the disc surface with coordinates $(x_{id}^i, y_{id}^i, z_{id}^i)$ in a body-fixed frame. Its physical coordinates $(X^i, Y^i, Z^i)$ in the laboratory frame has a relation to the generalized coordinates of the disc $(x, y, z, \psi, \theta, \phi)$, where $(x, y, z)$ is the coordinates of the disc’s centre of mass, and $(\psi, \theta, \phi)$ are the Euler angles of describing the disc’s orientation.

$$
\begin{bmatrix}
X^i \\
Y^i \\
Z^i
\end{bmatrix} = R(\psi, \theta, \phi) 
\begin{bmatrix}
x_{id}^i \\
y_{id}^i \\
x_{id}^i
\end{bmatrix} + 
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix},
$$
where $R$ is the transformation matrix between the body-fixed frame to the laboratory-space frame. As camera matrices is determined based on calibration, the pixel coordinates of the dotted marker $i$ in camera $K$, $(u_i^k, v_i^k)$ are then related to the generalized coordinates of the disc by

$$
\begin{align*}
{u_i^k} &= U_i^k(x, y, z, \psi, \theta, \phi) \\
{v_i^k} &= V_i^k(x, y, z, \psi, \theta, \phi).
\end{align*}
$$

(A 3)

The motions in all six degrees of freedom were then found as follows:

(a) The coordinates $(u_i^k, v_i^k)$ of each mark in each individual camera were obtained first. In order to improve the accuracy in image recognition, a circle template was used to find the values of the pixel coordinates.

(b) The projection relationships $U_i^k$ and $V_i^k$ between pixel coordinates and the generalized coordinates of the disc was then used to generate a set of linear equations for obtaining a least-square solution. This is performed using a Levenberg–Marquardt algorithm [23] to solve the nonlinear optimization as follows:

$$
J = \min \sum \left( (u_i^k - U_i^k)^2 + (v_i^k - V_i^k)^2 \right).
$$

(A 4)

Except the system error induced by the uncertainty in our experimental set-up, the numerical algorithm only generates small error in the data processing of image recognition. As one pixel in image corresponds to about 0.25 mm in space, numerical algorithm reveals that the final recognition error in image is about 0.02 pixel, which is a much smaller quantity.

References