Laplace’s equation on convex polyhedra via the unified method

A. C. L. Ashton

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0W A, UK

We provide a new method to study the classical Dirichlet problem for Laplace’s equation on a convex polyhedron. This new approach was motivated by Fokas’ unified method for boundary value problems. The central object in this approach is the global relation: an integral equation which couples the known boundary data and the unknown boundary values. This integral equation depends holomorphically on two complex parameters, and the resulting analysis takes place on a Banach space of complex analytic functions closely related to the classical Paley–Wiener space. We write the global relation in the form of an operator equation and prove that the relevant operator is bounded below using some novel integral identities. We give a new integral representation to the solution to the underlying boundary value problem which serves as a concrete realization of the fundamental principle of Ehrenpreis.

1. Introduction

In the forthcoming book [1], the editors list the three most important open problems in the context of the unified transform method for boundary value problems. The third of these states:

Three dimensional problems . . . the implementation of the unified transform to elliptic PDEs in three dimensions remains open.

In this paper, we address this problem. We offer an implementation of the unified method for the Dirichlet problem for Laplace’s equation in three-dimensional polyhedrons.

The Dirichlet problem of determining a harmonic function $u$ on a domain $\Omega \subset \mathbb{R}^3$ given its behaviour on $\partial \Omega$ has a long and rich history. One popular
approach to this problem is through the use of Layer potentials, a detailed account of which can be found in [2,3]. Within this treatment is the direct approach, where Green’s third identity is used

\[
 u(x) = \int_{\partial \Omega} \left[ G(x, y) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial G}{\partial n}(x, y) \right] \, d\sigma(y), \quad x \in \Omega, \tag{1.1}
\]

where \( n \) denotes the outward normal to \( \Omega \), \( d\sigma \) is the Lebesgue measure on \( \partial \Omega \) and \( G \) is a Green’s function for the Laplacian. One seeks to find the relationship between \( u|_{\partial \Omega} \) and \( \partial u/\partial n|_{\partial \Omega} \) by means of the Dirichlet–Neumann map. This map is determined by taking non-tangential limits towards \( \partial \Omega \) in (1.1) and deriving a boundary integral equation that relates the Dirichlet and Neumann data. The solution to the boundary integral equation for the unknown Neumann data is realized by the Steklov–Poincaré operator \( S \):

\[
 S : u|_{\partial \Omega} \mapsto \frac{\partial u}{\partial n}|_{\partial \Omega}.
\]

This then gives rise to an integral representation for the solution to the underlying boundary value problem via (1.1): if \( u|_{\partial \Omega} = f \) is given then

\[
 u(x) = \int_{\partial \Omega} \left[ G(x, y) Sf(y) - f(y) \frac{\partial G}{\partial n}(x, y) \right] \, d\sigma(y), \quad x \in \Omega,
\]

is the resulting solution to the underlying boundary value problem, assuming \( f \) has sufficient regularity.

Let us now discuss a more recent approach introduced by Fokas \([4,5]\) in the case when \( \Omega \subset \mathbb{R}^2 \) is a convex polygon. Let \( u = u(x) \) be harmonic in \( \Omega \) and set \( v_2(x) = \exp(-i \lambda \cdot x) \). If we restrict \( \lambda \in \mathbb{C}^2 \) so that \( \lambda_1^2 + \lambda_2^2 = 0 \) then \( v_2 \) is harmonic. Green’s second identity gives

\[
 0 = \int_{\Omega} \left[ v_2 \Delta u - u \Delta v_2 \right] \, dx = \int_{\partial \Omega} e^{-i \lambda \cdot y} \left[ \frac{\partial u}{\partial n}(y) + i(\lambda \cdot n)u(y) \right] \, d\sigma(y),
\]

which holds for all \( \lambda \in \mathbb{C}^2 \) for which \( \lambda_1^2 + \lambda_2^2 = 0 \). This is called the global relation: it is an integral identity that couples the Neumann and Dirichlet boundary values. It can be shown that the global relation completely determines the Dirichlet–Neumann map \([6–8]\). The global relation can be formally interpreted as a new representation of the Steklov–Poincare operator \( S \). The functional analytic properties of \( S \) can be derived through this representation using techniques from complex analysis \([7]\).

The unified method also gives rise to new integral representations for the solution to elliptic boundary value problems \([4,9]\). These integral representations are in direct accordance with the fundamental principle of Ehrenpreis \([10]\). The fundamental principle tells us that the solution to a constant coefficient, linear PDE \( P(-i \partial_1, \ldots, -i \partial_n)u = 0 \) on a convex domain \( \Omega \) can be written as a superposition of exponential solutions. More concretely, it says that there exists a complex Radon measure \( \mu(\lambda) \) with support contained within the algebraic variety \( Z(P) = \{ \lambda \in \mathbb{C}^n : P(\lambda) = 0 \} \) such that

\[
 u(x) = \int_{Z(P)} e^{i \lambda \cdot x} c(x, \lambda) \, d\mu(\lambda), \quad x \in \Omega,
\]

where \( c(x, \lambda) \) is a polynomial in \( x \). This is a highly abstract result and the proof is non-constructive. A particularly striking feature of the unified method is that for an important class of two-dimensional PDEs, it provides an explicit Ehrenpreis-type representation.

In this paper, we derive new results that extend the remit of the unified method to three-dimensional boundary value problems. This is, we believe, the first extension of this type in the context of the unified method for elliptic boundary value problems. We consider the classical Dirichlet problem for Laplace’s equation inside a convex polyhedron. In §2, we set up the necessary global relation and rewrite it in the form of an operator equation \( T\Phi = \Psi \), where the vector \( \Psi \) is known and \( \Phi \) is unknown. Then in §3, we set up the relevant function spaces \( X, Y \) to work on so that we can carry out the relevant analysis for the operator \( T : X \rightarrow Y \). The technical results appear in §4, where we prove that \( T \) is an injective operator with closed range.
This result can then be used to show that the global relation gives rise to a continuous linear map between the known Dirichlet data and the unknown Neumann boundary values. We briefly discuss the practical application of these results in §5 pertaining to the numerical solution of the boundary value problem. Finally in §6, we introduce a new Ehrenpreis-type representation for the solution of boundary value problem. We show that any solution \( q \) to Laplace’s equation in a convex polyhedron can be written in the form

\[
q(x) = \frac{1}{8\pi^2} \sum_{i=1}^{n} Z_i e^{i\mu \cdot x} \rho_i(\mu) \, d\nu_i(\mu), \quad x \in \Omega,
\]

where the \( Z_i \) are known subsets of the variety \( \{ \lambda \in \mathbb{C}^3 : \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \} \) and the functions \( \{ \rho_i \}_{i=1}^{n} \) and the measures \( \{ d\nu_i \}_{i=1}^{n} \) are known explicitly in terms of the boundary values of \( q \). It is immediately apparent that this gives a concrete realization of the Ehrenpreis principle for harmonic functions on convex polyhedra.

2. Global relation for convex polyhedrons

Let \( \Omega \subset \mathbb{R}^3 \) be a convex polyhedron with faces \( \{ \Sigma_i \}_{i=1}^{n} \). The centroids and outwards normals of the faces are denoted by \( \mathbf{m}_i \) and \( \mathbf{n}_i \) \((1 \leq i \leq n)\), respectively. We use \( x = (x_1, x_2, x_3) \) as an orthogonal coordinate system in \( \mathbb{R}^3 \) and \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) to denote an orthogonal coordinate system in \( \mathbb{C}^3 \). Throughout we will use the convention that an unbolded coordinate corresponds to the projection \( \mathbb{R} \) of a bolded coordinate in the first two variables. So \( x = (x_1, x_2) \) and \( \lambda = (\lambda, \lambda_3), \ dx = dx_1 \, dx_2 \, dx_3 \) and \( dx = dx_1 \, dx_2, \) etc.

We are given real-valued boundary data \( f_i \in H^1(\Sigma_i) \) for \( i = 1, \ldots, n \) that are compatible at the edges of the polyhedron, i.e. \( f_i = f_j \) on \( \Sigma_i \cap \Sigma_j \). We study the Dirichlet–Neumann map associated with the Dirichlet problem

\[
\Delta q = 0 \quad \text{in} \ \Omega, \tag{2.1a}
\]

and

\[
q = f_i \quad \text{on} \ \Sigma_i \text{ for } i = 1, \ldots, n. \tag{2.1b}
\]

We refer the reader to Dauge [11] for a classical analysis of this problem. The solution to (2.1) defines a map between the given Dirichlet data \( \{ f_i \}_{i=1}^{n} \) and the \emph{a priori} unknown Neumann boundary values. We shall characterize this map by analysing the global relation associated with (2.1)

\[
\sum_{n=1}^{n} \rho_n(\lambda) = 0, \quad \lambda \in Z(\Delta), \tag{2.2}
\]

where

\[
\rho_n(\lambda) = \int_{\Sigma_i} e^{-i\lambda \cdot x} \left[ \frac{\partial q}{\partial \mathbf{n}}(x) + i(\lambda \cdot \mathbf{n}_i) f_i(x) \right] \, d\sigma(x) \tag{2.3}
\]

and \( Z(\Delta) \) is the algebraic variety \( \{ \lambda \in \mathbb{C}^3 : \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \} \). The global relation can be seen as a consequence of Green’s second identity: assume a solution of (2.1) exists and set \( a \equiv q \) and \( b \equiv e^{-i\lambda \cdot x} \) in

\[
\int_{\Omega} [a \Delta b - b \Delta a] \, dx = \sum_{i=1}^{n} \int_{\Sigma_i} \left[ a \frac{\partial b}{\partial \mathbf{n}_i} - b \frac{\partial a}{\partial \mathbf{n}_i} \right] \, d\sigma.
\]

Then if \( \lambda \in Z(\Delta) \) the left-hand side vanishes and the right-hand side becomes the global relation (2.2) (figure 1).

It will be convenient to have a local description of the faces. To this end, set \( X = (X_1, X_2)^t \) introduce the maps \( \psi_i, 1 \leq i \leq n, \) defined by

\[
\psi_i : X \mapsto \mathbf{m}_i + R_i \begin{pmatrix} \chi \\ 0 \end{pmatrix},
\]

where \( R_i = R(\theta_i, \mathbf{n}_i) \in \text{SO}(3) \) is the orthogonal matrix representing a rotation about \( \mathbf{e}_z \wedge \mathbf{n}_i \) by an angle \( \theta_i = \arccos(\mathbf{n}_i \cdot \mathbf{e}_z) \). Then \( \psi_i^{-1} \) maps the face \( \Sigma_i \) into the \( x_3 = 0 \) plane, with its normal aligned
with the $x_3$-axis, its centroid mapped to the origin (see figure 1). For $f$ defined on the face $\Sigma_i$, we define its pull back by $\psi_i$ by

$$\psi_i^*(f)(X) = f(\psi_i(X)).$$

Using this change of variables the term $\int_{\Sigma_i} e^{-i \lambda \cdot X} f(X) d\sigma(x)$ becomes

$$e^{-i \lambda \cdot m_i} \int_{Q_i} e^{-i(\delta_i, \lambda, X_i - i(\delta_i, \lambda, X_2)} \psi_i^*(f)(X) dX,$$

where $\psi_i(Q_i) = \Sigma_i$. Set $\psi_i^*(q) = q_i$ and $\psi_i^*(\partial q / \partial n) = \partial_n q_i$. By noting

$$\lambda \cdot n_i = (\delta_i, \lambda)^t R_i n_i = (\delta_i, \lambda)^t e_z,$$

we find

$$\rho_i(\lambda) = e^{-i \lambda \cdot m_i} \int_{Q_i} e^{-i(\delta_i, \lambda, X_i - i(\delta_i, \lambda, X_2)} [\partial_n q_i + i(\delta_i, \lambda)_3 q_i] dX.$$ 

Now fix $i$, multiply the global relation by $e^{i \lambda \cdot m_i}$ and replace $\lambda$ with $R_i^* \lambda$. The latter operation is valid because the variety $Z(\Delta)$ is invariant under rotations $\lambda \mapsto R_i \lambda$. The global relation becomes

$$0 = \int_{Q_i} e^{-i \lambda \cdot X} [\partial_n q_i + i \lambda_3 q_i] dX + \sum_{j \neq i} e^{-i(\delta_i, \lambda, X_i - i(\delta_i, \lambda, X_2)} [\partial_n q_j + i(\delta_i, \lambda)_3 q_j] dX,$$

where we have defined $\Delta_{ij} = R_i R_j^t \equiv R_i R_j^{-1}$.

The global relation is to hold for all $\lambda \in Z(\Delta)$. We can parametrize one part of this algebraic variety by setting $\lambda_3 = i(\lambda_1^2 + \lambda_2^2)^{1/2}$, where the square root is defined to have positive real part when $\lambda_1^2 + \lambda_2^2$ is positive. It is sufficient that the global relation only holds on this part, as it can be shown that it must also hold on the remaining part by taking complex conjugates and changing the sign of $\lambda_1$ and $\lambda_2$. So from now on, we aim to solve the global relation for

$$Z_+(\Delta) = \left\{ \lambda : \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_3 = i(\lambda_1^2 + \lambda_2^2)^{1/2} \right\}.$$ 

Let us denote the Fourier transform by

$$\hat{f}_i(\lambda) = \int_{Q_i} e^{-i \lambda \cdot X} f(X) dX.$$
and introduce the vectors
\[ \Theta(\lambda) = \begin{bmatrix} \hat{q}_1(\lambda) \\ \vdots \\ \hat{q}_n(\lambda) \end{bmatrix}, \quad \Phi(\lambda) = \begin{bmatrix} (\partial_{\lambda} q_1)(\lambda) \\ \vdots \\ (\partial_{\lambda} q_n)(\lambda) \end{bmatrix}. \]

Define the linear operator \( T \) and \( S \) by
\[ (T \Phi)_i(\lambda) = \sum_{j=1}^n e^{-i(m_j \cdot m_i) \Delta i j (\lambda)} \Phi_j((\Delta i j \lambda)_1, (\Delta i j \lambda)_2) \]
and
\[ (S \Theta)_i(\lambda) = \sum_{j=1}^n i e^{-i(m_j \cdot m_i) \Delta i j (\lambda)} \Theta_j((\Delta i j \lambda)_1, (\Delta i j \lambda)_2), \]
for \( i = 1, \ldots, n \), where from now onward we abuse notation by using \( \lambda \) to denote the projection of \( \lambda \) on \( Z_+(\Delta) \). Then the global relation becomes
\[ [T \Phi](\lambda) + [S \Theta](\lambda) = 0, \quad \lambda \in \mathbb{C}^2. \]

Our aim will be to solve this equation for the unknown vector \( \Phi \). To do this, we will study the abstract operator equation \( T \Phi = \Psi \) where \( \Psi \equiv -S \Theta \) is known. We first establish appropriate function spaces to work with.

### 3. Paley–Wiener spaces on convex polygons

The components \( \hat{q}_i \) of the known vector \( \Theta \) are related to the pull back of Dirichlet data \( \psi_i^*(f_i) \in H^1(Q_i), i = 1, \ldots, n \) via the two-dimensional Fourier transform. It is natural, then, to work with the classical Paley–Wiener spaces
\[ \text{PW}_2^{Q_i} = \mathcal{F}L^2(Q_i), \]
which contain the Fourier transforms of square integrable functions whose support lies in the convex polygon \( Q_i \). If \( H_K \) denotes the supporting function of the convex set \( K \subset \mathbb{R}^2 \),
\[ H_K(x) = \sup_{y \in K} x \cdot y, \]
then the Paley–Wiener(Schwartz) theorem states
\[ \text{PW}_2^{K} = \{ f : C^2 \to \mathbb{C} \text{ entire}, f|_{\mathbb{R}^2} \in L^2(\mathbb{R}^2), |f(z)| \lesssim e^{H_K(|Im z|)} \}, \]
where \( z = (z_1, z_2) \). Paley–Wiener functions satisfy the pointwise estimate
\[ |f(z)| \leq \|f\|_2 e^{H_K(|Im z|)}, \quad f \in \text{PW}_2^{K}, \tag{3.1} \]
where here and throughout \( \|f\|_2 \) refers to the \( L^2 \) norm of \( f|_{\mathbb{R}^2} \). As we are dealing with real valued data, we need to work on a closed subspace of the classical Paley–Wiener spaces which consist of Fourier transforms of real-valued functions. The elements of this subspace are identified by the fact
\[ f(z) = f(-\bar{z}). \]

We denote this closed subspace of \( \text{PW}_2^{K} \) by \( \text{PW}_2^{K_{\text{sym}}} \). By setting \( \lambda = 0 \) in the global relation, we obtain a necessary condition for a solution to exist
\[ \sum_{i=1}^n \Phi_i(0) = 0. \]
This observation is a restatement of the fact the integral over \( \partial \Omega \) of the Neumann data must vanish. It is now natural to work on the Hilbert space

\[
X = \left\{ \Phi \in \text{PW}_{Q_1,\text{sym}}^2 \times \cdots \times \text{PW}_{Q_n,\text{sym}}^2 : \sum_i \Phi_i(0) = 0 \right\},
\]

which has the \( L^2 \) inner product

\[
(\Phi, \Psi) = \sum_{i=1}^n \int_{\mathbb{R}^2} \Phi_i(\lambda) \overline{\Psi_i}(\lambda) \, d\lambda \equiv \sum_{i=1}^n \int_{\mathbb{R}^2} \Phi_i(\lambda) \overline{\Psi_i}(-\lambda) \, d\lambda
\]

and norm

\[
\|\Phi\|_X^2 = \int_{\mathbb{R}^2} |\Phi(\lambda)|^2 \, d\lambda \equiv \sum_{i=1}^n \int_{\mathbb{R}^2} |\Phi_i(\lambda)|^2 \, d\lambda.
\]

We will need the following pseudo-compactness lemma.

**Lemma 3.1.** Every bounded sequence in \( X \) contains a subsequence that converges locally uniformly in \( C^2 \) to an element of \( X \) which obeys the same norm-bound.

**Proof.** Let \( \{\Phi_m\}_{m \geq 1} \) be our sequence and assume \( \|\Phi_m\| \leq C \) for some \( C > 0 \) and each \( m \). By the Paley–Wiener estimate (3.1), we see that this sequence is locally uniformly bounded in \( C^2 \), so by Montel’s theorem we can extract locally uniformly convergent subsequence. Denote the limit of this subsequence by \( \Phi \). Then \( \Phi \) must be an entire function on \( C^2 \) with \( \sum_i \Phi_i(0) = 0 \) and Fatou’s lemma implies \( \|\Phi\|_X \leq C \). Also, for each \( i = 1, \ldots, n \) and each \( z \in C^2 \) the Paley–Wiener estimate in (3.1) gives

\[
|\Phi_i(z)| = \lim_{m \to \infty} |(\Phi_m)_i(z)| \leq \lim_{m \to \infty} e^{H_{Q_i}(\text{Im} z)}\|\Phi_m\|_i \leq C e^{H_{Q_i}(\text{Im} z)},
\]

so \( \Phi \in X \) by the Paley–Wiener theorem. \( \square \)

We note the isomorphism \( X \simeq \{ v \in L^2(\partial \Omega), \int_{\partial \Omega} v \, d\sigma = 0 \} \), with real-valued data assumed. We set \( Y = L^2(\mathbb{R}^2)^{\times n} \). Assuming that the Dirichlet data \( f_i \in H^1(\Sigma_i), i = 1, \ldots, n \), is compatible at the edges of the polyhedron, i.e. \( f_i = f_j \) on \( \Sigma_i \cap \Sigma_j \), then it can be shown, via integration by parts and a similar argument to the proof of lemma 4.3, that \( \Psi \equiv -S\Theta \in Y \). This further justifies our decision to work with the function space \( X \), since we show in lemma 4.3 that \( T\Phi \in Y \) if \( \Phi \in X \) (it is also well known [15, Theorem 4.24] that if the Dirichlet data has the observed regularity then the Neumann data will be square integrable). Since \( X \) and \( Y \) share the same inner product and norm, we will simply use the notation \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) throughout. In §4, we will establish that \( T : X \to Y \) defines a bounded linear operator which is injective and has closed range.

**4. Analysis of the operator \( T \)**

We consider the abstract operator equation \( T\Phi = \Psi \), where \( \Phi \in X \) and \( \Psi \in Y \) as defined in §3. Here we will establish the following results.

**Theorem 4.1.** The operator \( T : X \to Y \) defines a bounded and injective linear map with closed range.

Using this result, it is relatively straightforward establish the following.

**Theorem 4.2.** The solution to the global relation (2.2) associated with (2.1) exists, is unique and depends continuously on the given Dirichlet data.

The majority of the work in this section will be establishing the closedness of the range of \( T : X \to Y \). With this in hand, it is a relatively straightforward to prove the existence result stated in theorem 4.2. Indeed, using Banach’s closed range theorem, the result in theorem 4.1 implies

\[
R(T) = N(T^*)^\perp,
\]

where the right-hand side denotes the perpendicular compliment of the null space of the adjoint operator \( T^* \). It can be shown that with boundary data \( \{f_i\}_{i=1}^n \) as described in the boundary value
problem (2.1), the known quantity $\Psi \equiv -S\Theta$ belongs to $N(T^*)$. Alternatively, and perhaps more straightforwardly, one can appeal to the results in Ashton [6].

The continuity result in theorem 4.2 follows directly from theorem 4.1. Indeed, since $R(T)$ is closed and $T$ is injective, the map $T : X \to R(T)$ is a bijection between Banach spaces, so its inverse is also bounded, i.e. continuous, by Banach’s bounded inverse theorem.

To establish the result in theorem 4.1, we will show that $T : X \to Y$ is bounded below, i.e.

$$\|T\Phi\| \geq \|\Phi\|,$$

for all $\Phi \in X$. If this is the case, then it is clear that $T$ is injective and the closedness of $R(T)$ is a standard operator theory result, e.g. [12, §2.1].

We first establish the simplest of these results.

**Lemma 4.3.** $T : X \to Y$ defines a bounded linear map.

**Proof.** We consider the $i$th component of $T\Phi(\lambda)$. Clearly,

$$|T\Phi_i(\lambda)|^2 \leq \sum_{j=1}^n |e^{-i(m_i - m_j)\cdot R_j} \Phi_i((\Delta_{ij}\lambda)_1, (\Delta_{ij}\lambda)_2)|^2,$$

so it suffices to treat each term in the sum separately. The $j = i$ term in this sum simply $\Phi_i(\lambda)$, so the $L^2$ norm of this term is certainly bounded above by $\|\Phi\|^2$. Now we consider the terms with $j \neq i$. Using the Fourier inversion theorem these terms can be written as

$$\frac{1}{(2\pi)^2} \int_{Q_i} \Phi_i(-X) \exp \left[ i \left( m_i - m_j - R_j \right) \cdot \frac{X}{0} \right] dX.$$

Recalling that $\lambda = (\lambda, i|\lambda|)$, the absolute value of the exponential term is

$$\exp \left[ -|\lambda| \left( m_i - m_j - R_j \right) \cdot \frac{X}{0} \right].$$

where we have used the fact $R_j e_z = n_i$. For fixed $X \in Q_j$, the vector $v_j(X) = m_i - m_j - R_j(X, 0)^t$ represents an orientated line from the point $\psi_j(X)$ on the $j$th face to the centroid $m_i$ of the $i$th face. In particular, if the $i$th and $j$th faces are not adjacent then there is some positive $\delta_{ij}$ with

$$\sup_{X \in Q_j} v_{ij}(X) \cdot n_i \geq \delta_{ij} > 0,$$

and the corresponding exponential is bounded by $\exp(-\delta_{ij}|\lambda|)$. So, for the $j$th side not adjacent to the $i$th side, by Cauchy–Schwarz and Parseval

$$\int_{Q_i} \Phi_i(-X) \exp \left[ i \left( m_i - m_j - R_j \right) \cdot \frac{X}{0} \right] dX \leq e^{-\delta_{ij}|\lambda|}\|\Phi\|.$$

So the $L^2$ norm of these terms is again bounded above $\|\Phi\|$. When the $i$th and $j$th sides are adjacent, we will have $v_{ij}(X) \cdot n_i = 0$ precisely when $X$ lies on the edge or vertex that connects the two faces. In this case, after a change of coordinates, the relevant integral over $Q_j$ can be written

$$\int_0^\infty \int_0^\infty e^{-((\lambda_1 + a_1|\lambda|)X_1 - (\lambda_2 + a_2|\lambda|)X_2)} f_j(X) dX$$

where $a_1$ and $a_2$ are suitable positive constants and $f_j \in L^2(\mathbb{R}^+ \times \mathbb{R}^+)$ has compact support with $\|f\|_2 \leq \|\Phi\|_2$. Indeed, the transformations we made essentially align the $i$th face of the polygon in the $x_3 = 0$ plane, with the remaining parts lying below. The faces of the adjacent sides can be
described in the form $a_1(x_1 - \beta_1) + a_2(x_2 - \beta_2) + x_3 = 0$ for appropriate positive constants $a_1, a_2$ and constants $\beta_1$ and $\beta_2$. We take $a_1 = a_2 = 1$ for notational simplicity—their actual values are largely irrelevant. If we label the expression in (4.1) by $F_j(\lambda)$ then Cauchy–Schwarz gives

$$|F_j(\lambda)|^2 \leq \int_0^\infty \int_0^\infty |f_j(X)|^2 (X_1 + X_2)^{\delta} e^{-|\lambda|(X_1 + X_2)} dX \int_0^\infty (X_1 + X_2)^{-\delta} e^{-|\lambda|(X_1 + X_2)} dX,$$

for any $1 < \delta < 2$. The second integral is proportional to $|\lambda|^{-\delta - 2}$, and so

$$\|F_j\|_2^2 \leq \int_0^\infty \int_0^\infty |f_j(X)|^2 (X_1 + X_2)^{\delta} \left[ \int_{\mathbb{R}^2} |\lambda|^{-\delta - 2} e^{-|\lambda|(X_1 + X_2)} d\lambda \right] dX.$$

The inner integral is proportional to $(X_1 + X_2)^{-\delta}$, so we arrive at

$$\|F_j\|_2 \leq \|f\|_2 \leq \|\Phi\|_2.$$

So the $L^2$ norm of each terms in the sum defining $T\Phi_j(\lambda)$ is bounded above by some constant multiple of $\|\Phi\|$. Hence $\|T\Phi\| \leq \|\Phi\|$ as desired. \hfill \Box

Now we establish the fact that $T : X \to Y$ is bounded below. To do this, it is enough to show that if $\{\Phi_m\}_{m \geq 1}$ is a sequence in $X$ such that $T\Phi_m \to 0$ in $Y$, then $\Phi_m \to 0$ in $X$. The simplest way to do this is to establish some sort of integral identity that relates $T\Phi$ to $\|\Phi\|$.

**Lemma 4.4.** For $\Phi \in X$ define the linear operators $M$ and $L$ by

$$(M\Phi)_i(\lambda) = i(m_i \cdot R_i^\lambda)\Phi_i(\lambda) \quad \text{and} \quad (L\Phi)_i(\lambda) = \lambda_1 \frac{\partial \Phi_i}{\partial \lambda_1} + \lambda_2 \frac{\partial \Phi_i}{\partial \lambda_2},$$

for $i = 1, \ldots, n$. Then the related linear maps

$$M\Phi(\lambda) := \frac{1}{|\lambda|}M\Phi(\lambda) \quad \text{and} \quad L\Phi(\lambda) := \frac{1}{|\lambda|}L\Phi(\lambda),$$

constitute bounded linear maps from $X$ to $Y$. Also the following integral identity is valid

$$\int_{\mathbb{R}^2} [T\Phi(\lambda) \cdot L\Phi(-\lambda) - LT\Phi(\lambda) \cdot \Phi(-\lambda) - 2MT\Phi(\lambda) \cdot \Phi(-\lambda)] \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} \Phi(\lambda) \cdot L\Phi(-\lambda) - L\Phi(\lambda) \cdot \Phi(-\lambda) - 2M\Phi(\lambda) \cdot \Phi(-\lambda) \frac{d\lambda}{|\lambda|}. \quad (4.2)$$

We note that $T\Phi(-\lambda) = \overline{T\Phi(\lambda)}$ and similarly for $L\Phi$ and $M\Phi$.

Establishing this integral identity takes a significant amount of work, some of which has been relegated to the appendix. The sceptical reader will be pleased to know that this identity has been computationally verified (as much as can be). The integral identity is similar to that derived by the author in Ashton [3, Lemma 4.3] which was of importance in the two-dimensional problem.

**Lemma 4.5.** For $\Phi \in X$, we have the commutator relation

$$[L, T]\Phi = MT\Phi - E\Phi,$$

where the linear operator $E$ is defined by

$$(E\Phi)_i(\lambda) = \sum_{j=1}^n i(m_j \cdot R_j^\lambda) e^{-i(m_j \cdot m_0) R_0^\lambda} \Phi_j((\Delta_{ij} \lambda)\lambda, (\Delta_{ij} \lambda)\lambda),$$

for $i = 1, \ldots, n$ and $\Phi \in X$.

The proof is just a matter of computation so we omit it. We will also require the following technical lemma, the proof of which is in the appendix.
Lemma 4.6. For $\Phi \in X$ and $\lambda = (\lambda, i|\lambda|)^i$ define the functions

$$A_{ij}(\lambda) = e^{-i(m_i - m_j)|\lambda|^2} \Phi((\Delta_{ij}\lambda)_1, (\Delta_{ij}\lambda)_2)(L\Phi)_i(-\lambda)$$

and

$$B_{ij}(\lambda) = i(m_j - m_i)|\lambda|^2 e^{-i(m_i - m_j)|\lambda|^2} \Phi((\Delta_{ij}\lambda)_1, (\Delta_{ij}\lambda)_2)\Phi(-\lambda),$$

For $i \neq j$, we have

$$\int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} A_{ij}(\lambda)\frac{d\lambda}{|\lambda|}$$

(4.3a)

and

$$\int_{\mathbb{R}^2} B_{ij}(\lambda) \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} B_{ij}(\lambda)\frac{d\lambda}{|\lambda|}$$

(4.3b)

To establish our integral (4.2) identity, we first consider

$$\int_{\mathbb{R}^2} T\Phi(\lambda) \cdot L\Phi(-\lambda) \frac{d\lambda}{|\lambda|} = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|}$$

It is straightforward to show that this integral is well defined for $\Phi \in X$. Using the identity (4.3a) in lemma 4.6, we find

$$\sum_{i,j=1}^{n} \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|} = \sum_{i=1}^{n} \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|} + \sum_{i=1}^{n} \sum_{j \neq i} \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|}$$

$$= \int_{\mathbb{R}^2} \Phi(\lambda) \cdot L\Phi(-\lambda) \frac{d\lambda}{|\lambda|} + \sum_{i=1}^{n} \sum_{j \neq i} \int_{\mathbb{R}^2} A_{ij}(\lambda)\frac{d\lambda}{|\lambda|}$$

$$= \int_{\mathbb{R}^2} L\Phi(\lambda) \cdot \Phi(-\lambda) \frac{d\lambda}{|\lambda|}$$

This gives the integral identity

$$\int_{\mathbb{R}^2} [(T\Phi)(\lambda) \cdot (L\Phi)(-\lambda) - (TL\Phi)(\lambda) \cdot \Phi(-\lambda)] \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} [\Phi(\lambda) \cdot L\Phi(-\lambda) - L\Phi(\lambda) \cdot \Phi(-\lambda)] \frac{d\lambda}{|\lambda|}.$$
So on using this expression in place of (4.5) in (4.4), we have the identity

\[
\int_{\mathbb{R}^2} \left[ T\Phi(\lambda) \cdot L\Phi(-\lambda) - LT\Phi(\lambda) \cdot \Phi(-\lambda) - 2MT\Phi(\lambda) \cdot \Phi(-\lambda) \right] \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} \left[ \Phi(\lambda) \cdot L\Phi(-\lambda) - L\Phi(\lambda) \cdot \Phi(-\lambda) - 2M\Phi(\lambda) \cdot \Phi(-\lambda) \right] \frac{d\lambda}{|\lambda|},
\]

which is precisely the identity (4.2). We can now prove the easy part of theorem 4.1.

**Lemma 4.7.** \( T : X \to Y \) is injective.

This is a direct consequence of lemma 4.4. Indeed, note that if \( T\Phi = 0 \) then the left-hand side of (4.2) vanishes. Note also that the term

\[
\Phi(\lambda) \cdot L\Phi(-\lambda) - L\Phi(\lambda) \cdot \Phi(-\lambda)
\]

is purely imaginary. So upon taking real parts we find

\[
0 = -\text{Re} \left( \sum_{i=1}^{n} \int_{\mathbb{R}^2} (im_i \cdot \xi_i^r |\xi_i^r\rangle |\Phi_i(\lambda)\rangle)^2 \frac{d\lambda}{|\lambda|} \right) = \sum_{i=1}^{n} (m_i \cdot \xi_i^r) \|\Phi_i\|_2^2 = \sum_{i=1}^{n} (m_i \cdot n_i) \|\Phi_i\|_2^2. \tag{4.6}
\]

We observe that the analysis carried out in the proof to lemma 4.4 is invariant under affine transformations of the \( \{m_i\}_{i=1}^{n} \), this being a consequence of the fact that the operator \( T \) only involves differences of \( m_i \) and \( m_j \) when \( i \neq j \). This is not surprising; these affine transformations correspond to a simple translation of the domain \( \Omega \). Consequently, we may assume the origin lies within \( \Omega \), so by convexity \( \min_i (m_i \cdot n_i) \geq 1 \). So if \( T\Phi = 0 \), equation (4.6) gives

\[
\|\Phi\|^2 \lesssim \sum_{i=1}^{n} (m_i \cdot n_i) \|\Phi_i\|_2^2 = 0,
\]

so \( T : X \to Y \) is injective. We now prove the remaining part of theorem 4.1.

**Proof of theorem 4.1.** If we assume that the result is false, then there must exist a sequence \( \{\Phi_m\}_{m \geq 1} \) in \( X \) for which \( \|\Phi_m\| = 1 \) but \( T\Phi_m \to 0 \) in \( Y \). We first establish that this sequence contains a subsequence that converges locally uniformly to zero.

Using lemma 3.1 and going to a subsequence if necessary, we may assume that the \( \Phi_m \to \Phi \) locally uniformly for some \( \Phi \in X \) with \( \|\Phi\| \leq 1 \). It quickly follows that \( T\Phi_m \to T\Phi \) locally uniformly. So for any bounded subset \( D \subset \mathbb{R}^2 \)

\[
\int_{D} |T\Phi(\lambda)|^2 d\lambda = \lim_{m \to \infty} \int_{D} |T\Phi_m(\lambda)|^2 d\lambda \leq \lim_{m \to \infty} \|T\Phi_m\|^2 = 0.
\]

So \( T\Phi(\lambda) = 0 \) on \( D \), and since \( D \) was arbitrary we must have \( T\Phi = 0 \), hence \( \Phi = 0 \) by lemma 4.7. Hence, we are free to assume \( \Phi_m \to 0 \) locally uniformly and from this it follows that \( T\Phi_m \to 0 \) locally uniformly also.

From the integral identity (4.2) in lemma 4.4 and our discussion following lemma 4.7 we see that

\[
\|\Phi_m\|^2 \lesssim \text{Re} \int_{\mathbb{R}^2} \left[ T\Phi_m(\lambda) \cdot L\Phi_m(-\lambda) - LT\Phi_m(\lambda) \cdot \Phi_m(-\lambda) - 2MT\Phi_m(\lambda) \cdot \Phi_m(-\lambda) \right] \frac{d\lambda}{|\lambda|}
\]

\[
\lesssim |\langle T\Phi_m, \Phi_m \rangle| + |\langle LT\Phi_m, \Phi_m \rangle| + 2|\langle M\Phi_m, \Phi_m \rangle|. \tag{4.7}
\]

The first term clearly tends to zero using Cauchy–Schwarz. Also, the third term is

\[
\left| \sum_{i=1}^{n} \int_{\mathbb{R}^2} (T\Phi_m)_{\langle i \rangle}(\lambda) \left[ \left( i \frac{1}{|\lambda|} \right) \frac{1}{|\lambda^2|} \frac{1}{1} \right] (\Phi_m)(-\lambda) d\lambda \right| \lesssim |\langle T\Phi_m, \Phi_m \rangle| \lesssim \|T\Phi_m\| \|\Phi_m\|,
\]
where we used Cauchy–Schwarz and the fact that the absolute value of the term in square brackets is bounded uniformly in \( \lambda \). So this term also vanishes in the limit. Finally, going on to polar coordinates \((r, \theta)\), noting \( \mathcal{L} \equiv \partial / \partial r \) and integration by parts, the second term in (4.7) is

\[
\left| \int_{\mathbb{R}^2} \mathcal{L} \Phi_m(\lambda) \cdot \Phi_m(-\lambda) \, d\lambda \right| = \left| \int_0^{2\pi} \int_0^\infty \frac{\partial}{\partial r} [T\Phi_m(\lambda)] r \Phi_m(-\lambda) \, dr \, d\theta \right|
\]

\[
= \left| \int_{\mathbb{R}^2} T\Phi_m(\lambda) \cdot \Phi_m(-\lambda) \frac{d\lambda}{|\lambda|} \right| + \left| \int_{\mathbb{R}^2} T\Phi_m(\lambda) \cdot \mathcal{L} \Phi_m(-\lambda) \, d\lambda \right|
\]

\[
\leq \left| \int_{\mathbb{R}^2} T\Phi_m(\lambda) \cdot \Phi_m(-\lambda) \frac{d\lambda}{|\lambda|} \right| + \| \langle T\Phi_m, \mathcal{L}\Phi_m \rangle \|.
\]

Clearly,

\[
\left| \int_{|\lambda| > 1} T\Phi_m(\lambda) \cdot \Phi_m(-\lambda) \frac{d\lambda}{|\lambda|} \right| \leq \int_{|\lambda| > 1} |T\Phi_m(\lambda) \cdot \Phi_m(-\lambda)| \, d\lambda \leq \| T\Phi_m \| \| \Phi_m \|
\]

and since \( T\Phi_m \rightarrow 0 \) and \( \Phi_m \rightarrow 0 \) locally uniformly we have

\[
\lim_{m \rightarrow \infty} \int_{|\lambda| < 1} T\Phi_m(\lambda) \cdot \Phi_m(-\lambda) \frac{d\lambda}{|\lambda|} = 0.
\]

So the second term in (4.7) also vanishes in the limit. To conclude

\[
\lim_{m \rightarrow \infty} \| \Phi_m \| = 0,
\]

which contradicts the fact \( \| \Phi_m \| = 1 \). So \( T : X \rightarrow Y \) is bounded below, so \( R(T) \subset Y \) is closed.  

\section{5. New numerical treatments}

The technical results established in §4 can be used to create new numerical treatments for the solution to the Dirichlet–Neumann map defined by the boundary value problem (2.1).

It is first beneficial to construct a weak version to the problem \( T\Phi = \Psi \), where \( \Psi \) is given and \( \Phi \in X \) is unknown. Consider the linear and bilinear forms defined by

\[
\ell : X \rightarrow \mathbb{C} : \Phi' \mapsto \langle \Psi, T\Phi' \rangle,
\]

\[
a : X \times X \rightarrow \mathbb{C} : (\Phi, \Phi') \mapsto \langle T\Phi, T\Phi' \rangle.
\]

A weak version of the problem \( T\Phi = \Psi \) is to find \( \Phi \in X \) such that

\[
a(\Phi, \Phi') = \ell(\Phi') \quad \text{for all} \quad \Phi' \in X.
\]  

(5.1)

We can quickly establish the following result.

\textbf{Theorem 5.1.} The weak problem (5.1) admits a unique solution.

\textit{Proof.} This will be an immediate corollary of the Lax–Milgram lemma if we can establish the following:

\begin{itemize}
  \item[(a)] \( \ell : X \rightarrow \mathbb{C} \) is bounded;
  \item[(b)] \( a : X \times X \rightarrow \mathbb{C} \) is bounded; and
  \item[(c)] \( a : X \times X \rightarrow \mathbb{C} \) is coercive.
\end{itemize}

Parts (a) and (b) are established using Cauchy–Schwarz. For (b)

\[
|a(\Phi, \Phi')| = |\langle T\Phi, T\Phi' \rangle| \leq \| T\Phi \| \| T\Phi' \| \| \Phi \| \| \Phi' \|.
\]

For (a), we find \( |\ell(\Phi)| \leq \| T\Phi \| \| \Psi \| \| \Phi \| \), since \( \Psi \in Y \) by the regularity of the Dirichlet data. To show that \( a \) is coercive one appeals to theorem 4.1:

\[
|a(\Phi, \Phi)| = \| T\Phi \|^2 \geq \| \Phi \|^2.
\]

So by the Lax–Milgram lemma, the problem (5.1) has a unique solution. \( \blacksquare \)
This well-posed weak problem lends itself to a simple Galerkin scheme. Let \( X_N \subset X \) be an \( N \)-dimensional subspace of \( X \) with basis \( \{ e_m \}_{m=1}^N \). Consider now the finite-dimensional problem: find \( \Phi_N \in X_N \) such that

\[
 a(\Phi_N, e_i) = \ell(e_i), \quad i = 1, \ldots, N.
\]

This problem is also well posed and a solution \( \Phi_N = \sum_{i=1}^N c_i e_i \) can be found by solving the linear system of equations

\[
 \sum_{j=1}^N A_{ji} c_j = \ell_i, \quad i = 1, \ldots, N,
\]

where \( A_{ji} = a(e_j, e_i) \) and \( \ell_i = \ell(e_i) \). The error between the solution to the full problem \( \Phi \) and \( \Phi_N \) is controlled by Céa’s lemma.

The explicit integrations that take place to compute \( A_{ji} \) and \( \ell_i \) involve an integral over \( \mathbb{R}^2 \) of an holomorphic function. Computational experiments show that these can be computed very quickly and efficiently using standard Monte Carlo integration methods. A more detailed numerical analysis of this new numerical approach will be presented elsewhere.

It is interesting that our approach gives an opportunity to use results from signal analysis. For example, it is known that for symmetric convex polygons \( M \) the Paley–Wiener space \( PW^2_M \) admits an explicit Riesz basis (see Lyubarskii [13]), so the same is also true for \( X \).

6. Ehrenpreis-type representation

In this section, we provide a new integral representation for the solution to the boundary value problem (2.1). This integral representation is a concrete realization of the fundamental principle of Ehrenpreis, being a superposition of exponential solutions.

**Theorem 6.1.** For \( x \in \Omega \) define the function

\[
 q(x) = \frac{1}{8\pi^2} \sum_{i=1}^n \int_{Z_i} e^{i\mu \cdot x} \rho_i(\mu) \, dv_i(\mu),
\]

where the \( \{ \rho_i \}_{i=1}^n \) are as in equation (2.3) and \( Z_i \subset Z(\Delta) \) and \( dv_i(\mu) \) are defined by

\[
 Z_i = \left\{ \mu \in Z(\Delta) : \mu = -R_{i}^t \left( \frac{\lambda}{|\lambda|} \right), \, \lambda \in \mathbb{R}^2 \right\}
\]

and

\[
 dv_i(\mu)|_{Z_i} = \frac{d(R_i \mu) \wedge (R_i \mu)_2}{i(R_i \mu)_3} \bigg|_{Z_i} \equiv \frac{d\lambda}{|\lambda|}.
\]

Then \( q \in C^\infty(\Omega) \) and \( q \) is harmonic in \( \Omega \). If the \( \{ \rho_i \}_{i=1}^n \) satisfy the global relation, then \( q \) is the unique solution to (2.1).

It is clear that this integral representation is precisely in accordance with the fundamental principle of Ehrenpreis. Formally differentiating under the integral sign gives

\[
 -\Delta q(x) = \frac{1}{8\pi^2} \sum_{i=1}^n \int_{Z_i} (\mu_1^2 + \mu_2^2 + \mu_3^2) e^{i\mu \cdot x} \rho_i(\mu) \, dv_i(\mu) = 0
\]

since each \( Z_i \) is contained in \( Z(\Delta) \). A large proportion of the proof to theorem 6.1 will be spent showing that \( q \) has the appropriate behaviour on the boundary of \( \Omega \). For economy of presentation we assume the Dirichlet data has sufficient regularity so that \( q \in C(\bar{\Omega}) \).
Proof of theorem 6.1. First we note that if \( x \in \Omega \) then the integrals converge exponentially fast. Indeed, if \( y \in \Sigma_i \) and \( \mu \in \mathbb{Z}_i \) then

\[
|e^{\mu \cdot x} \rho_j(\mu)| = \int_{\Sigma_i} e^{\mu \cdot (x-y)} \left[ \frac{\partial q}{\partial n}(y) + i(\mu \cdot n_i) f_i(y) \right] d\sigma(y)
\]

and the absolute value of the exponential term is

\[
\exp(\text{Im} \rho_j^* \cdot (x-y)) = \exp((\text{Re} \rho_j^* \cdot (x-y))) = \exp(\text{Re} \cdot n_i \cdot (x-y)).
\]

As \( y \in \Sigma_i \) and \( x \in \Omega \), the vector \( x - y \) points into the polyhedron from the \( i \)th face, and consequently the \( n_i \) component of this vector is strictly negative. So we have the desired exponential convergence. This allows us to differentiate \( q \) under the integral sign, and since \( \mu \in \mathbb{Z}(\Delta) \) we have \( \Delta q = 0 \) for \( x \in \Omega \). It remains to show that \( q \) has the desired behaviour on the boundary of \( \Omega \). Fix \( x_i \in \Sigma_i \) and consider \( q(x_i) \). For \( j \neq i \), let us consider

\[
\int_{Z_i} e^{\mu \cdot x_i} \rho_j(\mu) \ d\nu_j(\mu) = \int_{\mathbb{R}^2} e^{-i\gamma \lambda} \rho_j(-\Delta \chi \lambda) \ d\lambda \equiv \int_{\mathbb{R}^2} C_{ij}(\lambda) \ d\lambda
\]

By applying arguments entirely analogous to those used to prove lemma 4.4 one can show that

\[
\int_{\mathbb{R}^2} C_{ij}(\lambda) \ d\lambda = \int_{\mathbb{R}^2} C_{ij}(\Delta \chi \lambda) \ d\lambda
\]

The term on the right-hand side can be rewritten as

\[
\int_{Z_i} e^{-i\mu \cdot x_i} \rho_j(\mu) \ d\nu_j(\mu).
\]

So, upon using the global relation we find

\[
\sum_{j \neq i} \int_{Z_i} e^{\mu \cdot x_i} \rho_j(\mu) \ d\nu_j(\mu) = \sum_{j \neq i} \int_{Z_i} e^{-i\mu \cdot x_i} \rho_j(-\mu) \ d\nu_j(\mu)
\]

\[
= \int_{Z_i} e^{-i\mu \cdot x_i} \left( \sum_{j \neq i} \rho_j(-\mu) \right) \ d\nu_j(\mu)
\]

\[
= - \int_{Z_i} e^{-i\mu \cdot x_i} \rho_i(-\mu) \ d\nu_j(\mu).
\]

Hence

\[
q(x_i) = \frac{1}{8\pi^2} \int_{Z_i} e^{i\mu \cdot x_i} \rho_i(\mu) \ d\nu_j(\mu) - \frac{1}{8\pi^2} \int_{Z_i} e^{-i\mu \cdot x_i} \rho_i(-\mu) \ d\nu_j(\mu).
\]

Note that

\[
\int_{Z_i} \rho_i(\mu) \ d\nu_j(\mu) = \int_{\mathbb{R}^2} e^{-i\gamma \lambda \cdot \chi_i - y} \left[ \frac{\partial q}{\partial n}(y) - i(\hat{e} \cdot \hat{n}) f_i(y) \right] d\sigma(y) \ d\lambda
\]

and also

\[
\int_{Z_i} \rho_i(-\mu) \ d\nu_j(\mu) = \int_{\mathbb{R}^2} e^{-i\gamma \lambda \cdot \chi_i - y} \left[ \frac{\partial q}{\partial n}(y) + i(\hat{e} \cdot \hat{n}) f_i(y) \right] d\sigma(y) \ d\lambda
\]

\[
= \int_{\mathbb{R}^2} e^{i\gamma \lambda \cdot \chi_i - y} \left[ \frac{\partial q}{\partial n}(y) - |\lambda| f_i(y) \right] d\sigma(y) \ d\lambda
\]

\[
= \int_{\mathbb{R}^2} e^{-i\gamma \lambda \cdot \chi_i - y} \left[ \frac{\partial q}{\partial n}(y) - |\lambda| f_i(y) \right] d\sigma(y) \ d\lambda
\]
where in the final line we made the substitution $\lambda' = -\lambda$ and noted that $R_i(x_i - y) \perp e_z$. In conclusion, we find

$$q(x_i) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left[ \int_{\Sigma_i} e^{-i\lambda \cdot R_i(x_i - y)} f_i(y) \, d\sigma(y) \right] d\lambda$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left[ \int_{\Sigma_i} e^{-i\lambda \cdot (X_i - Y)} \psi_i^*(f_i(Y)) \, dY \right] d\lambda$$

$$= \psi_i^*(f_i)(X_i)$$

$$= f_i(x_i).$$

where in the second line we defined $\psi_i(X_i) = x_i$, $\psi_i(Y) = y$, and in the third, we used the Fourier inversion theorem (so the equality is to be interpreted for almost all $x_i \in \Sigma_i$). ■

7. Conclusion

In this paper, we have made the first extension of Fokas’ unified method for elliptic boundary value problems to three dimensions. This is important not just from a theoretical perspective, but more so from a practical perspective. Important problems in PDE theory are more often than not set in three dimensions, reflecting the world we live in.

It has been shown that the relevant global relation completely classifies the relevant Dirichlet–Neumann map, and that it can be used effectively to compute the unknown Neumann boundary values. Our theoretical results have also given rise to new numerical treatments that can be used to obtain approximate solutions to the underlying boundary value problem.

We have also given a new integral representation for the solution to Laplace’s equation on convex polyhedra. This integral representation is in direct accordance with the fundamental principle of Ehrenpreis in that it is a superposition of exponential solutions. The integrals within this representation converge exponentially fast at interior points of the domain and can be evaluated quickly using standard numerical integrators.

While our analysis has only concerned Laplace’s equation, one only needs a small perturbation so that these arguments can be used for the Helmholtz and modified-Helmholtz equations. In this variety, $Z(\Delta)$ is replaced by

$$Z(\beta) = \{ \lambda \in \mathbb{C}^3 : \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \beta^2 = 0 \},$$

where $\beta^2$ is negative (positive) for Helmholtz (modified–Helmholtz). We note $Z(0) = Z(\Delta)$. If $T(\beta)$ corresponds to the linear operator arising from the global relation, we also have $T = T(0)$. A natural way to analyse $T(\beta)$ is to write

$$T(\beta) = T(0) + K(\beta),$$

where $K(\beta) := T(\beta) - T(0)$. It can be shown that $K(\beta) : X \to Y$ is compact, so that $T(\beta)$ is a compact perturbation from $T(0)$. The relevant results for $T(\beta)$ can now be obtained by appealing to standard results from perturbation theory of linear operators [14]. A more detailed treatment of these results is presented in Ashton [15], as well as new Ehrenpreis-type integral representations.

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Appendix A

We must elaborate on the validity of deformation of the region of integration which was used to establish the integral identities appearing in lemma 4.4. We will only demonstrate the veracity of
this claim for (4.3a) as the analysis for (4.3b) is the same. We are required to show

\[ \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda}{|\lambda|} = \int_{\mathbb{R}^2} A_{ij}(-\Delta^t_{ij}\lambda) \frac{d\lambda}{|\lambda|}. \]  

We will make extensive use of the exterior calculus, so we write \(d\lambda = d\lambda_1 \wedge d\lambda_2\). It is useful to make use of the \(\xi\)-coordinates defined by

\[ \lambda_1 = \xi_1^2 - \xi_2^2 \]

and

\[ \lambda_2 = 2\xi_1\xi_2. \]

In these coordinates, we see that \(|\lambda| = \xi_1^2 + \xi_2^2\). If \(\lambda = |\lambda|(\cos \theta, \sin \theta)\) and \(\xi = |\xi|(\cos \phi, \sin \phi)\), we note the relationships

\[ \tan \theta = \tan 2\phi, \quad |\lambda| = |\xi|^2. \]

So for \(\lambda \in \mathbb{Z}_+(\Delta)\), we have the parametrization

\[ \lambda_1 = \xi_1^2 - \xi_2^2, \]

\[ \lambda_2 = 2\xi_1\xi_2 \]

and

\[ \lambda_3 = i\xi_1^2 + i\xi_2^2. \]

The vector \(\xi = (\xi_1, \xi_2)^T\) is called the 2-spinor associated with the isotropic vector \(\lambda = (\lambda_1, \lambda_2, \lambda_3)^T\) (i.e. \(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0\)). An elementary account of spinors is given in Cartan [16]. Spinor coordinates possess the following salient feature: if \(\xi \mapsto \xi' = U(n, \theta)\xi\) where

\[ U(n, \theta) = I \cos \left( \frac{1}{2} \theta \right) - i(n \cdot \sigma) \sin \left( \frac{1}{2} \theta \right) \in \text{SU}(2), \]

then the corresponding isotropic vector transforms as \(\lambda \mapsto \lambda' = R(n, \theta)\lambda\), where \(R(n, \theta) \in \text{SO}(3)\) is the rotation about \(n\) through an angle \(\theta\). Here \(\sigma_i\) is the \(i\)th Pauli matrix and \(\sigma = (\sigma_1, -\sigma_3, \sigma_2)\).

In the spinor coordinates, we find

\[ 4 d\xi_1 \wedge d\xi_2 = \frac{d\lambda_1 \wedge d\lambda_2}{|\lambda|}, \]

so it follows that

\[ \int_{\mathbb{R}^2} A_{ij}(\lambda) \frac{d\lambda_1 \wedge d\lambda_2}{|\lambda|} = 2 \int_{\mathbb{R}^2} A_{ij}(\lambda(\xi)) \frac{d\xi_1 \wedge d\xi_2}{2} \equiv 2 \int_{\mathbb{R}^2} G_{ij}(\xi) \frac{d\xi_1 \wedge d\xi_2}{2}, \]

where each of the integrals runs over \(\mathbb{R}^2\) and is \(G_{ij}\) defined accordingly. Now introduce \(f_{ij} : \mathbb{R}^2 \to \mathbb{C}^2\) with \(f_{ij}(\xi) = U_{ij}V_{ij}\xi\) (no sum) where \(U_{ij} \in \text{SU}(2)\) corresponds to the rotation matrix \(\Delta^t_{ij} \in \text{SO}(3)\) and \(V_{ij} \in \text{SU}(2)\) corresponds to the rotation matrix \(R(n_{ij}, \pi) \in \text{SO}(3)\) where \(n_{ij}\), at this

\[ \text{Here } \sigma_2 \text{ and } -\sigma_3 \text{ swap roles. This is because our parametrization of the isotropic vector } \lambda \text{ is slightly different, the standard being } (\lambda_1, \lambda_2, \lambda_3) = (\xi_1^2 - \xi_2^2, i(\xi_1^2 + \xi_2^2), -2\xi_1\xi_2). \]
stage, is any unit vector in the \((x, y)\)-plane. If we set \(\omega_{ij} = G_{ij}(\xi) \, d\xi_1 \wedge d\xi_2\), then our initial claim is equivalent to

\[
\int_{\mathbb{R}^2} \omega_{ij} = \int_{\mathbb{R}^2} f_{ij}^* \omega_{ij}. \tag{A 2}
\]

Indeed, the left-hand side of (A 2) is just one-half the left-hand side of (A 1) and the integrand on the right-hand side is

\[
f_{ij}^* \omega_{ij}(\xi_1, \xi_2) = G_{ij}(U_{ij}V_{ij}^*) \det(U_{ij}V_{ij}) \, d\xi_1 \wedge d\xi_2
= G_{ij}(U_{ij}V_{ij}) d\xi_1 \wedge d\xi_2
= \lambda_{ij}(\Delta_{ij}^t \Phi(n_{ij}, \pi) \lambda(\xi)) \frac{d\lambda_1(\xi) \wedge d\lambda_2(\xi)}{4|\lambda(\xi)|}, \tag{A 3}
\]

where we used the fact that if \(\xi \mapsto f_{ij}(\xi)\) then the corresponding isotropic vector transforms via the map \(\lambda \mapsto \Delta_{ij}^t \Phi(n_{ij}, \pi) \lambda\). Labelling the entries of \(\Phi(n_{ij}, \pi)\) by \(\alpha_{mn}\), we have

\[
\int_{\mathbb{R}^2} f_{ij}^* \omega_{ij} = 2 \int_{\mathbb{R}^2} A_{ij} \left[ \Delta_{ij} \left( \begin{array}{ccc} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & -1 \end{array} \right) \frac{1}{\lambda_1 \lambda_2} \right] \frac{d\lambda_1 \wedge d\lambda_2}{4|\lambda|} = \frac{1}{2} \int_{\mathbb{R}^2} A_{ij}(-\Delta_{ij}^t \lambda) \frac{d\lambda}{|\lambda|},
\]

where in the final line we made the change of variables \(\lambda \mapsto -\alpha^t \lambda\), where \(\alpha\) is the \(2 \times 2\) orthogonal matrix with entries \(\alpha_{mn}\). As claimed, the right-hand side is precisely one half of the right-hand side of (A 1), so it is enough for us to establish (A 2).

Since each \(G_{ij}\) is an entire function of \(\xi\) the two-form \(\omega_{ij}\) is closed:

\[
d\omega_{ij} = \frac{\partial G_{ij}}{\partial \xi_1} d\xi_1 \wedge d\xi_1 \wedge d\xi_2 + \frac{\partial G_{ij}}{\partial \xi_2} d\xi_2 \wedge d\xi_1 \wedge d\xi_2 = 0.
\]

Now let \(h_{ij} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{C}^2\) be a smooth homotopy between the identity and \(f_{ij}\), i.e. \(h_{ij}(0, \xi) = \xi\) and \(h_{ij}(1, \xi) = f_{ij}(\xi)\). If \(B_R\) is the ball of radius \(R\) in \(\mathbb{R}^2\), we have by Stokes theorem\(^2\) and that \(d(h_{ij}^* \omega_{ij}) = h_{ij}^*(d\omega_{ij}) = 0\)

\[
0 = \int_{B_R \times [0, 1]} d(h_{ij}^* \omega_{ij}) = \int_{B_R \times [0, 1]} h_{ij}^* \omega_{ij} + \int_{B_R \times [1]} h_{ij}^* \omega_{ij} - \int_{B_R \times [0]} h_{ij}^* \omega_{ij} = \int_{B_R \times [0, 1]} h_{ij}^* \omega_{ij} + \int_{B_R} f_{ij}^* \omega_{ij} - \int_{B_R} \omega_{ij}.
\]

If the homotopy is such that \(h_{ij}^* \omega_{ij}\) is rapidly decaying as \(|\xi| \rightarrow \infty\) (or equivalently as \(|\lambda| \rightarrow \infty\)) uniformly on \([0, 1]\) then we obtain

\[
\lim_{R \rightarrow \infty} \int_{B_R} \omega_{ij} = \lim_{R \rightarrow \infty} \int_{B_R} f_{ij}^* \omega_{ij}, \tag{A 4}
\]

which is the result we wish to prove. So it remains to check that such a homotopy exists. Before we prove this result some preparation is in order.

To make our lives easier, we will work on a dense subspace of \(X\), denoted \(\mathcal{X}\), in which each \(\Phi_i\), \(1 \leq i \leq n\), is the Fourier transform of a smooth function with compact support in \(Q_i\). This is preferable because the functions \(\Phi_i(\lambda)\) will then have rapid decay as \(|\lambda| \rightarrow \infty\) in \(\mathbb{R}^2\). If we label the expression in (A 1) by \(\ell_{ij}(\Phi, \Phi)\) then it is straightforward to show that \(\ell_{ij} : X \times X \rightarrow \mathbb{C}\) is bilinear and continuous. Consequently, for each \(\Phi \in X\) we can pick a bounded sequence \(\{\Phi_m\}_{m \geq 1}\) in \(\mathcal{X}\) for

\[\int_M d\omega = \int_{\partial M} \omega \text{ with } M = B_R \times [0, 1]\text{ so } \partial M = (\partial B_R \cup [0, 1]) \cup (B_R \cup \partial [0, 1]).\]
which
\[ \ell_{ij}(\Phi, \Phi) = \lim_{m \to \infty} \ell_{ij}(\Phi_m, \Phi_m), \]
so it suffices to work on this dense subspace. Using the Fourier inversion theorem, the components of the elements of \( X \) take the form
\[ \Phi_i(\lambda) = \int_{Q_i} e^{-i\lambda \cdot X} \Phi_i(-X) \, dX \quad (A \, 5) \]
with \( \Phi_i \in C_c^\infty(Q_i) \) for \( 1 \leq i \leq n \). Given some \( \epsilon > 0 \), we can use a simple partition of unity argument to write \( \Phi_i = \sum_k \hat{\Phi}^k_i \), where \( \text{supp} \hat{\Phi}^k_i \) is contained in a ball in the interior of \( Q_i \) with radius less than \( \epsilon \). Writing \( \Phi = \sum_k \Phi^k \in X \), where the \( i \)th component of \( \Phi^k \) is \( \Phi^k_i \), we have by bilinearity
\[ \ell_{ij}(\Phi, \Phi) = \sum_k \sum_l \ell_{ij}(\Phi^k, \Phi^l), \]
so without loss of generality we need only prove our result when components of \( \Phi \) are the Fourier transform of smooth functions whose support is contained inside arbitrarily small balls.

We shall use a homotopy of the form
\[ h_{ij}(\epsilon, \xi) = U_{ij}(\epsilon)V_{ij}(\epsilon)\xi, \]
where \( U_{ij}(0) = I, U_{ij}(1) = U_{ij} \) and \( V_{ij}(\epsilon) \in SU(2) \) corresponds to the rotation matrix \( R(n_{ij}, \epsilon \pi) \). This implies that \( h(0, \xi) = \xi \) and \( h(1, \xi) = f_{ij}(\xi) \). For simplicity, we set \( \xi_{ij} = h_{ij}(\epsilon, \xi) \), suppressing the \( i,j \)-dependence. In terms of the \( \lambda \)-coordinates, the isotropic vector transforms as
\[ \lambda \mapsto \lambda_{ij} = \Delta_{ij}(\epsilon)R(n_{ij}, \epsilon \pi)\lambda, \]
where \( \Delta_{ij}(0) = I, \Delta_{ij}(1) = \Delta_{ij}^t \). By our previous discussion, we would like to show that for some choice of \( U_{ij}(\epsilon) \) the term \( U_{ij}(\epsilon) \Delta_{ij}^t \) has rapid decay as \( |\lambda| \to \infty \) for each \( \epsilon \in [0, 1] \).

Using the identity in (A 5), we may write \( U_{ij}(\epsilon) \Delta_{ij}^t \) on \( \partial B_R \times [0, 1] \) in terms of the \( \lambda \)-coordinates
\[ \int_{Q_i} \int_{Q_j} dX \, dY \exp \left[ i \left( m_i - m_j + R_i^\epsilon \left( X^i_0 \right) - R_j^\epsilon \left( Y^j_0 \right) \right) \cdot R_i^\epsilon \lambda \right] \times \left\{ (-iX_1 - iX_2)\Phi_i(-X)\Phi_j(-Y) \right\} \frac{d\lambda}{|\lambda|}. \]

By our previous discussion, we may assume without loss of generality that the \( X \) - and \( Y \)-integrals run over arbitrarily small balls contained in the interior of \( Q_j \) and \( Q_i \), respectively. The large \( |\lambda| \) behaviour of the integrand is determined by the exponential term in this expression: if absolute value of the exponential term remains bounded then the integrand will have rapid decay as \( |\lambda| \to \infty \) for each \( \epsilon \in [0, 1] \). In addition, since the \( X \)- and \( Y \)-integrals can be taken over arbitrarily small balls, the vector
\[ w_{ij}(X, Y) := m_i + R_i^\epsilon \left( X^i_0 \right) - m_j - R_j^\epsilon \left( Y^j_0 \right), \]
representing an orientated line from a point \( \psi_j(Y) \) on the \( j \)th face to a point \( \psi_i(X) \) on the \( i \)th face is approximately constant. The absolute value of the exponential term is given by
\[ \exp(-w_{ij}(X, Y) \cdot \text{Im} R_i^\epsilon \lambda_{ij}), \]
for \( \epsilon \in [0, 1] \). Now the map
\[ \epsilon \mapsto \text{Im} R_i^\epsilon \lambda_{ij} \equiv R_i^\epsilon \Delta_{ij}^t(\epsilon)R(n_{ij}, \epsilon \pi)\epsilon e_z, \]
corresponds to path on the unit sphere which continuously transforms the vector \( R_i^\epsilon e_z = n_i \) (\( \epsilon = 0 \)) to the vector \( -R_i^\epsilon e_z = -n_i \) (\( \epsilon = 1 \)) (Figure 2). We can choose a homotopy so that this path is such that that the \( w_{ij}(X, Y) \) component of these vectors is always positive. Indeed, for fixed \( (X, Y) = (X_0, Y_0) \) and fixed \( \theta \) we can certainly do this as we are free to choose \( n_j \) and \( U_{ij}(\epsilon) \) and hence \( \Delta_{ij}(\epsilon) \). So such a homotopy certainly exists for fixed \( (X, Y) = (X_0, Y_0) \). However, because \( w_{ij}(X, Y) \) depends continuously on \( X \) and \( Y \), the same must hold for \( (X, Y) \) is sufficiently small neighbourhoods of
Figure 2. The vector $w_{ij}(X, Y)$ and the path $\epsilon \mapsto R^j_i \Delta_i^j(\epsilon) R(e_i, \pi \epsilon)e_z$.

$(X_0, Y_0)$ and as we are free to choose the components of $\Phi$ to have arbitrarily small support, we are done. In conclusion, the homotopy $h_{ij}$ can be chosen so that

$$\lim_{R \to \infty} \int_{\partial B_R \times [0,1]} h_{i,j}^\ast \omega_{ij} = 0,$$

from which (A 4), and hence (A 1), follows. We emphasize that the corresponding results for the remaining identity in lemma A 1 are entirely analogous, as is the result for $C_{ij}(\lambda)$ which appears in the proof to theorem 6.1.

References