Space–time fractional Zener wave equation

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The space–time fractional Zener wave equation, describing viscoelastic materials obeying the time-fractional Zener model and the space-fractional strain measure, is derived and analysed. This model includes waves with finite speed, as well as non-propagating disturbances. The existence and the uniqueness of the solution to the generalized Cauchy problem are proved. Special cases are investigated and numerical examples are presented.

1. Introduction

Here, we study a class of generalized wave equations. The wave equation can be generalized within the theory of fractional calculus by replacing the second-order derivative (space and/or time) with the fractional ones, as done in [1–7]. The space–time fractional Zener wave equation represents a generalization of the classical wave equation obtained as a system consisting of the equation of motion of the deformable (one-dimensional) body, the time-fractional Zener constitutive equation and the space-fractional strain measure. Our generalization is done by the fractionalization in both space and time variables, based on the physically acceptable concepts. More details on the formulation and the mechanical background will be given in this section, which finishes with the remark related to the analysis of our generalization of the wave equation.
In §2, we show the existence and uniqueness of a distributional and a classical solution to the distributional (2.1) and classical space–time fractional Zener wave equation (1.17), (1.19). For this purpose, we use the Fourier and Laplace transforms in the spaces of distributions. In §2b–d, we examine quite different estimates of double integrals proving their absolute convergence. The analysis presented in §2b implies that the distributional solution of the distributional space–time fractional Zener wave equation (2.1), with \( u_0, v_0 \in L^1(\mathbb{R}) \), is a distribution of the second order with respect to \( t \); see theorem 2.1. In §2c, with the assumption \( u_0 = 0 \) and \( v_0 \) regular enough, we obtain the classical solution to the space–time fractional Zener wave equation (1.17), (1.19); see theorem 2.2. On the other hand, in §2d, by the regularization of distribution, we show that the solution to (2.1) is given by a distributional limit of a net of approximated solutions, which are continuous with respect to \( x \in \mathbb{R}, t > 0 \), bounded with respect to \( x \in \mathbb{R} \) and exponentially bounded with respect to \( t > 0 \). The results in §2 are justified in §3 by discussing the influence of parameters \( \alpha \) and \( \beta \) (orders of the time- and space-fractional derivatives) on the solution to (2.1) and in §4 by the numerical examples. Mathematical background is given in appendix A.

(a) Model

Recall that the classical wave equation describes the waves that occur in an elastic medium. It is obtained from the equations of the deformable body \([8]\). The wave equation can be written in the form of a system which consists of three equations: equation of motion, constitutive equation and strain measure. Unknown functions depending on time, \( x \in \mathbb{R} \), variables are: displacement \( u \), stress \( \sigma \), measured in m, Pa, respectively, and (dimensionless) strain \( \varepsilon \). We consider an infinite viscoelastic rod (one-dimensional body), positioned along the \( x \)-axis, that is not under the influence of body forces. Then, the equation of motion reads

\[
\partial_x \sigma(x, t) = \rho \partial_t^2 u(x, t), \quad x \in \mathbb{R}, \ t > 0, \tag{1.1}
\]

where \( \rho > 0 \) denotes the (constant) density of the rod. The constitutive equation gives the relation between stress and strain, and in the case of elastic media it is the Hooke law. As we consider waves occurring in viscoelastic media, we choose the constitutive equation to be the time-fractional Zener model. The time-fractional Zener model was introduced in \([9]\) in the form

\[
\sigma(x, t) + \tau_\varepsilon \partial_0^{RL} D_t^\alpha \sigma(x, t) = E(\varepsilon(x, t) + \tau_\varepsilon \partial_0^{RL} D_t^\alpha \varepsilon(x, t)), \quad x \in \mathbb{R}, \ t > 0, \tag{1.2}
\]

where \( \tau_\varepsilon \) and \( \tau_c \) are relaxation and creep times (measured in s), respectively, \( E \) is the generalized Young modulus (measured in Pa) and \( \partial_0^{RL} D_t^\alpha \) denotes the left Riemann–Liouville operator of fractional differentiation of order \( \alpha \in [0, 1) \). It has been shown in \([10]\) that the Riemann–Liouville fractional derivative in the above constitutive model can be replaced by the Caputo derivative yielding the same model. The fractional Zener model containing the Caputo fractional derivative was independently introduced in \([11]\), where one also finds a discussion on the Mittag–Leffler function. We refer to \([12, 13]\) for the background of using the fractional derivatives in the theory of viscoelastic materials. For \( \alpha = 0 \), the constitutive equation (1.2) reduces to the Hooke law

\[
\sigma = E_r \varepsilon, \quad \text{with} \quad E_r = E \frac{1 + \tau_\varepsilon}{1 + \tau_\sigma}.
\]

For \( \alpha = 1 \), the constitutive equation (1.2) reduces to the classical Zener model. For more details on fractional derivatives see \([14, 15]\). We refer to \([16]\) for a review on the fractional models in viscoelasticity and to \([17]\) for a systematic analysis of the thermodynamical restrictions on parameters in such models. The strain measure gives the connection between strain and displacement. The classical strain

\[
\varepsilon_{cl}(x, t) = \partial_x u(x, t), \quad x \in \mathbb{R}, \ t > 0,
\]
describes local deformations. As we consider non-local effects in a material, we use, at all points of the body, the fractional model of strain measure, assumed in the form of spatially averaged (weighted) classical strain as

\[ \varepsilon(x,t) = \frac{1}{t^{\alpha}} \mathcal{E}_x^\beta u(x,t) = \frac{1}{2t^{1-\beta} \Gamma(1-\beta)} |x|^{-\beta} * x \varepsilon_{cl}(x,t), \quad x \in \mathbb{R}, \ t > 0, \quad (1.3) \]

where \( t \) (measured in m) denotes the length-scale parameter and \( \mathcal{E}_x^\beta \) is the symmetrized Caputo fractional derivative of order \( \beta \in [0, 1) \); see appendix A. We assume that the rod is of finite cross-section and that the cross-sectional area \( A \) characterizes the length-scale parameter \( \ell \), such that \( \ell = \sqrt{A} \). Note that both strain measure \( \varepsilon \) and classical strain \( \varepsilon_{cl} \) are dimensionless quantities. For the derivation of the multi-dimensional fractional strain measure using the standard continuum mechanics approach and its geometrical interpretation see [18] and references therein. The use of strain measures other than classical strain is acceptable in continuum field theory [19, p. 268]. In [20], it was shown that (1.3) can be used as a strain measure, as the displacement field is only a function of time (corresponding to rigid body motion), if and only if \( \varepsilon \) from (1.3) equals zero. The Fourier transform of the strain measure (1.3) reads \( \hat{\varepsilon}(\xi, t) = (1/t^{1-\beta}) \hat{u}(\xi/|\xi|^{1-\beta}) \sin(\beta \pi/2) \hat{\varepsilon}(\xi, t) \), see appendix A, which for \( \beta = 1 \) becomes \( \hat{\varepsilon}(\xi, t) = i \xi \hat{u}(\xi, t) \), which is the Fourier transform of the classical strain \( \varepsilon_{cl} \). Regarding the fractionalization of the strain measure, we follow the approach presented in [20], where the symmetrized fractional derivative is introduced in order to describe the non-local effects of the material. Note that, in [21], the same type of fractional derivative is used in the framework of the heat conduction problem of the space–time fractional Cattaneo-type equation.

One may also treat the non-locality in viscoelastic media by a different approach. Namely, contrary to (1.3), one may retain the classical strain measure and introduce the non-locality in the constitutive equation. In the classical setting, this was done by Eringen [22]. In the framework of the fractional calculus, this approach is followed in [23–27]. The wave equation, obtained from a system consisting of the equation of motion, the fractional Eringen-type constitutive equation and the classical strain measure, is studied in [28,29]. The approach of a construction of spatially fractional viscoelastic wave equations in three dimensions based on fractional constitutive equations in the spirit of Kunin’s stress–strain relations for crystalline solids, or Edelen’s and Eringen’s approach to non-locality, can be found in [30]. Regarding the applications of the anisotropic multi-dimensional operator considered in [30] in diffusion in biological tissues, we refer to [31,32]. We also refer to [33–36] for some recent mechanical models of fractional-order viscoelasticity.

Following the approach where the classical strain measure is retained and non-locality is introduced in the constitutive equation, the system of equations (1.1)–(1.3) may be reformulated. Using the procedure described in §1b, the constitutive equation (1.2) solved with respect to \( \sigma \) reads

\[ \sigma(x,t) = E \left( \left( \frac{\tau_\sigma}{\tau_\sigma} \right)^\alpha \delta(t) + \left( \frac{\tau_\sigma}{\tau_\sigma} \right)^\alpha - 1 \right) \varepsilon'_\sigma(t) * t \varepsilon(x,t), \quad x \in \mathbb{R}, \ t > 0, \quad (1.4) \]

where \( \delta \) is the Dirac distribution, \( \varepsilon'_\sigma = (d/dt)\varepsilon_\sigma \), with \( \varepsilon_\sigma \) being the Mittag–Leffler function \( \varepsilon_\sigma(t) = E_\alpha (-t/\tau_\sigma) \alpha \); see also (1.22). When the constitutive equation, written in the form (1.4), is combined with the strain measure (1.3), a single constitutive relation between stress \( \sigma \) and classical strain \( \varepsilon_{cl} \), non-local in space and time, is obtained as \( x \in \mathbb{R}, \ t > 0 \)

\[ \sigma(x,t) = \frac{E}{t^{1-\beta} \Gamma(1-\beta)} \left( \left( \frac{\tau_\sigma}{\tau_\sigma} \right)^\alpha |x|^{-\beta} \delta(t) + \left( \frac{\tau_\sigma}{\tau_\sigma} \right)^\alpha - 1 \right) |x|^{-\beta} \varepsilon'_\sigma(t) * x \varepsilon_{cl}(x,t). \quad (1.5) \]

Constitutive equation (1.5) corresponds to constitutive modelling of materials that show both history-dependent properties [37] and length-scale or non-local properties [22]. Model parameters, corresponding to materials non-local in space and time described by (1.5), can be experimentally evaluated by fitting data obtained by standard measurement techniques to constitutive relation (1.5). We refer to, for example, [38] for an estimation of the model parameters appearing in history-dependent models of materials used in dentistry.
The system of equations, equivalent to (1.1)–(1.3), consisting of the equation of motion, the space–time non-local constitutive equation and the classical strain \( \varepsilon_{\text{cl}} \) reads \((x \in \mathbb{R}, t > 0)\)
\[
\partial_x \sigma(x, t) = \rho \partial_t^2 u(x, t),
\]
\[
\sigma(x, t) = \frac{E}{2(1+\beta)} \left( \left( \frac{\tau_x}{\tau_{x_t}} \right)^{\alpha} |x|^{-\beta} \epsilon(t) + \left( \frac{\tau_x}{\tau_{x_t}} \right)^{\alpha} - 1 \right) |x|^{-\beta} \epsilon_{\alpha}'(t) \right) *_{x_t} \varepsilon_{\text{cl}}(x, t)
\]
and \( \varepsilon_{\text{cl}}(x, t) = \partial_x u(x, t) \).

The initial conditions corresponding to the system of equations (1.1)–(1.3) are
\[
u(x, 0) = \nu_0(x), \quad \partial_t \nu(x, 0) = \nu_0(x), \quad \sigma(x, 0) = 0 \quad \text{and} \quad \varepsilon(x, 0) = 0, \quad x \in \mathbb{R},
\]
where \( \nu_0 \) and \( \nu_0 \) are the initial displacement and velocity, while the boundary conditions are
\[
\lim_{x \to \pm \infty} \nu(x, t) = 0 \quad \text{and} \quad \lim_{x \to \pm \infty} \sigma(x, t) = 0, \quad t > 0.
\]

Alternatively, if the system of equations (1.6)–(1.8) is analysed, then it is subject to initial conditions (1.9)1,2 and boundary condition (1.10)1.

Note that boundary conditions (1.10) are the natural choice for the case of the unbounded domain, while in the case of the bounded domain there can be a large variety of different boundary conditions depending on the type of problem one is faced with. In the case of the local, time-fractional wave equation on a bounded domain we refer to [16,17,39] and references therein.

(b) System of equations

Introducing the dimensionless quantities
\[
\tilde{x} = \frac{x}{\ell}, \quad \tilde{t} = \frac{t}{\tau_x}, \quad \tilde{\nu} = \frac{\nu}{\tilde{\nu}}, \quad \tilde{\sigma} = \frac{\sigma}{\tau_x}, \quad \tau = \left( \frac{\tau_x}{\tau_{x_t}} \right)^{\alpha}, \quad \tilde{\nu}_0 = \frac{u_0}{\tilde{\nu}}, \quad \tilde{\nu}_0 = \frac{\nu_0}{\tilde{\nu}}, \quad \text{and} \quad \tilde{c} = \frac{c}{\ell} \sqrt{\frac{\beta}{E}},
\]
where \( \ell = \sqrt{A} \), in either (1.1)–(1.3) or (1.6)–(1.8), and, omitting the bar, we obtain
\[
\partial_x \sigma(x, t) = \tilde{c}^2 \partial_t^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
\[
\sigma(x, t) + \tau \bar{D}_t^\gamma \sigma(x, t) = \epsilon(x, t) + \bar{C} \bar{D}_t^\gamma \epsilon(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
and
\[
\epsilon(x, t) = \epsilon_{\text{cl}}^\beta(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
or \((x \in \mathbb{R}, t > 0)\)
\[
\partial_x \sigma(x, t) = \tilde{c}^2 \partial_t^2 u(x, t),
\]
\[
\sigma(x, t) = \frac{1}{2\Gamma(1-\beta)} \left( \frac{1}{\tau} |x|^{-\beta} \epsilon(t) + \left( \frac{1}{\tau} - 1 \right) |x|^{-\beta} \epsilon_{\alpha}'(t) \right) *_{x_t} \epsilon_{\text{cl}}(x, t)
\]
and
\[
\varepsilon_{\text{cl}}(x, t) = \partial_x u(x, t).
\]

In the sequel, we take \( \tilde{c} = 1 \).

The system of equations (1.11)–(1.13), or alternatively (1.14)–(1.16), can be reduced to the space–time fractional Zener wave equation
\[
\partial_t^2 u(x, t) = L_t^\alpha \partial_x \epsilon_{\text{cl}}^\beta(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
where \( L_t^\alpha \) is a linear operator (of convolution type) given by
\[
L_t^\alpha = \mathcal{L}^{-1} \left[ \frac{1+\alpha}{1+\tau s^\alpha} \right] *_{t} = \left( \frac{1}{\tau} \delta(t) + \left( \frac{1}{\tau} - 1 \right) \epsilon_{\alpha}'(t) \right) *_{t}, \quad t > 0,
\]
and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform (see appendix A). The dimensionless quantities give that initial and boundary conditions, (1.10) and (1.9), for the space–time fractional Zener wave equation (1.17) again become
\begin{equation}
  u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad \sigma(x,0) = 0 \quad \text{and} \quad \varepsilon(x,0) = 0, \quad x \in \mathbb{R},
\end{equation}
and
\begin{equation}
  \lim_{x \to \pm \infty} u(x,t) = 0 \quad \text{and} \quad \lim_{x \to \pm \infty} \sigma(x,t) = 0, \quad t > 0.
\end{equation}

The procedure of obtaining (1.17) is as follows. Applying the Laplace transform to (1.12) with respect to time variable $t$, one obtains
\begin{equation}
(1 + \tau s^\alpha)\tilde{\sigma}(x,s) = (1 + s^\alpha)\tilde{\varepsilon}(x,s), \quad x \in \mathbb{R}, \quad \Re s > 0.
\end{equation}
The inverse Laplace transform, as $\mathcal{L}^{-1}[(1 + \tau s^\alpha)/(1 + s^\alpha)]$ is a well-defined element in $\mathcal{S}'_+$ [40], gives
\begin{equation}
\sigma = \mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] \ast_t \varepsilon.
\end{equation}
Setting $L_0^\alpha = \mathcal{L}^{-1}[(1 + s^\alpha)/(1 + \tau s^\alpha)] \ast_t$, inserting $\varepsilon$, given by (1.13), into (1.21) and then inserting the obtained $\sigma$ into (1.11), we obtain (1.17). Note that $L_0^\alpha = \mathcal{L}^{-1}[(1 + s^\alpha)/(1 + \tau s^\alpha)] \ast_t$ can be explicitly expressed via the Mittag–Leffler function. Recall that for the Mittag–Leffler function $e_\alpha$, defined by
\begin{equation}
  e_\alpha(t) = E_\alpha \left( -\frac{t^\alpha}{\tau} \right), \quad t > 0, \quad \alpha \in (0,1),
\end{equation}
where $E_\alpha(z) = \sum_{k=0}^{\infty} c^k / \Gamma(\alpha k + 1)$, $z \in \mathbb{C}$, we have that $e_\alpha \in C^\infty((0,\infty)) \cap C([0,\infty))$ and $e'_\alpha \in C^\infty((0,\infty)) \cap L_1 \ast \mathcal{S}'(0,\infty))$, where $e'_\alpha(t) = (d/dt)e_\alpha(t)$, $t > 0$, and
\begin{equation}
  \mathcal{L}[e_\alpha(t)](s) = \frac{s^\alpha - 1}{s^\alpha + 1/\tau}, \quad \Re s > 0;
\end{equation}
see [41]. Therefore,
\begin{equation}
\mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] \ast_t (t) = \mathcal{L}^{-1} \left[ \frac{1 + (1 - \tau)s^\alpha}{\tau(s^\alpha + 1/\tau)} \right] \ast_t (t) = \frac{1}{\tau} \delta(t) + \left( \frac{1}{\tau} - 1 \right) e'_\alpha(t), \quad t > 0,
\end{equation}
and thus we obtain $L_0^\alpha$ as given by (1.18).

For $\alpha = 0$ and $\beta = 1$, i.e. when the Hooke law and the classical strain measure are used, equation (1.17) is the classical wave equation
\begin{equation}
\partial_t^2 u = c^2 \partial_x^2 u, \quad \text{with} \quad c = \sqrt{\frac{2}{1 + \tau}}.
\end{equation}
Therefore, the system of equations (1.11)–(1.13), or equivalently (1.17), generalize the classical wave equation. We collect other special cases of (1.17) in the following remark.

**Remark 1.1.** Generalizations of the classical wave equation, given by the system of equations (1.11)–(1.13), or (1.17), are distinguished and classified according to parameter $\beta$ as follows.

(i) Case $\beta = 0$. We obtain the non-propagating disturbance if $v_0 = 0$. Namely, for $\beta = 0$, we obtain $\varepsilon = 0$, owing to (1.13) and the property of the symmetrized fractional derivative that $\varepsilon_0^\alpha u = 0$ (see appendix A). This and (1.12) imply $\sigma = 0$, so that from (1.11), (1.19) and (1.20) one obtains
\begin{equation}
  u(x,t) = u_0(x) + v_0(x)t, \quad x \in \mathbb{R}, \quad t \geq 0.
\end{equation}
Note, for $v_0 = 0$, we have $u(x,t) = u_0(x), x \in \mathbb{R}, t \geq 0$.

(ii) Case $\beta \in (0,1)$. For $\alpha = 0$, we obtain the space-fractional wave equation
\begin{equation}
\partial_t^2 u(x,t) = c^2 \partial_x^\beta \varepsilon_\alpha^\beta u(x,t) \quad \text{and} \quad c = \sqrt{\frac{2}{1 + \tau}}, \quad x \in \mathbb{R}, \quad t > 0,
\end{equation}
studied in [20]. Case $\alpha \in (0, 1)$, to the best of the authors’ knowledge, has not been studied in the literature, so it is the subject of the analysis presented in this work. For $\alpha = 1$, (1.17) becomes the space-fractional Zener wave equation
\[
\partial_t^2 u(x, t) = L_t^0 \partial_x \mathcal{L}^{\beta} u(x, t), \quad x \in \mathbb{R}, \quad t > 0.
\]

For all $\alpha \in [0, 1]$, when $\beta$ tends to zero, the solution to the system of equations (1.11)–(1.13), (1.19) and (1.20) tends to (1.23); see §3. This suggests that the parameter $\beta$ measures the resistance of the material to the propagation of initial disturbance.

(iii) In the case when $\beta = 1$, $\alpha \in (0, 1)$, equation (1.17) reduces to the time-fractional Zener wave equation
\[
\partial_t^2 u(x, t) = L_t^0 \partial_x^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
studied in [42,43]. For $\alpha = 0$, as already mentioned above, we obtain the classical wave equation and for $\alpha = 1$ the Zener wave equation
\[
\partial_t^2 u(x, t) = L_t^1 \partial_x^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0.
\]

### 2. Cauchy problems (2.1) and (1.17), (1.19)

(a) Framework

The framework for our analysis is the spaces of the distributions: $S'(\mathbb{R})$ (or shortly $S'$) and $K'(\mathbb{R})$ (or $K'$), the duals of the Schwartz space $S(\mathbb{R})$ (or $S$) and of the space $K(\mathbb{R})$ (or $K$); $K$ is the space of smooth functions $\varphi$ with the property that there exists $m \in \mathbb{N}_0$, such that $\sup_{x \in \mathbb{R}, \alpha \leq m} |\varphi^{(\alpha)}(x)| e^{m|x|} < \infty$. The elements of $S'$, respectively of $K'$, are of the form $f = \sum_{\alpha=0}^\infty \Phi_\alpha$, where $\Phi_\alpha$ are continuous functions on $\mathbb{R}$ and $|\Phi_\alpha(t)| \leq C(1 + |t|)^{\beta_0}$, respectively $|\Phi_\alpha'(t)| \leq C e^{\beta_0|t|}$, $\alpha \leq \alpha$, $t \in \mathbb{R}$, for some $C > 0$, $r \in \mathbb{N}_0$ and $k_0 \in \mathbb{N}_0$. The space $S'_+ (K'_+)$ is a subspace of $S' (K')$ consisting of elements supported by $[0, \infty)$. The elements of $S'_+$, respectively of $K'_+$, are of the form $f(t) = (\Phi(t) (1 + |t|)^{\beta_0})$, respectively $f(t) = (\Phi(t) e^{\beta_0 |t|})$, $t \in \mathbb{R}$, where $\Phi$ is a continuous bounded function such that $\Phi(t) = 0$, $t \leq 0$. Note that $S'$ and $S'_+$ are subspaces of $K'$ and $K'_+$, respectively. The elements of $K'_+$ have the Laplace transform, which are analytic functions in the domain $\text{Res} > s_0 > 0$. We also recall that for the Lebesgue spaces of integrable and bounded functions $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, $f * g \in L^\infty(\mathbb{R})$ if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$.

We shall apply the Fourier transform with respect to $x$ and the Laplace transform with respect to $t$. Actually, we shall consider the distributions within the space $S' \otimes K'_+$, which is the subspace of $K'(\mathbb{R}^2)$, consisting of distributions having support in $\mathbb{R} \times [0, \infty)$. For the background of the tensor product, we refer to [44]. We shall obtain the solution $u$ as an element of $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for fixed $\varphi \in K$, i.e. $(u(x, t), \varphi(t)) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and elements of $K'_+$ for fixed $\psi \in K$, i.e. $(u(x, t), \psi(x)) \in K'_+$.

(b) Existence and uniqueness of a generalized solution

We consider the existence and uniqueness of the solution to the Cauchy problem (1.17), (1.19) and (1.20). If $u_0 \in C^1(\mathbb{R})$ and $v_0 \in C(\mathbb{R})$, then the classical solution to the Cauchy problem (1.17), (1.19) and (1.20) is a function $u(x, t)$ of class $C^2$ for $t > 0$, of class $C^1$ for $t \geq 0$, which satisfies equation (1.17) for $t > 0$ and initial conditions (1.19) when $t = 0$, as well as the boundary conditions (1.20). If the function $u_{cl}$ is continued by zero for $t < 0$, then putting
\[
u(x, t) = u_{cl}(x, t) \delta(t), \quad x, t \in \mathbb{R},
\]
we obtain the distributional space–time fractional Zener wave equation
\[
\partial_t^2 u(x, t) = L_t^0 \partial_x \mathcal{L}^{\beta} u(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t), \quad \text{in } K'(\mathbb{R}^2).
\]

We shall refer to solution $u$ of (2.1) as a generalized, or distributional, solution.
Theorem 2.1. Let \( \alpha \in [0, 1), \beta \in [0, 1), \tau \in (0, 1) \) and let \( u_0, v_0 \in L^1(\mathbb{R}) \). Then there exists a unique
generalized solution \( u \in \mathcal{K}'(\mathbb{R}^2) \), supp \( u \subset \mathbb{R} \times [0, \infty) \), to the
distributional space–time fractional Zener wave equation (2.1).

More precisely, \( u \) is of the form

\[
    u(x, t) = \frac{1}{2\pi s} \left( \delta'(t)u_0(x) + \delta(t)v_0(x) \right) *_{x, t} P(x, t), \quad x \in \mathbb{R}, \quad t > 0, 
\]

and

\[
    P(x, t) = I(x, t) - \left( \frac{\partial}{\partial t} J_1(x, t) + \frac{\partial^2}{\partial t^2} J_2(x, t) \right) e^{\omega t}, \quad x \in \mathbb{R}, \quad t > 0, 
\]

where

\[
    J_1 = i\left( J^+_1 - J^-_1 \right), \quad J_2 = J^+_2 + J^-_2. 
\]

Functions \( I, J^+_1, J^-_1, J^+_2 \) and \( J^-_2 \), given by (2.12), (2.15), (2.19), (2.16) and (2.20), respectively, are
bounded and continuous on \( \mathbb{R} \), for fixed \( t \geq 0 \), and continuous, exponentially bounded on \( [0, \infty) \), for fixed \( x \in \mathbb{R} \).

Proof. Formally applying the Laplace transform to (2.1) with respect to \( t \), we obtain

\[
    \partial_x \mathcal{L}_s^\beta u(x, s) - s^2 \frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha} u(x, s) = -\frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha} (su_0(x) + v_0(x)), \quad x \in \mathbb{R}, \quad \text{Re} \, s > 0, \tag{2.3}
\]

for arbitrary, but fixed, \( s_0 > 0 \), where \( \tilde{u} \) is an analytic function with respect to \( s \), \( \text{Re} \, s > s_0 \). Equation
(2.3) is of the type

\[
    \partial_x \mathcal{L}_s^\beta u(x) - \omega u(x) = -\nu u_0(x) - \mu v_0(x), \quad x \in \mathbb{R}, \tag{2.4}
\]

where

\[
    \omega = \omega(s) = s^2 \frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha}, \quad \nu = \nu(s) = s \frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha} \quad \text{and} \quad \mu = \mu(s) = \frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha}, \quad \text{Re} \, s > s_0. \tag{2.5}
\]

We have shown in [42, theorem 4.2] that \( \omega(s) \in \mathbb{C} \setminus (-\infty, 0] \) for \( \text{Re} \, s > 0 \). For fixed \( s \), \( \text{Re} \, s > s_0 \), the
unique solution \( u \in \mathbb{C}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) to (2.4), given by

\[
    u(x) = \frac{1}{\pi} \left( \nu u_0(x) + \nu v_0(x) \right) * \int_0^\infty \frac{1}{\rho^{1+\beta} \sin(\beta \pi / 2)} \cos(\rho \xi) \, d\rho, \quad x \in \mathbb{R}, \tag{2.6}
\]

is obtained as the inverse Fourier transform of

\[
    \hat{u}(\xi) = \frac{\nu \hat{u}_0(\xi) + \nu \hat{v}_0(\xi)}{\xi^{1+\beta} \sin(\beta \pi / 2)} + \omega, \quad \xi \in \mathbb{R}. \tag{2.7}
\]

The previous expression, with \( \omega, \nu \) and \( \mu \) given by (2.5), takes the form

\[
    \hat{u}(\xi, s) = \frac{\hat{s} \hat{u}_0(\xi) + \hat{v}_0(\xi)}{s^2 + ((1 + \mathcal{s}^\alpha)/(1 + \tau \mathcal{s}^\alpha)) \xi^{1+\beta} \sin(\beta \pi / 2)}, \quad \xi \in \mathbb{R}, \quad \text{Re} \, s > s_0. \tag{2.8}
\]

The Fourier and Laplace transform of \( u \) (2.7), with \( u_0 = v_0 = 0 \), shows the uniqueness
of the distributional solution to (2.1). For fixed \( s \), \( \text{Re} \, s > s_0 \), \( \hat{u}(., s) \in L^\infty(\mathbb{R}) \cap \mathbb{C}(\mathbb{R}) \), as \( x \mapsto \int_0^\infty (1/\rho^{1+\beta} \sin(\beta \pi / 2) + \omega) \cos(\rho \xi) \, d\rho \) is a continuous bounded function and \( u_0, v_0 \in L^1(\mathbb{R}) \).

Therefore, by (2.6), we have the solution to (2.3) in the form

\[
    u(x, s) = \frac{1}{\pi} \left( \frac{1 + \tau \mathcal{s}^\alpha}{1 + \mathcal{s}^\alpha} (su_0(x) + v_0(x)) \right) * \int_0^\infty \frac{1}{\rho^{1+\beta} \sin(\beta \pi / 2) + s^2 ((1 + \tau \mathcal{s}^\alpha)/(1 + \mathcal{s}^\alpha))} \cos(\rho \xi) \, d\rho 
\]

\[
    = \frac{1}{\pi} (su_0(x) + v_0(x)) * \int_0^\infty \frac{1}{s^2 + ((1 + \mathcal{s}^\alpha)/(1 + \tau \mathcal{s}^\alpha)) \rho^{1+\beta} \sin(\beta \pi / 2)} \cos(\rho \xi) \, d\rho, \quad x \in \mathbb{R}, \quad \text{Re} \, s > s_0. \tag{2.8}
\]
Thus, the justification of the previously presented procedure is based on the analysis of the inverse Laplace transform. Formally, when applied to (2.8), the inverse Laplace transform gives

\[ u(x, t) = \frac{1}{2\pi i} (\delta(t)u_0(x) + \delta(t)v_0(x)) *_{x,t} P(x, t), \quad x \in \mathbb{R}, \quad t > 0, \tag{2.9} \]

where

\[ P(x, t) = 2\pi L^{-1} \left[ \int_{0}^{\infty} \frac{\cos(\rho x)}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} d\rho \right](x, t) \tag{2.10} \]

is a fundamental solution to (2.1). Consider the divergent integral (2.11). We introduce the parametrization \( s = s_0 + ip, \quad p \in \mathbb{R}, \) in (2.11), so that

\[ P(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{ip t}}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \left|_{s = s_0 + ip} \right| ds \, dp \]

where \( I, I^+ \) and \( I^- \) are given below.

The integral

\[ I(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{ip t}}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \left|_{s = s_0 + ip} \right| ds \, dp, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.12} \]

is absolutely convergent, as

\[ |I(x, t)| \leq e^{s_0 t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\left| \text{Re}(s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)) \right|_{s = s_0 + ip}} \rho \, d\rho \, dp \]

\[ \times \left| \text{d}d\rho \, dp < \infty, \quad x \in \mathbb{R}, \quad t > 0. \tag{2.13} \right. \]

In (2.13), \( p_0 \) is chosen so that

\[ \text{Re} \left( \left[ s^2 + \frac{1 + s^\alpha}{1 + \tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s = s_0 + ip} \right) \]

\[ = r^2 \cos(2\varphi) + \frac{1 + (1 + \tau) s^\alpha \cos(\alpha \varphi) + \tau r^{2\alpha}}{1 + 2\tau r^{\alpha} \cos(\alpha \varphi) + r^{2\alpha}} \rho^{1+\beta} \sin \frac{\beta \pi}{2} > 0, \]

with \( r = \sqrt{s_0^2 + p^2} \) and \( \tan \varphi = p/s_0. \) Note that for \( p = 0 \) and \( \rho = 0 \) the integrand is well defined, due to \( s_0 > 0. \) Thus, the integral \( I \) exists and belongs to \( C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with respect to \( x, \) and it is a continuously exponentially bounded function with respect to \( t \geq 0. \) Next, we consider

\[ I^+(x, t) = e^{s_0 t} \int_{p_0}^{\infty} \int_{0}^{\infty} \frac{\cos(\rho x) e^{ip t}}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \left|_{s = s_0 + ip} \right| ds \, dp \]

\[ = -\left( \frac{\partial}{\partial t} I^+(x, t) + \frac{\partial^2}{\partial t^2} I^+(x, t) \right) e^{s_0 t}, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.14} \]

where

\[ I^+_1(x, t) = \int_{p_0}^{\infty} \int_{0}^{\infty} \frac{\cos(\rho x) e^{ip t}}{p^2 s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \left|_{s = s_0 + ip} \right| ds \, dp, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.15} \]

and

\[ I^+_2(x, t) = \int_{p_0}^{\infty} \int_{0}^{\infty} \frac{\cos(\rho x) e^{ip t}}{p^2 s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \left|_{s = s_0 + ip} \right| ds \, dp, \quad x \in \mathbb{R}, \quad t > 0. \tag{2.16} \]
Function $I_1^+$, given by (2.15), is continuous on $\mathbb{R} \times [0, \infty)$, bounded with respect to $x$ and exponentially bounded with respect to $t$, as

$$
|I_1^+(x, t)| \leq \int_{p_0}^{\infty} \int_0^1 \frac{1}{p \operatorname{Im}((s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2))_{s=s_0+i\rho})} \, d\rho \, dp \\
\leq \frac{1}{2s_0} \int_{p_0}^{\infty} \int_0^1 \frac{1}{p^2} \, d\rho \, dp < \infty, \quad x \in \mathbb{R}, \ t > 0,
$$

where we used

$$
\operatorname{Im}\left(s^2 + \frac{1 + s^\alpha}{1 + \tau s^\alpha} \rho_1 + \beta \sin\left(\frac{\beta \pi}{2}\right)\right)_{s=s_0+i\rho} = r^2 \sin(2\varphi) + \frac{1 - \tau}{1 + \tau} \frac{r^\alpha \sin(\alpha \varphi)}{1 + 2\tau r^\alpha \cos(\alpha \varphi) + \tau^2 r^\alpha} \rho_1 + \beta \sin\left(\frac{\beta \pi}{2}\right) > 0
$$

$$
\sim r^2 \sin(2\varphi) + \frac{1 - \tau}{1 + \tau} \frac{r^\alpha \rho_1 + \beta \sin(\beta \pi/2)}{2}, \quad \text{as} \ r \to \infty,
$$

$$
\sim 2s_0 p + \frac{1 - \tau}{1 + \tau} \rho_1 + \beta \sin\left(\frac{\alpha \pi}{2}\right) \frac{\sin(\beta \pi/2)}{2}, \quad \text{as} \ p \to \infty. \quad (2.17)
$$

In obtaining (2.17), we used: $r^2 \sin(2\varphi) = r^2(2 \tan \varphi/(1 + \tan^2 \varphi)) = 2s_0 p$, as well as $\varphi \sim \pi/2$ and $r \sim p$, as $p \to \infty$. Function $I_2^+$, given by (2.16), is continuous on $\mathbb{R} \times [0, \infty)$, bounded with respect to $x$ and exponentially bounded with respect to $t$, as, by (2.17) and by the Fubini theorem,

$$
|I_2^+(x, t)| \leq \int_{p_0}^{\infty} \int_1^1 \frac{1}{p^2 \operatorname{Im}((s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2))_{s=s_0+i\rho})} \, d\rho \, dp \\
\leq \int_{p_0}^{\infty} \left(\int_1^1 \frac{1}{2s_0 p^3 + ((1 - \tau)/\tau^2) p^{-\alpha} \rho_1 + \beta \sin(\beta \pi/2) \sin(\alpha \pi/2)} \, d\rho \right) \, dp \\
\leq \frac{\tau^2}{1 - \tau} \frac{1}{\sin(\alpha \pi/2) \sin(\beta \pi/2)} \frac{1}{p^{-\alpha}} \left(\int_{p_0}^{\infty} \frac{1}{\rho_1 + \beta \sin(\beta \pi/2)} \, d\rho\right) \, dp < \infty, \quad x \in \mathbb{R}, \ t > 0.
$$

Thus, we have that $I^+$, given by (2.14), belongs to $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with respect to $x$, and it is a derivative of a continuous exponentially bounded function with respect to $t \geq 0$. Similarly as $I^+$, we analyse

$$
I^{-}(x, t) = e^{s_0 t} \int_{-\infty}^{0} \int_0^1 \frac{\cos(\rho x) e^{i\rho t}}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2)}_{s=s_0+i\rho} \, d\rho \, dp \\
e^{s_0 t} \int_{-\infty}^{\infty} \int_0^1 \frac{\cos(\rho x) e^{-i\rho t}}{s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2)}_{s=s_0-i\rho} \, d\rho \, dq \\
= \left(\frac{\partial}{\partial t} I_1^{-}(x, t) - \frac{\partial^2}{\partial t^2} I_2^{-}(x, t)\right) e^{s_0 t}, \quad x \in \mathbb{R}, \ t > 0, \quad (2.18)
$$

where

$$
I_1^{-}(x, t) = \int_{p_0}^{\infty} \int_0^1 \frac{\cos(\rho x) e^{-i\rho t}}{q(s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2))_{s=s_0-i\rho}} \, d\rho \, dq \quad x \in \mathbb{R}, \ t > 0, \quad (2.19)
$$

and

$$
I_2^{-}(x, t) = \int_{p_0}^{\infty} \int_1^1 \frac{\cos(\rho x) e^{-i\rho t}}{q^2(s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho_1 + \beta \sin(\beta \pi/2))_{s=s_0-i\rho}} \, d\rho \, dq, \quad x \in \mathbb{R}, \ t > 0. \quad (2.20)
$$
By (2.19), we have
\[
|J^-_1(x,t)| \leq \int_{p_0}^{\infty} \frac{1}{q} |\text{Im}(s^2 + (1 + s^\alpha)/(1 + s^\beta))\rho^{1+\beta}(s\pi/2)|_{s=s_0-\text{i}q} |\,d\rho\,dq, \\
x \in \mathbb{R}, \ t > 0.
\]
Function \(J^-_1\) is continuous on \(\mathbb{R} \times [0, \infty)\), bounded with respect to \(x\) and exponentially bounded with respect to \(t\), as
\[
|J^-_1(x,t)| \leq \frac{1}{2s_0} \int_{p_0}^{\infty} \frac{1}{q^3} |\text{Im}(s^2 + (1 + s^\alpha)/(1 + s^\beta))\rho^{1+\beta}(s\pi/2)|_{s=s_0-\text{i}q} |\,d\rho\,dq < \infty, \ \ x \in \mathbb{R}, \ t > 0,
\]
where \(r = \sqrt{s_0^2 + q^2}\), \(\tan \varphi = -q/s_0\) and where we used
\[
\text{Im}\left(s^2 + \frac{1 + s^\alpha}{1 + t s^\alpha} \rho^{1+\beta} \sin \left(\frac{s\pi}{2}\right)\right)_{s=s_0-\text{i}q} = \rho^{1+\beta} \cos(\alpha\varphi) + \frac{t^{\alpha}}{\tau^2} \rho^{1+\beta} \sin \left(\frac{s\pi}{2}\right), \ \text{as} \ \varphi \to \infty.
\]
Consider (2.20):
\[
|J^-_2(x,t)| \leq \int_{p_0}^{\infty} \frac{1}{q^2} |\text{Im}(s^2 + (1 + s^\alpha)/(1 + s^\beta))\rho^{1+\beta}(s\pi/2)|_{s=s_0-\text{i}q} |\,d\rho\,dq \\
\leq \int_{p_0}^{\infty} \frac{1}{q^2} |\text{Im}(s^2 + (1 + s^\alpha)/(1 + s^\beta))\rho^{1+\beta}(s\pi/2)|_{s=s_0-\text{i}q} |\,d\rho\,dq \\
\leq \frac{\tau^2}{1 - \tau} |\sin(\alpha\pi/2)\rho^{1+\beta}(s\pi/2)|_{s=s_0-\text{i}q} |\,d\rho\,dq, \ \ x \in \mathbb{R}, \ t > 0.
\]
Function \(J^-_2\) is continuous on \(\mathbb{R} \times [0, \infty)\), bounded with respect to \(x\) and exponentially bounded with respect to \(t\). Thus, we have that \(I^-\), given by (2.18), belongs to \(C(\mathbb{R}) \cap L^\infty(\mathbb{R})\) with respect to \(x\), and it is a derivative of a continuous exponentially bounded function with respect to \(t \geq 0\).

Thus, \(P\) has the form
\[
P(x,t) = I(x,t) - \left(\frac{\partial}{\partial t}(J^+_1(x,t) - J^-_1(x,t)) + \frac{\partial^2}{\partial t^2}(J^+_2(x,t) + J^-_2(x,t))\right) e^{\text{o} t}, \ \ x \in \mathbb{R}, \ t > 0,
\]
where \(I, J^+_1, J^-_1, J^+_2, J^-_2\) are bounded and continuous functions with respect to \(x\) and continuous exponentially bounded functions with respect to \(t\).

(c) Existence and uniqueness of a classical solution

If the initial condition \(u_0\) is assumed to be zero, one may also obtain the existence and uniqueness result for the classical solution.

**Theorem 2.2.** Assume \(u_0 = 0\) and \(v_0 \in C^5(\mathbb{R})\) is compactly supported. Then the space–time fractional Zener wave equation (1.17), subject to initial conditions (1.19)\(_{1,2}\), admits the classical solution of the \(C^1\) class with respect to \(x \in \mathbb{R}\) and of the \(C^2\) class with respect to \(t \in [0, \infty)\), given in the form
\[
u_\text{cl}(x,t) = -\frac{1}{2\pi^2} (\text{H}(t)) \ast \left(\left(1 - \frac{\partial^2}{\partial x^2}\right)^4 v_0(x)\right) \ast_x I(x,t) + v_0(x)t, \ \ x \in \mathbb{R}, \ t > 0. \ (2.21)
\]
Note that the assumptions on \( v_0 \) imply that \( ((1 - \partial^2/\partial x^2)^4 v_0(x)) \ast_x I(x,t) \) is a \( C^1 \) function with respect to \( x \). Moreover, we can assume less regular conditions on \( v_0 \), which would lead to the use of the pseudodifferential operators.

**Proof.** Assume that \( u_0 = 0 \) in (2.2), so that

\[
 u(x,t) = \frac{1}{2\pi^2} (\delta(t)v_0(x)) \ast_{x,t} P(x,t), \quad x \in \mathbb{R}, \ t > 0. \tag{2.22}
\]

Further assume that \( P \), given by (2.10), or alternatively by (2.11), with \( P(x,t) = 0, x \in \mathbb{R}, t > 0 \), is considered in the sense of distributions. Let

\[
 Q(x,t) = -i \int_{s_0-i\infty}^{s_0+i\infty} \int_0^\infty \frac{\cos(\rho x) e^{st}}{(1 + \rho^2)^4 (s^2 + ((1 + s^\alpha)/(1 + \tau s^\alpha)) \rho^{1+\beta} \sin(\beta \pi/2))} \, d\rho \, ds,
\]

so that

\[
 P(x,t) = \left( 1 - \frac{\partial^2}{\partial x^2} \right)^4 Q(x,t). \tag{2.23}
\]

In the sense of distributions, we have

\[
 \frac{\partial^2}{\partial t^2} Q(x,t) = -i \int_{s_0-i\infty}^{s_0+i\infty} \int_0^\infty \left( 1 - \frac{s^2}{1 + s^\alpha/(1 + \tau s^\alpha)} \rho^{1+\beta} \sin(\beta \pi/2) \right) \cos(\rho x) e^{st} \frac{d\rho}{(1 + \rho^2)^4} \, ds \tag{2.24}
\]

where

\[
 I(x,t) = -i \int_{s_0-i\infty}^{s_0+i\infty} \int_0^\infty \frac{\rho^{1+\beta} \sin(\beta \pi/2) \cos(\rho x) e^{st}}{s^2((1 + s^\alpha)/(1 + s^\alpha)) + \rho^{1+\beta} \sin(\beta \pi/2)} \frac{d\rho}{(1 + \rho^2)^4} \, ds, \quad x \in \mathbb{R}, \ t > 0, \tag{2.25}
\]

and \( g(x) = \int_0^\infty (\cos(\rho x)/(1 + \rho^2)^4) \, d\rho, \ x \in \mathbb{R}, \) is a continuous function. We used that \( 2\pi \delta(t) = -i \int_{s_0-i\infty}^{s_0+i\infty} e^{st} \, ds, t > 0, \) in the sense of distributions.

Our formal calculations will be justified by showing that the double integral (2.25) exists as the Lebesgue integral and that it is a continuous function for \( x \in \mathbb{R}, t > 0 \). We introduce the parametrization \( s = s_0 + ip, p \in \mathbb{R}, \) in (2.25), so that

\[
 I(x,t) = \int_{-\infty}^\infty \int_0^\infty \frac{\rho^{1+\beta} \sin(\beta \pi/2) \cos(\rho x) e^{st}}{s^2((1 + s^\alpha)/(1 + s^\alpha)) + \rho^{1+\beta} \sin(\beta \pi/2)} \frac{d\rho}{(1 + \rho^2)^4} \, dp \, ds
 = I_1(x,t) + I_2(x,t) + I_3(x,t), \quad x \in \mathbb{R}, \ t > 0,
\]

where \( I_1, I_2 \) and \( I_3 \) will be defined below.

We put

\[
 I_1(x,t) = \int_{s_0}^\infty \int_0^\infty \frac{\rho^{1+\beta} \sin(\beta \pi/2)}{s^2((1 + s^\alpha)/(1 + s^\alpha)) + \rho^{1+\beta} \sin(\beta \pi/2)} \frac{d\rho}{(1 + \rho^2)^4} \, dp \, ds, \quad x \in \mathbb{R}, \ t > 0.
\]
We put $s = r e^{i \psi}$, where $r = \sqrt{s_0^2 + p^2}$, tan $\varphi = p/s_0$ and also put $p_0 = s_0$, so that $\varphi \in (\pi/4, \pi/2)$. This implies

$$\text{Im} \left( s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha} + \rho^{1 + \beta} \sin \frac{\beta \pi}{2} \right)$$

$$= \text{Im} \left( (r^2 \cos(2\psi) + i r^2 \sin(2\psi)) \left( \frac{1 + (1 + \tau) s^\alpha \cos(\alpha \varphi) + r \tau^2}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2} + i \frac{(\tau - 1) s^\alpha \sin(\alpha \varphi) + r \tau^2}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2} \right) \right)$$

$$= r^2 \frac{(1 + (1 + \tau) s^\alpha \cos(\alpha \varphi) + r \tau^2) \sin(2\psi) + (\tau - 1) s^\alpha \sin(\alpha \varphi) \cos(2\varphi)}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2}.$$

Since $\varphi \in (\pi/4, \pi/2)$, we have $\cos(2\psi) < 0$. Moreover, by assumption, $\tau < 1$. Thus, we have

$$\left| \text{Im} \left( s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha} + \rho^{1 + \beta} \sin \frac{\beta \pi}{2} \right) \right| \geq Cr^{2-\alpha}, \quad r \geq s_0 \sqrt{2}.$$

(2.26)

Therefore,

$$|I_1(x,t)| \leq C e^{\alpha t} \int_{s_0}^\infty \int_0^\infty \frac{1}{\sqrt{s_0^2 + p^2}} \rho^{1 + \beta} \frac{\rho^{2 - \alpha} d\rho dp}{(1 + \rho^2)^4}, \quad x \in \mathbb{R}, \quad t > 0,$$

so that $I_1$ is a continuous, bounded function with respect to $x$ and a continuous, exponentially bounded function with respect to $t$.

We put

$$I_2(x,t) = \int_{s_0}^\infty \int_{-\infty}^\infty \frac{\rho^{1 + \beta} \sin(\beta \pi/2)}{s^2(1 + \tau s^\alpha)/(1 + s^\alpha) + \rho^{1 + \beta} \sin(\beta \pi/2)} \cos(\rho x) e^{\rho t} e^{ipt} \frac{d\rho dp}{(1 + \rho^2)^4}.$$

$$= \int_{s_0}^\infty \int_{-\infty}^\infty \frac{\rho^{1 + \beta} \sin(\beta \pi/2)}{s^2(1 + \tau s^\alpha)/(1 + s^\alpha) + \rho^{1 + \beta} \sin(\beta \pi/2)} \cos(\rho x) e^{\rho t} e^{-ipt} \frac{d\rho dp}{(1 + \rho^2)^4}, \quad x \in \mathbb{R}, \quad t > 0.$$

A similar estimate holds for (2.26). Thus,

$$|I_2(x,t)| \leq C e^{\alpha t} \int_{s_0}^\infty \int_0^\infty \frac{1}{\sqrt{s_0^2 + p^2}} \rho^{1 + \beta} \frac{d\rho dp}{(1 + \rho^2)^4}, \quad x \in \mathbb{R}, \quad t > 0,$$

so that $I_2$ is a continuous, bounded function with respect to $x$ and a continuous, exponentially bounded function with respect to $t$.

We put

$$I_3(x,t) = \int_{s_0}^\infty \int_{-\infty}^\infty \frac{\rho^{1 + \beta} \sin(\beta \pi/2)}{s^2(1 + \tau s^\alpha)/(1 + s^\alpha) + \rho^{1 + \beta} \sin(\beta \pi/2)} \cos(\rho x) e^{\rho t} e^{ipt} \frac{d\rho dp}{(1 + \rho^2)^4}, \quad x \in \mathbb{R}, \quad t > 0,$$

and write

$$\text{Re} \left( s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha} + \rho^{1 + \beta} \sin \frac{\beta \pi}{2} \right)$$

$$= \text{Re} \left( (r^2 \cos(2\psi) + i r^2 \sin(2\psi)) \left( \frac{1 + (1 + \tau) s^\alpha \cos(\alpha \varphi) + r \tau^2}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2} + i \frac{(\tau - 1) s^\alpha \sin(\alpha \varphi) + r \tau^2}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2} \right) \right)$$

$$= r^2 \frac{(1 + (1 + \tau) s^\alpha \cos(\alpha \varphi) + r \tau^2) \cos(2\psi) + (\tau - 1) s^\alpha \sin(\alpha \varphi) \cos(2\varphi)}{1 + 2 r^\alpha \cos(\alpha \varphi) + r^2}.$$
so that
\[ |I_3(x,t)| \leq Ce^{\gamma t} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\rho^{1+\beta} \sin(\beta \pi/2)}{(1+\rho^2)^4} \, d\rho \, dp, \quad x \in \mathbb{R}, \ t > 0. \]

The integral \( I_3 \) is a continuous, bounded function with respect to \( x \) and a continuous, exponentially bounded function with respect to \( t \).

By (2.24), in the sense of distributions, we have
\[
\frac{\partial^2}{\partial t^2} Q(x,t) = 2\pi g(x)\delta(t) - I(x,t), \quad x \in \mathbb{R}, \ t > 0, \tag{2.27}
\]
where \( g \) is a continuous function and \( I \) is a continuous, bounded function with respect to \( x \) and a continuous, exponentially bounded function with respect to \( t \). By (2.22) and (2.23), we have
\[
u(x,t) = \frac{1}{2\pi^2} (\delta(t)v_0(x)) *_{x,t} \left( 1 - \frac{\partial^2}{\partial x^2} \right)^4 Q(x,t)
\]
\[
= \frac{1}{2\pi^2} \delta(t) \left( 1 - \frac{\partial^2}{\partial x^2} \right)^4 v_0(x) *_{x,t} \frac{\partial^2}{\partial t^2} Q(x,t)
\]
\[
= \delta(t)v_0(x) - \frac{1}{2\pi^2} \left( 1 - \frac{\partial^2}{\partial x^2} \right)^4 v_0(x) *_{x,t} I(x,t), \quad x \in \mathbb{R}, \ t > 0,
\]
as \( 1 - \partial^2/\partial x^2 \right)^4 g(x) = \pi \delta(x) \). Using
\[ u(x,t) = u_{cl}(x,t)H(t), \quad \text{i.e.} \quad \frac{\partial^2}{\partial t^2} u(x,t) = \left( \frac{\partial^2}{\partial t^2} u_{cl}(x,t) \right) H(t) + \delta(t)v_0(x)
\]
and the previous equation, we have
\[
\frac{\partial^2}{\partial t^2} u_{cl}(x,t) = -\frac{1}{2\pi^2} \left( 1 - \frac{\partial^2}{\partial x^2} \right)^4 v_0(x) *_{x,t} I(x,t), \quad x \in \mathbb{R}, \ t > 0,
\]
yielding (2.21).

(d) Regularization of a generalized solution

We give a regularization of the generalized solution \( u \) to the space–time fractional Zener wave equation (2.1), which is of particular importance for the numerical analysis of the problem. We start from the Fourier and Laplace transform of the solution given by (2.7) and write it as
\[
\hat{u} (\xi,s) = \left( \hat{v}_0(\xi) + \frac{1}{s} \hat{v}_0(\xi) \right) \hat{K}(\xi,s), \quad \xi \in \mathbb{R}, \ \text{Re} \ s > s_0, \tag{2.28}
\]
where
\[
\hat{K}(\xi,s) = \frac{s}{s^2 + (1 + s^\alpha/(1 + s^\alpha))(\xi)^{1+\beta} \sin(\beta \pi/2)} = \frac{1}{s} - \hat{Q}(\xi,s), \quad \text{with} \tag{2.29}
\]
\[
\hat{Q}(\xi,s) = \frac{((1 + s^\alpha)/(1 + s^\alpha))(\xi)^{1+\beta} \sin(\beta \pi/2)}{s^3 + s((1 + s^\alpha)/(1 + s^\alpha))(\xi)^{1+\beta} \sin(\beta \pi/2)} , \quad \xi \in \mathbb{R}, \ \text{Re} \ s > s_0. \tag{2.30}
\]
Note that
\[
\hat{\dot{K}}(\xi,s) = \frac{1}{2\pi} \hat{\dot{P}}(\xi,s), \quad \text{i.e.} \quad K(x,t) = \frac{1}{2\pi} \frac{\partial}{\partial t} P(x,t), \quad x, \xi \in \mathbb{R}, \ t > 0,
\]
where \( P \) is given by (2.10). We already know from theorem 2.1 that \( P \) (and also therefore \( K \)) is a distribution. We regularize \( \hat{K} \) by multiplying it by the Fourier transform of the Gaussian

\[
\delta_\epsilon(x) = \frac{1}{\epsilon \sqrt{\pi}} e^{-x^2/\epsilon^2}, \quad x \in \mathbb{R}, \; \epsilon \in (0, 1],
\]

which is a \( \delta \)-net, i.e. the Gaussian in a limiting process \( \epsilon \to 0 \) represents the Dirac delta distribution. Thus, we have that

\[
\hat{K}_\epsilon(\xi, s) = \hat{K}(\xi, s) e^{-((\xi)^2+4s^2)/\epsilon^2}, \quad \text{where } \mathcal{F}[\delta_\epsilon(x)](\xi) = e^{-((\xi)^2+4s^2)/\epsilon^2}, \; \xi \in \mathbb{R}, \; \text{Re } \xi > \delta_0, \; \epsilon \in (0, 1], \quad (2.31)
\]

has the inverse Laplace and Fourier transforms which is a function and, in a distributional limit, gives the solution kernel \( K \) as a distribution.

We summarize these observations in the following theorem, which is given after we state the lemma.

**Lemma 2.3.** Let \( \alpha \in [0, 1) \), \( \tau \in (0, 1) \) and \( \theta > 0 \). Then

\[
\Psi_\alpha(s) = s^2 + \theta \frac{1 + s^\alpha}{1 + \tau s^\alpha}, \quad s \in \mathbb{C},
\]

admits exactly two zeros. They are complex-conjugate, located in the left complex half-plane and each of them is of multiplicity one.

**Theorem 2.4.** Let all conditions of theorem 2.1 be satisfied. Let \( u \in \mathcal{K}'(\mathbb{R}^2) \), with support in \( \mathbb{R} \times [0, \infty) \), be the distributional solution to the distributional space–time fractional Zener wave equation (2.1). Then \( u \) is of the form

\[
u(x, t) = (u_0(x) \delta(t) + v_0(x) H(t)) *_{x, t} K(x, t), \quad (2.32)
\]

where \( K \) is a distributional limit in \( \mathcal{K}'(\mathbb{R}^2) \),

\[
K(x, t) = \lim_{\epsilon \to 0} K_\epsilon(x, t) \quad \text{and} \quad K_\epsilon(x, t) = \frac{1}{\pi} \int_0^\infty S(\rho, t) \cos(\rho x) e^{-((\rho)^2+4s^2)/\epsilon^2} \, d\rho, \quad x \in \mathbb{R}, \; t > 0, \quad (2.33)
\]

with

\[
S(\rho, t) = \frac{1}{2\pi} \int_0^\infty \left( \frac{1}{\rho^2 + ((1 + \tau e^{i\alpha})/(1 + \tau e^{i\alpha})) \rho^2 \sin(\beta \pi/2)} \right. \left. - \frac{1}{\rho^2 + ((1 + \tau e^{i\alpha})/(1 + \tau e^{i\alpha})) \rho^2 \sin(\beta \pi/2)} \right) qe^{-q^2 \rho^2} dq + \frac{2s(\alpha - 1) e^{s} / (1 + s e^{s})^2 \rho \sin(\beta \pi/2)}{2s(\alpha - 1) e^{s} / (1 + s e^{s})^2 \rho \sin(\beta \pi/2)} \bigg|_{s = \rho_0(\rho)} \bigg|_{s = \rho_\epsilon(\rho)}, \quad (2.34)
\]

and \( \rho_\epsilon \) are zeros of \( \Psi_\alpha \) from lemma 2.3.

In particular, for suitable \( \delta_0 > 0 \), \( K_\epsilon(x, t) e^{-\rho_\epsilon t} \) is bounded and continuous with respect to \( x \in \mathbb{R}, \; t > 0 \), for every \( \epsilon \in (0, 1] \).

**Proof of lemma 2.3.** Let \( s = re^{i\varphi}, \; r > 0, \; \varphi \in (-\pi, \pi) \). We have

\[
\text{Re } \Psi_\alpha(s) = r^2 \cos(2\varphi) + \theta \frac{1 + (1 + \tau) r^\alpha \cos(\alpha \varphi) + \tau r^{2\alpha}}{1 + 2 \tau r^\alpha \cos(\alpha \varphi) + \tau^2 r^{2\alpha}}, \quad (2.35)
\]

and

\[
\text{Im } \Psi_\alpha(s) = r^2 \sin(2\varphi) + \theta (1 - \tau) \frac{r^{\alpha} \sin(\alpha \varphi)}{1 + 2 \tau r^\alpha \cos(\alpha \varphi) + \tau^2 r^{2\alpha}}, \quad (2.36)
\]

From (2.35) and (2.36), one can easily see that \( \Psi_\alpha(s_\epsilon) = 0 \) implies \( \Psi_\alpha(\bar{s}_\epsilon) = 0 \).
Next we show that if $\text{Re} s_2 > 0$, then such $s_2$ cannot be a zero of $\Psi_\alpha$ and therefore zeros must lie in the left complex half-plane. Suppose $\text{Re} s > 0$, i.e. $\varphi \in (-\pi/2, \pi/2)$. Since zeros appear in complex-conjugate pairs, we can suppose that $\varphi \in [0, \pi/2]$. For $\varphi = 0$, we have $\Psi_\alpha(s) > 0$. Since $\alpha \in [0, 1)$, we have that $\varphi \varphi, 2\varphi \in (0, \pi)$ and therefore $\sin(2\varphi) > 0$ and $\sin(\alpha \varphi) > 0$, which together with $\theta > 0$ and $\tau \in (0, 1)$ implies $\text{Im} \Psi_\alpha(s) > 0$, so such an $s$ cannot be a zero.

It is left to show that there is only one pair of zeros of $\Psi_\alpha$. We use the argument principle. Recall that if $f$ is an analytic function inside and on a regular closed curve $C$, and non-zero on $C$, then the number of zeros of $f$ (counted as many times as its multiplicity) inside the contour $C$ is equal to the total change in the argument of $f(s)$ as $s$ travels around $C$. For our purpose, we choose contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$, parametrized as

$$C_1: s = x e^{i\pi/2}; \ x \in [r, R], \quad C_2: s = Re^{i\varphi}; \ \varphi \in \left[\frac{\pi}{2}, \pi\right],$$

$$C_3: s = x e^{i\tau}; \ x \in [r, R], \quad C_4: s = r e^{i\varphi}; \ \varphi \in \left[\frac{\pi}{2}, \pi\right],$$

where $r < r_0$, $R > R_0$ and $r_0$, $R_0$ are chosen as follows: $r_0$ is small enough such that for all $r < r_0$ it holds that $\text{Re} \Psi(s) \sim \theta$ and $\text{Im} \Psi(s) \sim \theta/(1 - \tau)\mu \sin(\alpha \varphi)$ and, therefore, there are no zeros for $r < r_0$, and $R$ is large enough such that for all $R > R_0$ it holds that $\text{Re} \Psi(s) \sim R^2 \cos(2\varphi)$ and $\text{Im} \Psi(s) \sim R^2 \sin(2\varphi)$.

On the contour $C_1$, we have that $\text{Im} \Psi_\alpha(s) \geq 0$ (as $\tau \in (0, 1)$ and $\alpha \in [0, 1)$ implies $\sin(\alpha \pi/2) \geq 0$), and $\text{Im} \Psi_\alpha(s) \to 0$ for $r, x \to 0$, as well as for $R, x \to \infty$. The real part of $\Psi_\alpha$ varies from $\theta$ (for $r, x \to 0$) to $-\infty$ (for $R, x \to \infty$). Therefore, on $C_1$ we have $\Delta \Psi_\alpha(s) = -\pi$.

On the contour $C_2$, for $R > R_0$, $R_0$ large enough, we have

$$\text{Im} \Psi \sim R^2 \sin(2\varphi) + \frac{\theta(1 - \tau)\sin(\alpha \varphi)}{\tau^2} \frac{1}{R^\alpha} \sim R^2 \sin(2\varphi) \leq 0,$$

and we have $\text{Im} \Psi_\alpha(s) \to 0$ for both $\varphi = \pi/2$ and $\varphi = \pi$. The real part of $\Psi_\alpha$ changes from $-\infty$ (for $\varphi = \pi/2$) to $\infty$ (for $\varphi = \pi$) because

$$\text{Re} \Psi(s) \sim R^2 \cos(2\varphi) + \frac{\theta}{\tau} \sim R^2 \cos(2\varphi).$$

So, on $C_2$ the change of the argument is $\Delta \Psi_\alpha(s) = -\pi$.

On the contours $C_3$ and $C_4$ the argument does not change. On $C_3$ the imaginary part of $\Psi_\alpha$ is always positive and it tends to zero for both $R \to \infty$ and $r \to 0$, while the real part changes from $\infty$ (for $R \to \infty$) to $\theta$ (for $r \to 0$), and even if it changes the sign it does not change the argument of $\Psi_\alpha$. On $C_4$ it holds that $\Psi_\alpha(s) \sim \theta$ and so there is no change of argument.

Taking everything together we have $\Delta \Psi_\alpha(s) = -2\pi$ as $s \in C$, and by the argument principle there is one zero inside of the contour $C$. Therefore, there is a unique pair of complex-conjugate numbers in the left complex plane which are zeros of $\Psi_\alpha$.

**Proof of theorem 2.4.** Let

$$\hat{\tilde{K}}_\alpha(\xi, s) = \frac{s}{s^2 + ((1 + s^\alpha)/(1 + s^\alpha))|\xi|^{1+\beta} \sin(\beta \pi/2)} e^{-\langle \xi\rangle^2/4} = \frac{1}{s} e^{-\langle \xi\rangle^2/4} - \hat{Q}_\alpha(\xi, s), \quad \xi \in \mathbb{R}, \ \text{Re} s > s_0,$$

(2.37)

by (2.29) and (2.31). We shall prove that

$$\hat{\tilde{Q}}_\alpha(\xi, s) = \hat{Q}(\xi, s) e^{-\langle \xi\rangle^2/4} = \frac{((1 + s^\alpha)/(1 + s^\alpha))|\xi|^{1+\beta} \sin(\beta \pi/2)}{s^3 + s((1 + s^\alpha)/(1 + s^\alpha))|\xi|^{1+\beta} \sin(\beta \pi/2)} e^{-\langle \xi\rangle^2/4},$$

(2.37)
see (2.30), has the inverse Laplace and Fourier transforms by examining the convergence of the double integral \((x \in \mathbb{R}, t > 0, \varepsilon \in (0, 1])\)

\[
Q_x(x, t) = \left. \frac{1}{(2\pi)^2 i} \int_{\mathbb{R}^2} e^{st} e^{i\xi x} d\xi \right|_{\varepsilon=0+i\rho} e^{\xi t} ds
\]

\[
= \frac{1}{2\pi^2} (J_x(x, t) + J_x^+(x, t) + J_x^-(x, t)) e^{\xi t}, \tag{2.38}
\]

with \((x \in \mathbb{R}, t > 0, \varepsilon \in (0, 1])\)

\[
J_x(x, t) = \int_{-p_0}^{p_0} \int_{0}^{\infty} \left. \frac{((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)}{s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \right|_{s=s_0+i\rho} e^{-(\rho^2/4\pi) + ip^t} d\rho dp,
\]

\[
J_x^+(x, t) = \int_{p_0}^{\infty} \int_{0}^{\infty} \left. \frac{((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)}{s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \right|_{s=s_0+i\rho} e^{-(\rho^2/4\pi) + ip^t} d\rho dp,
\]

\[
J_x^-(x, t) = \int_{-\infty}^{0} \int_{0}^{\infty} \left. \frac{((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)}{s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)} \right|_{s=s_0+i\rho} e^{-(\rho^2/4\pi) + ip^t} d\rho dp,
\]

where we introduced the parametrization \(s = s_0 + ip, p \in (-\infty, \infty)\) in (2.38) and used the fact that \(\hat{Q}_x\) is an even function in \(\xi\). From (2.39), we have \((x \in \mathbb{R}, t > 0, \varepsilon \in (0, 1])\)

\[
|J_x(x, t)| \leq \int_{-p_0}^{p_0} \int_{0}^{\infty} \left. \frac{|(1 + s^\alpha)/(1 + ts^\alpha)|_{s=s_0+i\rho} \rho^{1+\beta}}{|s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)|_{s=s_0+i\rho}} e^{-(\rho^2/4\pi) + ip^t} d\rho dp < \infty.
\]

Let us estimate the integral given by (2.40) as \((x \in \mathbb{R}, t > 0, \varepsilon \in (0, 1])\)

\[
|J_x^+(x, t)| \leq \int_{p_0}^{\infty} \int_{0}^{\infty} \left. \frac{|(1 + s^\alpha)/(1 + ts^\alpha)|_{s=s_0+i\rho} \rho^{1+\beta}}{|s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)|_{s=s_0+i\rho}} e^{-(\rho^2/4\pi) + ip^t} d\rho dp
\]

\[
\leq \int_{p_0}^{\infty} \int_{0}^{\infty} \left. \frac{|(1 + s^\alpha)/(1 + ts^\alpha)|_{s=s_0+i\rho} \rho^{1+\beta}}{|\text{Im}(s^3 + s((1 + s^\alpha)/(1 + ts^\alpha))\rho^{1+\beta} \sin(\beta \pi/2)|_{s=s_0+i\rho}} e^{-(\rho^2/4\pi) + ip^t} d\rho dp.
\]

We have \(\text{Im}(s_0 + ip)^3 = -p^3 + 3s_0^2p, |(1 + s^\alpha)/(1 + ts^\alpha)|_{s=s_0+i\rho} \sim 1/\tau\) and

\[
\text{Im} \left( \left. \frac{1 + s^\alpha}{1 + ts^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right|_{s=s_0+i\rho} \right) \sim \frac{1}{\tau} p + \frac{1 - \tau}{\tau^2} \frac{1}{p^2} \sin \frac{\alpha \pi}{2}, \quad \text{as } p \to \infty,
\]

as

\[
\text{Re} \left( \frac{1 + s^\alpha}{1 + ts^\alpha} \right) = \frac{1 + (1 + \tau)\rho^\alpha \cos(\alpha \varphi) + \tau \rho^{2\alpha}}{1 + 2\tau \rho^\alpha \cos(\alpha \varphi) + \tau^2 \rho^{2\alpha}} \sim \frac{1}{\tau}, \quad \text{as } r \to \infty,
\]

\[
\text{Im} \left( \frac{1 + s^\alpha}{1 + ts^\alpha} \right) = (1 - \tau) \frac{\rho^\alpha \sin(\alpha \varphi)}{1 + 2\tau \rho^\alpha \cos(\alpha \varphi) + \tau^2 \rho^{2\alpha}} \sim \frac{1 - \tau}{\tau^2} \frac{1}{\rho^2} \sin(\alpha \varphi), \quad \text{as } r \to \infty,
\]

and thus for \(r = \sqrt{s_0^2 + p^2}, \tan \varphi = s_0/p,\)

\[
\text{Im} \left( \left. \frac{s^3 + s_0^2 + s^\alpha}{1 + ts^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right|_{s=s_0+i\rho} \right) \sim -p^3 + 3s_0^2p + \frac{1}{\tau} p + s_0 \frac{1 - \tau}{\tau^2} \frac{1}{p^2} \rho^{1+\beta} \sin \frac{\alpha \pi}{2} \sin \frac{\beta \pi}{2}, \quad \text{as } p \to \infty. \tag{2.43}
\]
We choose $p_0$ so that (2.43) becomes

$$\text{Im} \left( \int_{s=\sigma_0+ip} \left[ s^3 + s \frac{1 + s^2}{1 + rs^2} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right] \right) \sim -p^3, \quad \text{as } p \to \infty.$$ 

Thus, for (2.42) we have

$$|f_r^t(x, t)| \leq \int_{p_0}^{\infty} \int_0^\infty \frac{e^{-\rho^2 t}}{p^3} e^{-\rho} d\rho \, dp < \infty.$$ 

Using the same arguments as for (2.40), we can prove that $f_r^t$, given by (2.41), is also absolutely integrable.

We proved that $Q_{r,\varepsilon}$, given by (2.38), has the inverse Laplace and Fourier transforms and therefore, by (2.37), we have

$$\hat{K}_\varepsilon(\xi, s) = \frac{1}{s} e^{-\rho_\varepsilon s^2/4} - \hat{Q}_\varepsilon(\xi, s), \quad \xi \in \mathbb{R}, \quad \text{Re } s > s_0, \quad \varepsilon \in (0, 1], \quad \text{i.e.}$$

$$K_\varepsilon(x, t) = H(t)\delta_\varepsilon(x) - Q_\varepsilon(x, t), \quad x \in \mathbb{R}, \quad t > 0.$$ 

Thus, in (2.37) we can first invert the Laplace transform and subsequently the Fourier transform. The Fourier transform of the solution kernel $\hat{K}_\varepsilon$ is obtained by the use of the inversion formula of the Laplace transform

$$\hat{K}_\varepsilon(\rho, t) = \frac{1}{2\pi i} \int_{\rho_0-i\infty}^{\rho_0+i\infty} \hat{K}_\varepsilon(\rho, s) e^{\sigma t} \, ds, \quad \sigma \geq 0,$$  

(2.44)

where $\hat{K}_\varepsilon$ is given by (2.37) and the complex integration along the contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\gamma \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$. The contour $\Gamma$ is parametrized by

$$\Gamma_1: s = R e^{i\varphi}, \quad \varphi_0 < \varphi < \pi; \quad \Gamma_2: s = q e^{i\varphi}, \quad -R < -q < -r;$$

$$\Gamma_\gamma: r e^{i\varphi}, \quad -\pi < \varphi < -\pi; \quad \Gamma_3: s = q e^{-i\varphi}, \quad r < q < R;$$

$$\Gamma_4: s = R e^{i\varphi}, \quad -\pi < \varphi < \varphi_0; \quad \gamma_0: s = s_0(1 + i \tan \varphi), \quad -\varphi_0 < \varphi < \varphi_0,$$

for arbitrary chosen $R > 0$ and $0 < r < R$, and $\varphi_0 = \arccos(s_0/R)$. By the Cauchy residues theorem and the results of lemma 2.3 we obtain:

$$\frac{1}{2\pi i} \int_{\Gamma} \hat{K}_\varepsilon(\rho, s) e^{\sigma t} \, ds = \text{Res}(\hat{K}_\varepsilon(\rho, s) e^{\sigma t}, \bar{s}_\varepsilon(\rho)) + \text{Res}(\hat{K}_\varepsilon(\rho, s) e^{\sigma t}, \bar{s}_\varepsilon(\rho)).$$  

(2.45)

Now, one shows (see, for example, [42] for similar calculations) that in (2.45), when $R$ tends to infinity and $r$ tends to zero, integrals along contours $\Gamma_1, \Gamma_3$ and $\Gamma_\gamma$ tend to zero. The integrals along contours $\Gamma_2$ and $\Gamma_3$ in the limiting process (when $R$ tends to infinity and $r$ tends to zero) read ($\rho \geq 0, t > 0$)

$$\lim_{R \to \infty, r \to 0} \int_{\Gamma_2} \hat{K}_\varepsilon(\rho, s) e^{\sigma t} \, ds$$

$$= -e^{-(\rho s)^2/4} \int_0^\infty \frac{q}{\rho^2 + ((1 + q^2 e^{i\alpha \pi})/(1 + \tau q^2 e^{i\alpha \pi}))\rho^{1+\beta} \sin(\beta \pi/2)} \, dq,$$

$$\lim_{R \to \infty, r \to 0} \int_{\Gamma_3} \hat{K}_\varepsilon(\rho, s) e^{\sigma t} \, ds$$

$$= e^{-(\rho s)^2/4} \int_0^\infty \frac{q}{\rho^2 + ((1 + q^2 e^{-i\alpha \pi})/(1 + \tau q^2 e^{-i\alpha \pi}))\rho^{1+\beta} \sin(\beta \pi/2)} \, dq.$$
By lemma 2.3, we have that the residues in (2.45) read \((\rho \geq 0, t > 0)\)
\[
\text{Res}(\hat{K}_c(\rho, s) e^{st}, s_\rho(\rho)) = \frac{s e^{st}}{2s + (\alpha(1 - \tau) s^{\epsilon-1} / (1 + \tau s^{\epsilon})^2)\rho^{1+\beta} \sin(\beta \pi / 2)} e^{-s^2/4},
\]
\[
\text{Res}(\hat{K}_c(\rho, s) e^{st}, s_\rho(\rho)) = \frac{s e^{st}}{2s + (\alpha(1 - \tau) s^{\epsilon-1} / (1 + \tau s^{\epsilon})^2)\rho^{1+\beta} \sin(\beta \pi / 2)} e^{-s^2/4}.
\]

The integral along the contour \(\gamma_0\) in the limiting process tends to the integral on the right-hand side of (2.44) and therefore, putting all together in (2.45), we obtain
\[
\hat{K}_c(\rho, t) = S(\rho, t) e^{-s^2/4}, \quad x \in \mathbb{R}, \quad t > 0,
\]
with \(S\) given by (2.34). The inverse Fourier transform of such obtained \(\hat{K}_c\) reads
\[
K_c(x, t) = \frac{1}{\pi} \int_0^\infty S(\rho, t) \cos(\rho x) e^{-s^2/4} \mathrm{d}\rho, \quad x \in \mathbb{R}, \quad t > 0,
\]
which in the distributional limit when \(\epsilon \to 0\) gives the solution kernel \(K\) in the form (2.33).

3. Dependence of a solution on parameters \(\alpha\) and \(\beta\)

We examine the solutions in the limiting cases of the system of equations (1.11)–(1.13), or equivalently (1.17), subject to (1.19), (1.20) in view of remark 1.1. In all cases, we write the solution \(u\) of theorem 2.4 as
\[
u(x, t) = (\tilde{u}_0(x) \delta(t) + \tilde{v}_0(x) H(t)) \ast_{x, t} K_{\alpha, \beta}(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
where the inverse Fourier transform of \(\hat{K}_{\alpha, \beta}\), (2.29), is given by
\[
\tilde{K}_{\alpha, \beta}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + ((1 + s^{\epsilon}) / (1 + \tau s^{\epsilon}))\rho^{1+\beta} \sin(\beta \pi / 2)} \cos(\rho x) \mathrm{d}\rho, \quad x \in \mathbb{R}, \quad \Re s > 0. \tag{3.1}
\]

Note that the integral in (3.1) is written formally and it denotes the inverse Fourier transform. As will be seen below, it may either converge or diverge, representing a distribution.

We are interested in the behaviour of \(K_{\alpha, \beta}\) for \(\alpha\) and \(\beta\) tending to zero and one. We expect that the solution kernel \(K_{\alpha, \beta}\) tends to solution kernels in specific cases. From the form of \(K_{\alpha, \beta}\) given by (2.33), this cannot be easily seen. However, numerical examples (see §4) suggest that this holds true. What can be seen analytically is that the Laplace transform of \(K_{\alpha, \beta}\) tends to the Laplace transforms of solution kernels in specific cases.

When \(\beta \to 0\), then, in the sense of distributions,
\[
\tilde{K}_{\alpha, 0}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{1}{s} \cos(\rho x) \mathrm{d}\rho, \quad x \in \mathbb{R}, \quad \Re s > 0,
\]
and the solution kernel \(K_{\alpha, 0}\) is of the form
\[
K_{\alpha, 0}(x, t) = \delta(x) H(t), \quad x \in \mathbb{R}, \quad t > 0. \tag{3.2}
\]

This is the case of the non-propagating disturbance, if the initial velocity is zero and the solution is given by (1.23) with \(v_0 = 0\). Therefore, regardless of the parameter \(\alpha\), when \(\beta\) tends to zero, the solution kernel tends to (3.2). This supports the idea from remark 1.1 (ii) that our system of equations can be useful in modelling materials which resist the propagation of the initial disturbance.
When $\beta \to 1$, we obtain the case of the time-fractional Zener wave equation, studied in [42]. In this case,

$$K_{\alpha,1}(x,s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + (1 + s^\alpha)/(1 + \tau s^\alpha)} \cos(\rho x) \, d\rho, \quad x \in \mathbb{R}, \quad \Re s > 0,$$

and the calculation similar to one presented in [42] leads to

$$K_{\alpha,1}(x,t) = \frac{1}{4\pi i} \int_0^\infty (f_-(q) e^{i|x|q^\alpha} - f_+(q) e^{i|x|q^\alpha}) e^{-qt} \, dq, \quad x \in \mathbb{R}, \quad t > 0,$$

with

$$f_+(q) = \sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}} \quad \text{and} \quad f_-(q) = \sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}}, \quad q > 0.$$

We note that in [42] the solution is given in a slightly different form.

When $\alpha \to 0$, then

$$K_{0,\beta}(x,s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + (2/(1 + \tau)) \rho^{1+\beta} \sin(\beta \pi/2)} \cos(\rho x) \, d\rho, \quad x \in \mathbb{R}, \quad \Re s > 0.$$

Using $\mathcal{L}^{-1}[s/(s^2 + \omega^2)](t) = \cos(\omega t)$ one easily comes to

$$K_{0,\beta}(x,t) = \frac{1}{\pi} \int_0^\infty \cos \left( t\sqrt{\frac{2}{1 + \tau}} \rho^{1+\beta} \sin \left( \frac{\beta \pi}{2} \right) \right) \cos(\rho x) \, d\rho, \quad x \in \mathbb{R}, \quad t > 0,$$

in the sense of distributions, which can be transformed to

$$K_{0,\beta}(x,t) = \frac{1}{2\pi} \int_0^\infty \left( \cos \left( x + ct, \sqrt{\frac{1}{\rho^{1-\beta}} \sin \left( \frac{\beta \pi}{2} \right)} \rho \right) \right) \, d\rho, \quad x \in \mathbb{R}, \quad t > 0,$$

where $c = \sqrt{2/(1 + \tau)}$. This is the case of the space-fractional wave equation studied in [20]. Note that the solution in [20] is given in a different form. In this case, from (3.4), one can recover solution kernels for $\beta = 0$ and $\beta = 1$.

If we put $\beta = 0$ in (3.4), we obtain

$$K_{0,0}(x,t) = \frac{1}{\pi} \int_0^\infty \cos(x \rho) \, d\rho = \delta(x), \quad x \in \mathbb{R}, \quad t > 0,$$

i.e. (3.2).

For $\beta = 1$ in (3.4), we obtain, in the sense of distributions,

$$K_{0,1}(x,t) = \frac{1}{2\pi} \int_0^\infty (\cos((x + ct)\rho) + \cos((x - ct)\rho)) \, d\rho, \quad x \in \mathbb{R}, \quad t > 0,$$

$$= \frac{1}{2} (\delta(x + ct) + \delta(x - ct)),$$

with $c = \sqrt{2/(1 + \tau)}$. This is the solution kernel for the classical wave equation.

4. Numerical examples

We examine the qualitative properties of the solution to the space–time fractional Zener wave equation (1.17). Furthermore, we investigate the influence of the orders $\alpha$ and $\beta$ of the, respective, time and space fractionalization of the constitutive equation and strain measure. Also, we numerically compare the solution to (1.17) with the solutions to the time-fractional Zener wave
equation (1.26), which represents the limiting case $\beta = 1$ in (1.17). Both equations are subject to initial conditions $u_0 = \delta$, $v_0 = 0$. In this case, the solution to (1.17), given by (2.32), (2.33), becomes

$$u(x, t) = \delta(x) *_x K(x, t) = K(x, t), \quad x \in \mathbb{R}, \ t > 0. \quad (4.1)$$

Since $K$, and therefore $u$, is a distribution in $x$, it cannot be plotted. Thus, we use the regularization $K_\varepsilon$ of the solution kernel so that (4.1) becomes

$$u_\varepsilon(x, t) = \frac{1}{\pi} \int_0^\infty \hat{K}(\rho, t) e^{-(\varepsilon \rho)^2/4} \cos(\rho x) \, d\rho, \quad x \in \mathbb{R}, \ t > 0. \quad (4.2)$$

In all figures, we present the displacement field only on the half-axis $x \geq 0$, as the field is symmetric with respect to the displacement axis. Figure 1 presents the plot of the displacement versus coordinate, obtained according to (4.2) for several time instants, while the other parameters of the model are $\alpha = 0.25$, $\beta = 0.45$, $\tau = 0.1$ and $\varepsilon = 0.01$. From figure 1, we see that, as time increases, the height of the peaks decreases, as the energy introduced by the initial disturbance field is being dissipated. This is the consequence of the viscoelastic properties of the material.

In figure 2, we compare the displacements obtained as solutions for non-local (1.17) and local (1.26) wave equations, given by (4.2) and (3.3), respectively. Apart from $\beta = 0.45$ in (1.17) and $\beta = 1$ in (1.26) the other parameters in both models are as above. The effect of non-locality introduced in the strain measure is observed, as at a fixed time-instant, apart from the primary peak that exists in both models, in the non-local one there are secondary peaks in the displacement field. These secondary peaks reflect the influence of the oscillations of a certain material point on other material points in a medium. Thus, when the initial disturbance propagates (this is reflected by the existence of the primary peak), due to the non-locality, the secondary peaks reflect the residual influence of the disturbance transported by the primary peak. The non-locality changes not only the number of peaks at a certain time-instant but also the shape of the primary peak. The primary peak in the non-local model (compared with the local one) is higher and placed closer to the origin—the point where the initial Dirac-type disturbance field is introduced.

The aim of the following figures is to show the influence of changing the non-locality parameter $\beta$. All other parameters are as above. Figure 3 presents plots of displacements for various values of $\beta$. From figure 3, one sees that, as the non-locality parameter $\beta$ increases, the effects of non-locality decrease, as the height of the secondary peaks decreases and eventually the secondary peaks cease to exist. Also, as $\beta$ increases, the height of the primary peak decreases and its position increases, being further from the point of the initial Dirac-type disturbance. In the limiting case $\beta = 1$, the displacement curve of non-local model (1.17) overlaps with the displacement curve of the local model (1.26). Figures 4 and 5 present the displacement field for
Figure 2. Snapshots of the solution $u(x, t)$ at $t \in \{0.5, 1, 1.5, 2\}$ as a function of $x$: (1.17), solid line, and (1.26), dashed line.

Figure 3. Snapshots of the solution $u(x, t)$ at $t = 1$ as a function of $x$: (1.17), solid line, and (1.26), dashed line.

Figure 4. Snapshots of the solution $u(x, t)$ of (1.17) at $t = 1$ as a function of $x$. 
Figure 5. Snapshots of the solution $u(x,t)$ of (1.17) at $t = 1$ as a function of $x$.

smaller values of $\beta$. In Figure 4, one notices the significant influence of non-local effects. Namely, the secondary peaks are more prominent in height than the primary peak. Finally, in the limiting case when $\beta \to 0$ one expects to obtain the displacement field in the form (1.23), i.e. $u(x,t) = \delta(x)$, $x \in \mathbb{R}$, $t > 0$. This can be seen from Figure 5. We might, thus, say that these numerical examples support the claim that $\beta$, as the non-locality parameter, measures the resistance of material to the disturbance propagation. Namely, at the same time-instant, as $\beta$ decreases, the primary peak is placed closer to the point where the Dirac-type disturbance occurred and for $\beta \to 0$ we obtain the non-propagating disturbance. Also, the shape of the primary peaks changes and, as $\beta$ increases, they become more like the peak of the local model and for $\beta = 1$ the curves overlap.

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Appendix A. Mathematical background

This section serves as a mathematical survey needed in the analysis that we have presented. We single out definitions and properties of fractional derivatives and, as our main tools are integral transforms, we recall the, more or less well known, main definitions and properties used. For a detailed exposition of the theory of fractional calculus see [14,15], and for the spaces and integral transforms we refer to [15,45].

Let $0 \leq \alpha < 1$, $-\infty \leq a < b \leq \infty$. The left and right Caputo derivatives, of order $\alpha$, of an absolutely continuous function $u$ are defined by

$$\frac{C_a}{C_b} D^\alpha_x u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(\theta)}{(t-\theta)^\alpha} \, d\theta$$

and

$$\frac{C_a}{C_b} D^\alpha_x u(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{u'(\theta)}{(\theta-t)^\alpha} \, d\theta,$$

where $\Gamma$ is the Euler gamma function and $u' = (d/dt)u$. Note that $\frac{C_a}{C_b} D^\alpha_x u(t) = \frac{C_b}{C_a} D^\alpha_x u(t) = u(t)$, and for continuously differentiable functions and distributions we have that, as $\alpha \to 1$, $\frac{C_a}{C_b} D^\alpha_x u(t) \to u'(t)$, $\frac{C_a}{C_b} D^\alpha_x u(t) \to -u'(t)$. Therefore, the Caputo derivatives generalize integer order derivatives.

Let $0 \leq \beta < 1$, $-\infty \leq a < b \leq \infty$. The symmetrized fractional derivative of an absolutely continuous function $u$ is defined as

$$\frac{C_a}{C_b} E^\beta_x u(x) = \frac{1}{2} \left( \frac{C_a}{C_b} D^\beta_x - \frac{C_b}{C_a} D^\beta_x \right) u(x) = \frac{1}{2} \frac{1}{\Gamma(1-\beta)} \int_a^b \frac{u'(\theta)}{|x-\theta|^{\beta}} \, d\theta.$$
For $a = -\infty$ and $b = \infty$, we write $\mathcal{E}_x^\beta$ instead of $\mathcal{E}_{a}^\beta$ and then

$$\mathcal{E}_x^\beta u(x) = \frac{1}{2} \frac{1}{\Gamma(1 - \beta)} |x|^{-\beta} * u'(x).$$

Note that $\mathcal{E}_x^0 u(x) = 0$ and $\mathcal{E}_x^\beta u(x) \to u'(x)$, as $\beta \to 1$. So, the symmetrized fractional derivative generalizes the first derivative of a function. The zeroth order symmetrized fractional derivative of a function is zero (not a function itself).

For fractional operators in the distributional setting, one introduces a family $\{f_a\}_{a \in \mathbb{R}} \in S'_+$ as

$$f_a(t) = \begin{cases} H(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{d^N}{dt^N} f_{a+N}(t), & \alpha \leq 0, \alpha + N > 0, N \in \mathbb{N}, \end{cases}$$

and $\{f_a\}_{a \in \mathbb{R}} \in S'_-$ as $f_a(t) = f_a(-t)$, where $H$ is the Heaviside function. Then $f_a \ast$ and $f_a \ast$ are convolution operators and for $a < 0$ they are operators of left and right fractional differentiation, so that for $u$ absolutely continuous we have $\mathcal{D}_t^\alpha u = f_{1-\alpha} \ast u'$ and $\mathcal{D}_t^\beta u = -f_{1-\alpha} \ast u'$.

For $u \in S'$, the Fourier transform is defined as $\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$, $\phi \in S(\mathbb{R})$, where for $\phi \in S$

$$\hat{\phi}(\xi) = \mathcal{F}[\phi(x)](\xi) = \int_{-\infty}^{\infty} \phi(x) e^{-i\xi x} \, dx, \quad \xi \in \mathbb{R}.$$ 

The Laplace transform of $u \in S'$ is defined by

$$\tilde{u}(s) = \mathcal{L}[u(t)](s) = \mathcal{F}[e^{-\xi t} u(t)](\eta), \quad s = \xi + i\eta.$$ 

It is well known that the function $\tilde{u}$ is holomorphic in the half-plane $\text{Re } s > 0$ (e.g., [45]). In particular, for $u \in L^1(\mathbb{R})$ such that $u(t) = 0$, for $t < 0$, and $|u(t)| \leq A e^{At}$ ($a, A > 0$) the Laplace transform is

$$\tilde{u}(s) = \int_{0}^{\infty} u(t) e^{-st} \, dt, \text{ Re } s > 0.$$ 

We recall the main properties of the Fourier and Laplace transforms. Let $u, u_1, u_2 \in S'$

$$\mathcal{F}[u_1 \ast u_2](\xi) = \mathcal{F}[u_1](\xi) \ast \mathcal{F}[u_2](\xi), \quad \mathcal{F}[u^{(n)}](\xi) = (i\xi)^n \mathcal{F}[u](\xi), \quad n \in \mathbb{N}, \quad \mathcal{F}[\alpha^\beta u](\xi) = \mathcal{F}[\mathcal{D}_t^\beta u](s) = s^\alpha \mathcal{L}[u](s), \quad \alpha \geq 0, \quad \mathcal{L}[\delta](s) = 1,$$

where $(\cdot)^{(n)}$ denotes the $n$th derivative. For $\beta \in [0, 1)$ it holds that

$$\mathcal{F}[|x|^{-\beta}](\xi) = 2 \Gamma(1 - \beta) \sin \frac{\beta \pi}{2} \frac{1}{|\xi|^1 - \beta} \quad \text{and} \quad \mathcal{F}[\mathcal{E}_x^\beta u(x)](\xi) = i \frac{\xi}{|\xi|^1 - \beta} \sin \frac{\beta \pi}{2} \mathcal{F}[u](\xi).$$

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