A remark on constrained von Kármán theories

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We derive the Euler–Lagrange equation corresponding to ‘non-Euclidean’ convex constrained von Kármán theories.

1. Introduction

Föppl–von-Kármán theories arise as asymptotic theories modelling the behaviour of thin elastic films in an energy regime allowing only for very small deformations. The elastic energy of such deformations (with respect to the thickness of the film) is therefore much lower than that of generic nonlinear bending deformations. The asymptotic behaviour of the latter is modelled by the fully nonlinear Kirchhoff plate theory. We refer to [1,2] for a derivation and thorough discussion of these theories (cf. also [3]).

More precisely, the asymptotic behaviour of thin film deformations whose elastic energy lies in a regime just below the nonlinear bending regime is captured by so-called constrained von Kármán theories (cf. [2]). Their behaviour is essentially fully described by their out-of-plane displacement $v: S \to \mathbb{R}$, where $S \subset \mathbb{R}^2$ is the reference configuration of the sample. The asymptotic elastic energy of such a displacement $v$ is then given by

$$\frac{1}{24} \int_S Q_2(\nabla^2 v(x)) \, dx + \int_S f \cdot v \, dx,$$

subject to the constraint

$$\det(\nabla^2 v) = 0.$$  (1.2)

Here, $Q_2$ is the quadratic form of linearized elasticity and $f$ models applied forces.

Motivated by applications in non-Euclidean (or pre-stretched) elasticity (cf. e.g. [4,5]), we consider variants of functionals as in (1.1) by allowing a non-zero right-hand side in (1.2). For simplicity, we restrict to the canonical case when $Q_2 = | \cdot |^2$ and we do not consider forces. More general situations can be handled in the same way, as our main focus is on the constraint

$$\det \nabla^2 v = k.$$  (1.3)
itself. The problem is therefore to understand, on a bounded domain (i.e. open and connected set) $S \subset \mathbb{R}^2$, and for given $k: S \rightarrow \mathbb{R}$, the functional

$$W_k(v) = \begin{cases} \int_S |\nabla^2 v(x)|^2 \, dx & \text{if } v \in W^{2,2}_k(S) \\ +\infty & \text{otherwise.} \end{cases}$$

Here, $W^{2,2}_k = \{ v \in W^{2,2}(S) : \det \nabla^2 v = k \text{ pointwise almost everywhere} \}$.

We use a similar notation for other function spaces, such as $C^{2,\alpha}_k(\bar{S})$ or $C^2_k(\bar{S})$. The Monge–Ampère equation $\det \nabla^2 v = k$ has been studied extensively over the last decades. We refer to Gilbarg & Trudinger [6] for a list of references on the topic.

The functionals $W_k$ are scalar variants of the functionals studied in [7,8]. The purpose of this note is to show how the approach developed in those papers can be adapted to the simpler situation considered here. In passing, we provide here a classical functional analytic framework for this sort of problems. Our main focus is on the elliptic case ($k > 0$), which is the simplest one. The methods are, therefore, very basic. Indeed, in this situation, soft arguments readily yield the desired Euler–Lagrange equation.

In §2, we verify the existence of minimizers and then state our main results regarding the validity of a Lagrange multiplier rule. In §3, we collect some fundamental functional analytic facts and then we prove the main results. In §4, we discuss the cases when $k$ is a constant of arbitrary sign.

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2. Main results

For simplicity, we assume throughout this note that $S \subset \mathbb{R}^2$ is a simply connected, bounded domain with a smooth boundary, and we let $k \in C^\infty(\bar{S})$.

(a) Existence of minimizers

As in [7], the existence of minimizers can be proved by a robust and straightforward argument.

**Proposition 2.1.** The functional $W_k$ attains a minimum in the space $W^{2,2}_k(S)$.

**Proof.** We only need to consider the case when the infimum of $W_k$ is finite. But then the result is a straightforward application of the direct method on the space

$$X = \left\{ v \in W^{2,2} : \int_S v = 0 \text{ and } \int_S \nabla v = 0 \right\}.$$

$W_k$ is obviously $W^{2,2}$-coercive and lower semi-continuous under weak $W^{2,2}$-convergence. But the constraint is stable under weak $W^{2,2}$-convergence, because the expression $\det \nabla^2 v = -\frac{1}{2} \text{curl} \text{curl}(\nabla v \otimes \nabla v)$ is continuous under weak $W^{2,2}$-convergence. Applying Poincaré’s inequality, we obtain the existence of a minimizer in $X$. ■

It is clear that the same proof also works for general domain dimensions, other energy densities, additional force terms, boundary conditions, etc. As in [8], when $k > 0$, then one has a better existence result.

**Proposition 2.2.** Assume that $k > 0$ on $\bar{S}$. Then the functional $W_k$ attains a minimum on the set

$$W^{2,2}_k(S) \cap C^\infty(\bar{S}). \quad (2.1)$$
Proof. First note that functions $v$ belonging to the set (2.1) are either convex or concave and that the infimum of $\mathcal{W}_k$ is the same on both of these components of the set (2.1). This is because $\mathcal{W}_k(-v) = \mathcal{W}_k(v)$, and if $v$ is convex then $-v$ is concave. So we will prove that the minimum is attained on the set

$$X = \{ v \in W^{2,2}_k(S) \cap C^\infty(S) : v \text{ is convex} \}.$$ 

By interior regularity for convex Monge–Ampère equations, we have

$$X = \{ v \in W^{2,2}_k(S) : v \text{ is convex} \}.$$ 

More precisely, the theorem in [10, ch. VIII, Section 6] (cf. also [11]) asserts that any convex solution of the Monge–Ampère equation with uniformly positive right-hand side in $C^\infty$ is itself $C^\infty$ in the interior of the domain.

By (2.2), the space $X$ is closed under weak $W^{2,2}$-convergence, because this convergence preserves convexity, due to the fact that the set of positive semi-definite symmetric matrices is convex. Therefore, we can find a minimizer in $X$ by the same arguments as in proof of proposition 2.1. 

(b) Lagrange multiplier rule for the elliptic case

The formal Lagrange multiplier rule asserts that critical points of $\mathcal{W}_k$ are critical for the functional

$$v \mapsto \int_S |\nabla^2 v|^2 - \lambda \det \nabla^2 v$$

without additional constraints, cf. e.g. [12] for a related situation. Here, $\lambda$ is some Lagrange multiplier. The Euler–Lagrange equation corresponding to (2.3) is

$$\text{div}\text{ div}(\nabla^2 v - \lambda \text{ cof } \nabla^2 v) = 0$$

or, as $\text{div}\text{ cof } \nabla^2 v = 0$,

$$\Delta^2 v - \text{cof } \nabla^2 v : \nabla^2 \lambda = 0.$$ 

We will show that, under suitable regularity assumptions, this formal Lagrange multiplier rule can be justified by means of very soft functional analytic arguments.

For a rigorous approach, we introduce the following notions, which are variants of those introduced in [7]: a function $v \in W^{2,2}_k(S)$ is said to be stationary for $\mathcal{W}_k$ if

$$\left. \frac{d}{dt} \right|_{t=0} \int_S |\nabla^2 u(t)|^2 = 0$$

for all (strongly $W^{2,2}$-continuous, say) maps $t \mapsto u(t)$ from a neighbourhood of zero in $\mathbb{R}$ into $W^{2,2}_k(S)$ such that $u(0) = v$ and such that the derivative $u'(0)$ exists.

A function $v \in W^{2,2}_k(S)$ is said to be formally stationary for $\mathcal{W}_k$ if

$$\int_S \nabla^2 v : \nabla^2 h = 0$$

for all $h \in W^{2,2}_k(S)$ with $\text{cof } \nabla^2 v : \nabla^2 h = 0$ a.e. in $S$.

Our main results for the elliptic case are the following two remarks.

**Proposition 2.3.** Let $\alpha \in (0,1)$ and let $k > 0$ on $\tilde{S}$. Then the set

$$\mathcal{C}^{2,\alpha}_k(\tilde{S}) := \{ u \in C^{2,\alpha}(\tilde{S}) : \det \nabla^2 u = k \text{ in } S \}$$

is a $C^\infty$-submanifold of $C^{2,\alpha}(\tilde{S})$.

**Proposition 2.4.** Let $\alpha \in (0,1)$ and let $k > 0$ on $\tilde{S}$. If $v \in \mathcal{C}^{2,\alpha}_k(\tilde{S})$ is stationary for $\mathcal{W}_k$, then there exists a unique Lagrange multiplier $\lambda \in (C^\infty \cap L^2)(S)$ such that

$$\text{div}\text{ div}(\chi_S(\nabla^2 v + \lambda \text{ cof } \nabla^2 v)) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

In particular,

$$\Delta^2 v + \text{cof } \nabla^2 v : \nabla^2 \lambda = 0$$

in the classical sense on $S$.
Remarks.

(i) Solutions of (2.4) clearly are critical points. The non-trivial assertion is the existence of a Lagrange multiplier.

(ii) The natural biharmonic boundary conditions encoded in (2.4) can of course be written classically by trivial integration by parts (cf. e.g. [3]).

3. The elliptic case

(a) Functional analysis background

In this section, $X$ and $Y$ denote real Banach spaces. Recall that a closed subspace $E$ of $X$ is said to split $X$ if $E$ has a closed complement, i.e. there exists a closed subspace $F$ of $X$ such that $X = E ⊕ F$.

For a linear operator $G: X → Y$, we denote by $N(G)$ its kernel and by $R(G)$ its range. The following result is standard (cf. e.g. [13]).

**Lemma 3.1.** Let $G: X → Y$ and $F: X → ℝ$ be bounded linear operators, and assume that the range of $G$ is closed. Then

$$Fh = 0 \quad \text{for all } h ∈ X \text{ with } Gh = 0$$

if and only if there exists $Λ ∈ Y'$ such that $F = Λ ◦ G$. If, moreover, $R(G) = Y$ then $Λ$ is unique.

Let $M ⊂ X$. A vector $h ∈ X$ is called a tangent vector to $M$ at $v ∈ M$ provided there exists a map $u$ from a neighbourhood of zero in $ℝ$ into $M$ such that $u(0) = v$, and such that the derivative $u'(0)$ at 0 exists and equals $h$. We denote the set of all tangent vectors $h ∈ X$ at $v$ by $T_vM$. We recall the following basic result.

**Lemma 3.2.** Let $G: X → Y$ be continuously Fréchet differentiable on $X$ and set

$$M = \{u ∈ X : G(u) = 0\}. \quad (3.1)$$

Assume that, for all $v ∈ M$, the derivative $G'(v): X → Y$ is surjective, and the kernel $N(G'(v))$ splits $X$. Then $M$ is a $C^1$-manifold.

In particular, for all $v ∈ M$, we have $T_vM = N(G'(v))$ and there exists a continuously Fréchet differentiable homeomorphism $φ$ from a neighbourhood of zero in $T_vM$ onto an open neighbourhood of $v$ in $M$ that satisfies

$$φ(h) = v + h + o(∥h∥_X) \quad \text{as } h → 0 \text{ in } T_vM.$$  

If $G$ is $C^m$ on $X$, then $M$ is a $C^m$-manifold.

**Remark.** In the context of surfaces, the existence of $φ$ as in the conclusion of lemma 3.2 amounts to the so-called continuation of infinitesimal bendings. We refer to [14] for the elliptic case and to [15] for the intrinsically flat case.

**Proof.** This result is classical (cf. [16]). For the reader’s convenience, we recall the proof of the existence of $φ$.

Let $E ⊂ X$ be a closed complement of $N(G'(v))$ in $X$. Define $H: N(G'(v)) × E → Y$ by setting $H(h, z) = G(v + h + z)$. As the partial derivative $D^2H(0, 0)$ is just the restriction of $G'(v)$ to $E$, the hypotheses show that we can apply the implicit function theorem to $H$. This yields a $C^1$-map $φ$ as in the statement, because $D^1H(0, 0) = 0$. ■
(b) Linear elliptic operators

In this paragraph, we recall some basic functional analytic properties of linear elliptic operators of the form

\[ Lu := A : \nabla^2 u + B \cdot \nabla u + Cu \]

on a bounded \( C^{2,\alpha} \) domain \( \Omega \subset \mathbb{R}^n \) for some \( \alpha \in (0, 1) \). Here, \( A \in C^0(\overline{\Omega}, \mathbb{R}^{n \times n}) \), \( B \in L^\infty(\Omega, \mathbb{R}^n) \) and \( C \in L^\infty(\Omega) \). We assume \( A \) to be strictly elliptic, i.e. there exists \( c > 0 \) such that

\[ A(x) : (\xi \otimes \xi) \geq c|\xi|^2 \quad \text{for all} \ x \in \Omega, \ \xi \in \mathbb{R}^n. \]

For simplicity, we only consider the case when \( C \leq 0 \).

**Lemma 3.3.** Let \( p \in (1, \infty) \), let \( A \in C^0(\overline{\Omega}), \ B, C \in L^\infty(\Omega) \), assume that \( C \leq 0 \) and define \( G : W^{2,p}(\Omega) \to L^p(\Omega) \) by \( Gu = Lu \). Then \( G \) is surjective and \( N(G) \) splits \( W^{2,p}(\Omega) \).

**Proof.** Under the above hypotheses on the coefficients, and for any \( f \in L^p(\Omega) \), the Dirichlet problem,

\[ Lu = f \]
\[ u \in W_0^{1, p}(\Omega), \]

has a unique solution \( u \in W^{2,p}(\Omega) \) (cf. [6, theorem 9.15]). Hence, \( G \) is surjective.

For each \( f \in L^p(\Omega) \) denote by \( T_f \) the solution \( u \) of (3.2). Then \( T : L^p(\Omega) \to W^{2,p}(\Omega) \) is bounded.

And obviously it is a right inverse of \( G \). As \( G \) is surjective and admits a bounded right inverse, we conclude that \( N(G) \) splits \( W^{2,p}(\Omega) \). \( \blacksquare \)

Similarly, this time using Schauder theory, one proves the following lemma.

**Lemma 3.4.** Assume \( A, B, C \in C^{0,\alpha}(\overline{\Omega}) \), assume that \( C \leq 0 \) and define \( G : C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}) \) by \( Gu = Lu \). Then \( G \) is surjective and \( N(G) \) splits \( C^{2,\alpha}(\overline{\Omega}) \).

**Lemma 3.5.** Assume \( A, B, C \in C^{0,\alpha}(\overline{\Omega}) \), assume that \( C \leq 0 \) and define \( G : W^{2,2}(\Omega) \to L^2(\Omega) \) by \( Gu = Lu \). Then \( C^{2,\alpha}(\overline{\Omega}) \cap N(G) \) is strongly \( W^{2,2} \)-dense in \( N(G) \).

**Proof of lemma 3.5.** Let \( u \in N(G) \) and let \( u_n \in C^{2,\alpha}(\overline{\Omega}) \) be such that \( u_n \to u \) in \( W^{2,2}(\Omega) \). Then by continuity

\[ Lu_n \to 0 \quad \text{in} \ L^2(\Omega). \]

As in the proof of lemma 3.3, there exists a unique solution \( \rho_n \in (W^{2,2} \cap W_0^{1, 2})(\Omega) \) of

\[ L\rho_n = - Lu_n \quad \text{in} \ \Omega. \]

Moreover, \( \rho_n \in C^{2,\alpha}(\overline{\Omega}) \) because \( Lu_n \in C^{0,\alpha}(\overline{\Omega}) \), and \( \rho_n \to 0 \) in \( W^{2,2}(\Omega) \) by (3.3). Thus, \( u_n + \rho_n \in N(G) \cap C^{2,\alpha}(\overline{\Omega}) \) and \( u_n + \rho_n \to u \) in \( W^{2,2}(\Omega) \). \( \blacksquare \)

(c) Proofs of the main results

**Proof of proposition 2.3.** Let \( m \in \mathbb{N} \) and set \( X = C^{2,\alpha}(\overline{\Omega}) \) and \( Y = C^{0,\alpha}(\overline{\Omega}) \). We must show that the map \( G : X \to Y \) defined by

\[ G(u) = \det \nabla^2 u - k \]

satisfies the hypotheses of lemma 3.2 with \( M \) given by (3.1). But of course \( G \) is in \( C^m \), because it is quadratic. More precisely, for all \( h \in X \), we have

\[ G(v + h) = G(v) + \cof \nabla^2 v : \nabla^2 h + \det \nabla^2 h. \]

As

\[ \|\det \nabla^2 h\|_{C^{0,\alpha}} \leq \|\nabla^2 h\|_{C^{0,\alpha}}^2 \leq \|h\|_{C^{2,\alpha}}^2, \]

we have

\[ G'(v)h = \cof \nabla^2 v : \nabla^2 h \quad \text{for all} \ h \in X. \]
Next, we claim that \( G'(v) : X \rightarrow Y \) is surjective for all \( v \in M \). But since \( \det \nabla^2 v = k \), by the assumptions on \( k \) we see that \( v \) is either strictly convex or concave. Hence, \( \cof \nabla^2 v \) is strictly elliptic. So lemma 3.4 shows that \( G'(v) \) is surjective and that \( N(G'(v)) \) splits \( X \).

**Lemma 3.6.** Assume that \( k > 0 \) on \( \tilde{S} \). If \( v \in C^2_{\alpha}(\tilde{S}) \) is stationary for \( \mathcal{W}_k \), then \( v \) is formally stationary for \( \mathcal{W}_k \).

**Proof.** Define \( F : W^{2,2}(S) \rightarrow L^2(S) \) by \( G = \cof \nabla^2 v : \nabla^2 u \) and \( \tilde{G} : C^2(\tilde{S}) \rightarrow C^0(\tilde{S}) \) by \( \tilde{G} = \cof \nabla^2 v : \nabla^2 u \), and define \( F : W^{2,2}(S) \rightarrow \mathbb{R} \) by \( F(v) = \int_{\tilde{S}} |\nabla^2 v|^2 \).

As \( F \) is continuously Fréchet differentiable, the fact that \( v \in C^2_{\alpha}(\tilde{S}) \) is stationary for \( \mathcal{W}_k \), combined with proposition 2.3 (in particular with the existence of \( \varphi \) as in the conclusion of lemma 3.2), implies that

\[
F'(v)h = 0 \quad \text{for all} \ h \in N(\tilde{G}).
\]

Now \( N(\tilde{G}) = N(G) \cap C^2(\tilde{S}) \). Lemma 3.5 implies that \( N(\tilde{G}) \) is strongly \( W^{2,2} \)-dense in \( N(G) \). Thus, by continuity of \( F'(v) \), formula (3.4) is in fact equivalent to

\[
F'(v)h = 0 \quad \text{for all} \ h \in N(G).
\]

And this means that \( v \) is formally stationary for \( \mathcal{W}_k \).

For formal stationary points, we use the basic Lagrange multiplier rule to prove the following lemma.

**Lemma 3.7.** Assume that \( k > 0 \) on \( \tilde{S} \). If \( v \in C^2_{\alpha}(\tilde{S}) \) is formally stationary for \( \mathcal{W}_k \), then there exists a Lagrange multiplier \( \lambda \in L^2(S) \) such that \( (2.4) \) holds.

**Proof.** Define \( F : W^{2,2}(S) \rightarrow \mathbb{R} \) by \( Fh = \int_{\tilde{S}} \nabla^2 v : \nabla^2 h \) and \( G : W^{2,2}(S) \rightarrow L^2(S) \) by \( Gh = \cof \nabla^2 v : \nabla^2 h \). By hypothesis, we know \( Fh = 0 \) for all \( h \in N(G) \). As \( \cof \nabla^2 v \in C^0(\tilde{S}) \), we can apply lemma 3.3 to see that \( G \) is surjective. Now lemma 3.1 implies that there exists a unique \( A \) in the dual of \( L^2(S) \) such that \( F = \Lambda \circ G \), i.e. there exists \( \lambda \in L^2(S) \) such that

\[
\int_{\tilde{S}} \nabla^2 v : \nabla^2 h = \int_{\tilde{S}} \lambda \cof \nabla^2 v : \nabla^2 h \quad \text{for all} \ h \in W^{2,2}(S).
\]

**Proof of proposition 2.4.** Combine lemma 3.7 with lemma 3.6, and observe that \( (2.4) \) implies that

\[
\text{div} \ \text{div}(\lambda \cof \nabla^2 v) = \Delta^2 v \quad \text{in} \ D'(S),
\]

which by standard elliptic regularity proves that \( \lambda \in C^\infty(S) \) because \( v \in C^\infty(S) \).

4. **Further remarks and the case of constant \( k \)**

For \( A \in \mathbb{R}^{2 \times 2}_{\text{sym}} \), we have \( |A|^2 = (\text{Tr} A)^2 - 2 \det A \) and \( |A|^2 = 2|A^\circ|^2 + 2 \det A \), where \( A^\circ = A - \frac{1}{2}(\text{Tr} A)I \) denotes the trace-free part of \( A \). Thus,

\[
|\nabla^2 v|^2 = (\Delta v)^2 - 2k
\]

and

\[
|\nabla^2 v|^2 = 2|\nabla^2 v - \frac{1}{2} \Delta v I|^2 + 2k.
\]

And so

\[
\mathcal{W}_k(v) = \int_{\tilde{S}} ((\Delta v)^2 - 2k) = 2 \int_{\tilde{S}} \left( \left| \nabla^2 v - \frac{1}{2} \Delta v I \right|^2 + k \right). \quad (4.2)
\]

The first equality in (4.2) leads to the following immediate observation.

**Remark 4.1.** If the class \( W^{2,2}_k(S) \) contains a harmonic function, then \( \mathcal{W}_k \) attains its absolute minimum \(-2 \int_{\tilde{S}} k \) precisely at the harmonic functions in \( W^{2,2}_k(S) \).
An obvious necessary condition for $W^{2,2}_k(S)$ to contain a harmonic function is that $k \leq 0$; this follows e.g. from (4.1). This equality trivially shows that harmonic functions $v \in W^{2,2}_k(S)$ satisfy $|\nabla^2 v|^2 = -2k \in C^\infty(S)$.

(a) The case of constant $k$

In this subsection, we assume that $k$ is constant. The second equality in (4.2) leads to the following observation: an absolute minimum of $W_k$ is attained if $\nabla^2 v - \frac{1}{2} \Delta v I$ vanishes identically. This can be achieved only when $k \geq 0$, and then it is the case precisely if $\nabla^2 v = \sqrt{k}I$ almost everywhere in $S$, (4.3)

Alternatively (and this includes the case of negative constant $k$), one minimizes $F \mapsto |F|^2$ among all $F \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ with $\det F = k$. The finite dimensional Lagrange multiplier rule then implies that for all critical $F$ there exists $\mu \in \mathbb{R}$ such that $F = \mu \operatorname{cof} F$. As $F$ is symmetric, this readily shows that $\mu = \pm 1$. And if $\mu = 1$ then $F = \sqrt{k}I$, whereas if $\mu = -1$ then $F^T F = -kI$. The latter means that $F = \sqrt{|k|}R$ for some $R \in O(2)$ with $\det R = -1$. In either case, $F$ is uniquely determined up to similarity transformations, which do not affect $|\cdot|^2$ under the constraint.

Thus, for constant $k \geq 0$, we have recovered (4.3). For constant $k < 0$, we have shown that the absolute minimizers of $W_k$ satisfy

$$\nabla^2 v(x) \in \left\{ \sqrt{|k|}R : R \in O(2) \right\} \quad \text{for almost every } x \in S.$$

for almost every $x \in S$. Now we apply the Liouville theorem asserting that a map in $W^{1,2}(S, \mathbb{R}^2)$ whose gradient takes values in $SO(2)$ is a rigid motion. A very short proof of this result (for arbitrary dimensions) can be found in [1, p. 1469].

It shows that (4.4) implies the existence of a constant rotation $R \in O(2)$ with $\det R = -1$ such that $\nabla^2 v = \sqrt{|k|}R$ almost everywhere in $S$. Summarizing, we have shown the following.

**Remark 4.2.** If $k$ is constant, then

$$v(x) = \begin{cases} \sqrt{k} \frac{|x|^2}{2} & \text{if } k \geq 0 \\ \sqrt{|k|} \left( x_1^2 - x_2^2 \right) & \text{if } k < 0 \end{cases}$$

is the unique absolute minimizer of $W_k$ (up to addition of affine maps and up to global rotations of the independent variables).

**Remarks.**

(i) The relation to remark 4.1 is that, by remark 4.2, for constant negative $k$ the absolute minimizers are precisely the harmonic quadratic polynomials $v(x) = x \cdot Ax$ with $\det A = k$.

(ii) A non-trivial problem for the case $k = 0$ results if one imposes boundary conditions or includes force terms. This situation is covered by the results in [15,17].

Indeed, the problem addressed there was to study minimizers of the Willmore functional

$$u \mapsto \int_S |A|^2$$

among all $W^{2,2}$ isometric immersions $u$ of $(S, \delta)$ into $\mathbb{R}^3$, where $\delta$ denotes the standard flat metric in $\mathbb{R}^2$ and $A$ denotes the second fundamental form of $u$. But by the Gauss–Codazzi–Mainardi equations $A$ is a possible second fundamental form for such an isometric
immersion \( u \) if and only if

\[
A \in \{ \nabla^2 v : v \in W^{2,2}(S) \text{ with } \det \nabla^2 v = 0 \text{ a.e. in } S \}.
\]

Related to this is the fact that, if \( u \in W^{2,2}(S, \mathbb{R}^3) \), then \( u \) is an isometric immersion if and only if the function \( v = u \cdot e \) satisfies the Darboux equation \( \det \nabla^2 v = 0 \) for any constant \( e \in \mathbb{S}^2 \).

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**References**


