Suppression of limit cycle oscillations using the nonlinear tuned vibration absorber

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The objective of this study is to mitigate, or even completely eliminate, the limit cycle oscillations in mechanical systems using a passive nonlinear absorber, termed the nonlinear tuned vibration absorber (NLTVA). An unconventional aspect of the NLTVA is that the mathematical form of its restoring force is not imposed a priori, as it is the case for most existing nonlinear absorbers. The NLTVA parameters are determined analytically using stability and bifurcation analyses, and the resulting design is validated using numerical continuation. The proposed developments are illustrated using a Van der Pol–Duffing primary system.

1. Introduction

Limit cycle oscillations (LCOs) are encountered in a number of real engineering applications, including aircraft [1,2], machine tools [3,4] and automotive disc brakes [5]. LCOs often limit the performance and can also endanger the safety of operation [6].

Active control strategies have been proposed as a means of counteracting LCOs [7–9]. These references have shown that active control can be used to raise the threshold above which LCOs occur. However, active control is also limited by its requirements in terms of energy or space for actuators. Furthermore, delay in the feedback loop can generate unexpected instabilities [10], whereas saturation of the actuators can limit the robustness of stability [11].

Passive vibration absorbers represent another alternative for mitigating undesired LCOs. Specifically, the linear tuned vibration absorber (LTVA), which comprises a small mass attached to the host system through a damper and a spring, has been widely studied in the literature [12–16]. In most of these works, the system under investigation is the classical Van der Pol
References [13,14] provide simple rules to properly tune the LTVA parameters, while references [15,16] study the post-bifurcation behaviour of the coupled system. In [17], a nonlinear damping element was added in parallel with the LTVA to decrease the maximum LCO amplitude. Other nonlinear vibration absorbers, including the autoparametric vibration absorber [18,19], the nonlinear energy sink (NES) [20–28] and the hysteretic tuned vibration absorber [29], have also been considered to increase the effectiveness of vibration attenuation. In particular, the NES exhibited three different mechanisms to suppress LCOs, namely complete, partial and intermittent suppression. These mechanisms were studied numerically [23], experimentally [24] and analytically [27].

The main idea of this study is to use the NLTVA for LCO suppression. This absorber, first introduced in reference [30], possesses a linear spring and a nonlinear spring whose mathematical form is determined according to the nonlinearity in the host system. Following references [12–16], the linear spring coefficient is determined to maximize the stable region of the trivial solution of the host system. Subsequently, the nonlinear spring is designed to ensure supercritical behaviour and to mitigate the LCOs in the post-critical range. A fundamental result of this paper is that, if properly designed, the linear and nonlinear springs of the NLTVA can complement each other giving rise to a very effective LCO suppression and management strategy. The example that will serve to validate the proposed developments is the Van der Pol–Duffing (VdPD) oscillator [31], which is a paradigmatic model for the description of self-excited oscillations. The bifurcation behaviour of the VdPD oscillator was studied in [32], whereas its stabilization using active control was proposed in [33,34].

The paper is organized as follows. Section 2 introduces the three design objectives pursued in this paper. In §3, optimal values for the linear parameters of the NLTVA are determined using stability analysis of the coupled system. Section 4 investigates the bifurcations occurring at the loss of stability and proposes an analytical tuning rule for the nonlinear coefficient of the NLTVA. In §5, the reduction of the LCO amplitude in the post-critical range is discussed. In §6, the analytical results are validated numerically using the MATCONT software, and the global behaviour of the system is also carefully discussed. The NLTVA is then compared with the NES, highlighting the better performance of the former absorber. Finally, conclusions are drawn in §7.

2. Problem formulation

The primary system considered throughout this work is the VdPD oscillator:

\[ m_1 q''_1 + c_1 (q^2_1 - 1)q'_1 + k_1 q_1 + k_{nl} q^3_1 = 0, \]

where \( m_1, c_1, k_1 \) and \( k_{nl} \) are the oscillator’s mass, damping and the coefficients of the linear and cubic springs, respectively. For instance, for an in-flow wing, the terms \( c_1 (q^2_1 - 1)q'_1 \) and \( k_{nl} q^3_1 \) would model the fluid–structure interaction and the structural nonlinearity, respectively. Stability analysis demonstrates that the trivial equilibrium point of the system loses stability when \( \mu_1 = c_1 / (2 \sqrt{k_1 m_1}) = 0 \). Loss of stability occurs through either a supercritical or a subcritical Hopf bifurcation. This latter scenario is dangerous, because stable large-amplitude LCOs can coexist with the stable equilibrium point [35].

The objective of this study is to mitigate, or even completely eliminate, the LCOs of the VdPD oscillator through the attachment of a fully passive nonlinear vibration absorber, termed the NLTVA [30]. One salient feature of the NLTVA compared with existing nonlinear absorbers is that the absorber’s load–deflection curve is not imposed a priori, but it is rather synthesized according to the nonlinear-restoring force of the primary system. The equations of motion of the coupled VdPD and NLTVA system are

\[
\begin{align*}
    m_1 q''_1 + c_1 (q^2_1 - 1)q'_1 + k_1 q_1 + k_{nl} q^3_1 + c_2 (q'_1 - q'_2) + G(q_d) &= 0, \\
    m_2 q''_2 + c_2 (q'_2 - q'_1) - G(q_d) &= 0,
\end{align*}
\]
where \( m_2 \) and \( c_2 \) are the absorber’s mass and viscous damping, respectively. The NLTVA is assumed to have a generic smooth elastic force \( G(q_d) \), where \( q_d = q_1 - q_2 \) with \( G(0) = 0 \).

The design problem is as follows. The mass ratio \( \varepsilon = m_2/m_1 \) (and, hence, the absorber mass) is prescribed by obvious practical constraints; \( \varepsilon = 0.05 \) is considered in the numerical examples of this paper. The damping coefficient \( c_2 \) and the absorber’s stiffness \( G(q_d) \) should be determined so as to

1. maximize the region where the trivial equilibrium is stable by displacing the Hopf bifurcation towards large positive values of \( \mu_1 \) (figure 1a);
2. avoid a catastrophic bifurcation scenario by transforming the potentially subcritical Hopf bifurcation of the VdPD into a supercritical Hopf bifurcation of the coupled system (figure 1b); and
3. reduce the amplitude of the remaining LCOs (figure 1c).

These three design objectives are studied in detail in the next three sections.

3. Maximization of the stable region of the trivial solution

The first design objective is to stabilize the trivial solution of the VdPD oscillator for values of \( \mu_1 \) as great as possible. Because the stability of an equilibrium point of a nonlinear system is governed only by the local underlying linear system, the NLTVA should comprise a linear spring for increased flexibility, i.e. \( G(q_d) = k_2(q_d) + G_{nl}(q_d) \). The system of interest for the stability analysis is therefore

\[
\begin{align*}
    m_1 q_1'' - c_1 q_1' + k_1 q_1 + c_2 (q_1' - q_2') + k_2 (q_1 - q_2) &= 0 \\
    m_2 q_2'' + c_2 (q_2' - q_1') + k_2 (q_2 - q_1) &= 0
\end{align*}
\]

Introducing the parameters \( \omega_{n1}^2 = k_1/m_1, \omega_{n2}^2 = k_2/m_2, \mu_2 = c_2/(2m_2\omega_{n2}), \gamma = \omega_{n2}/\omega_{n1} \) and the dimensionless time \( \tau = t\omega_{n1} \), we transform the system into first-order differential equations

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_3 \\
    \dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    -1 & 2\mu_1 & -\gamma^2\varepsilon & -2\mu_2\gamma\varepsilon \\
    0 & 0 & 0 & 1 \\
    -1 & 2\mu_1 & -\gamma^2(1+\varepsilon) & -2\mu_2\gamma(1+\varepsilon)
\end{bmatrix}\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}
\]
Figure 2. Stability chart in the $\mu_1, \mu_2, \gamma$ space for $\varepsilon = 0.05$. The surface indicates the stability boundary. (Online version in colour.)

or in compact form $\dot{x} = Wx$, where $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_d$, and $x_4 = \dot{q}_d$. The dot indicates derivation with respect to the dimensionless time $\tau$.

As reported for a similar system in [15], the trivial solution of equation (3.2) is asymptotically stable if and only if the roots of the characteristic polynomial $\det(W - zI) = 0$ have negative real parts. The roots are computed by solving $z^4 + 2((\varepsilon + 1)\gamma \mu_2 - \mu_1)z^3 + ((\varepsilon + 1)\gamma^2 - 4\gamma \mu_1 \mu_2 + 1)z^2 + 2\gamma(\mu_2 - \gamma \mu_1)z + \gamma^2 = 0$, which is rewritten as $a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 = 0$. Figure 2 depicts the stability chart in the $\mu_1, \mu_2, \gamma$ space obtained from direct evaluation of the roots. The surface, which represents the stability boundary, peaks along the $\mu_1$-axis at point C, meaning that the trivial solution can no longer be stable beyond this point. The corresponding maximum value of $\mu_1$ is around 0.1, and the other parameters are $\mu_2 \approx 0.1$ and $\gamma \approx 1$.

In order to calculate analytically the coordinates of point C, thus defining optimal parameters for maximizing stability, the Routh–Hurwitz stability criterion is exploited. The characteristic polynomial has roots with negative real parts if and only if the coefficients $a_i > 0$, $i = 1, \ldots, 4$, $e_2 = (a_3a_2 - a_4a_1)/a_3 > 0$ and $e_3 = a_2a_1 - a_3a_0 > 0$, i.e.

\[
a_4 = 1 > 0
\]
\[
a_3 = 2((\varepsilon + 1)\gamma \mu_2 - \mu_1) > 0 \iff \mu_1 < (\varepsilon + 1)\gamma \mu_2
\]
\[
a_2 = (\varepsilon + 1)\gamma^2 - 4\gamma \mu_1 \mu_2 + 1 > 0 \iff \mu_1 < \frac{(\varepsilon + 1)\gamma^2 + 1}{4\gamma \mu_2}
\]
\[
a_1 = 2\gamma(\mu_2 - \gamma \mu_1) > 0 \iff \mu_1 < \frac{\mu_2}{\gamma}
\]
\[
a_0 = \gamma^2 > 0
\]
\[
e_2 = \frac{2((\varepsilon + 1)\gamma \mu_2 - \mu_1)((\varepsilon + 1)\gamma^2 - 4\gamma \mu_1 \mu_2 + 1) + 2\gamma(\gamma \mu_1 - \mu_2)}{2((\varepsilon + 1)\gamma \mu_2 - \mu_1)} > 0
\]

and
\[
e_3 = \frac{\gamma(\mu_2 - \gamma \mu_1)(2((\varepsilon + 1)\gamma \mu_2 - \mu_1)((\varepsilon + 1)\gamma^2 - 4\gamma \mu_1 \mu_2 + 1) + 2\gamma(\gamma \mu_1 - \mu_2))}{(\varepsilon + 1)\gamma \mu_2 - \mu_1}
- 2\gamma^2((\varepsilon + 1)\gamma \mu_2 - \mu_1) > 0.
\]

It is not trivial to interpret the physical meaning of these coefficients; however, they give important information regarding stability. Figure 3 represents two-dimensional sections of the
Figure 3. Sections of the stability chart for different values of $\mu_2$. (a) $\mu_2 = 0.07$; (b) $\mu_2 = 1/2\sqrt{\varepsilon/(1 + \varepsilon)} = 0.1091$; and (c) $\mu_2 = 0.12$. Shaded area: stable, clear area: unstable. Encircled numbers 2 and 4 indicate regions with 2 or 4 eigenvalues (two pairs of complex conjugate eigenvalues) with positive real parts.

three-dimensional stability chart for different values of $\mu_2$. The curves $a_3 = 0$ and $a_1 = 0$ intersect at point A, whose coordinates in the $(\mu_1, \gamma)$ space are $A = (\mu_2 \sqrt{1 + \varepsilon}, 1/\sqrt{1 + \varepsilon})$. Substituting $\gamma = 1/\sqrt{1 + \varepsilon}$ in $e_3 = 0$, point $B = (\varepsilon/(4\mu_2 \sqrt{1 + \varepsilon}), 1/\sqrt{1 + \varepsilon})$ is defined. As presented in figure 3a,c, A and B mark alternatively the maximal value of $\mu_1$ for stability. Points A and B coincide if and only if $\mu_2 = 1/2\sqrt{\varepsilon/(1 + \varepsilon)}$, which is therefore the optimal condition for stability; this scenario is illustrated in figure 3b.

Summarizing, optimal values of the linear parameters are

$$\gamma_{\text{opt}} = \frac{1}{\sqrt{1 + \varepsilon}} \quad \text{and} \quad \mu_{2\text{opt}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{1 + \varepsilon}}, \quad (3.10)$$

and the corresponding maximal value of $\mu_1$ that guarantees stability is

$$\mu_{1\text{max}} = \sqrt{\frac{\varepsilon}{2}}. \quad (3.11)$$

4. Enforcement of supercritical Hopf bifurcations through normal form analysis

The second design objective is to ensure the robustness of the trivial solution, i.e. no stable LCO can coexist with the stable equilibrium, as depicted in figure 1b. Because supercritical Hopf bifurcations are sought in the coupled system, a detailed investigation of the bifurcations occurring at the loss of stability is the main focus of this section. Because bifurcation characterization depends on the nonlinear coefficient of the NLTVA, this analysis will allow us to define the optimal value of this coefficient, whereas the linear coefficients of the NLTVA should remain close to their optimal values (3.10), i.e. $\gamma = 0.976$ and $\mu_2 = 0.109$ for $\varepsilon = 0.05$.

Another key element that remains to be determined is the mathematical expression of the NLTVA’s elastic force $G(q_d)$. Applying the ‘principle of similarity’ between the primary and secondary systems, first proposed for linear systems [36,37] and extended to forced nonlinear vibrations in [30], the mathematical form of the nonlinear spring of the NLTVA is chosen to be also cubic. Hence, the coupled system writes

$$m_1 q''_1 + c_1 (q_1^2 - 1)q'_1 + k_1 q_1 + k_{nl1} q_1^3 + c_2 (q'_1 - q'_2) + k_2 (q_1 - q_2) + k_{nl2} (q_1 - q_2)^3 = 0 \quad (4.1)$$

and

$$m_2 q''_2 + c_2 (q'_2 - q'_1) + k_2 (q_2 - q_1) + k_{nl2} (q_2 - q_1)^3 = 0.$$
Considering dimensionless coordinates, we transform the system into first-order differential equations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 2\mu_1 & -\gamma^2\epsilon & -2\mu_2\gamma\epsilon \\
0 & 0 & 0 & 1 \\
-1 & 2\mu_1 & -\gamma^2(1+\epsilon) & -2\mu_2\gamma(1+\epsilon)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

or in compact form \( \dot{x} = Wx + b \). The variables \( \alpha_3 = k_{nl1}/k_1 \) and \( \beta_3 = k_{nl2}/(k_1\epsilon) \) have been introduced in these equations.

When stability is lost, one or two pairs of complex conjugate eigenvalues of \( W \) leave the left half plane, which corresponds to single or double Hopf bifurcation, respectively. As reported in [15], a double Hopf bifurcation is likely to occur when \( \gamma = \gamma_{\text{opt}} \) and \( \mu_2 \leq \mu_{\text{opt}} \). This assertion is confirmed in figure 3a, where it is shown that losing stability through point A, the system winds up in a region with two pairs of complex conjugate eigenvalues with positive real part. Referring to the three-dimensional stability chart, the locus of double Hopf bifurcations is depicted by the black line in figure 2. The analysis of the double Hopf bifurcation is of practical importance because it can generate quasi-periodic solutions, which might compromise the robustness of the stable trivial solution. However, a detailed study of the double Hopf bifurcations is beyond the scope of this paper.

\((a)\) Single Hopf bifurcation

The analysis is first focused on the single Hopf bifurcation scenario for which \( W \) has a pair of complex conjugate eigenvalues with zero real part \( \lambda_{1,2} = k_1 \pm j\omega_1 \) and two other eigenvalues \( \lambda_3 \) and \( \lambda_4 \) with negative real parts. Vectors \( s_1, s_2, s_3 \) and \( s_4 \) are the corresponding eigenvectors. In order to decouple the linear part of the system, we define the transformation matrix

\[
T = [\text{Re}(s_1) \ \text{Im}(s_1) \ \text{Re}(s_3) \ \text{Im}(s_3)],
\]

where if \( \lambda_3 \) and \( \lambda_4 \) are real, \( \text{Re}(s_3) \) and \( \text{Im}(s_3) \) are substituted with \( s_3 \) and \( s_4 \), respectively. Applying the transformation \( x = Ty \), equation (4.2) becomes

\[
\dot{y} = Ay + T^{-1}b,
\]

where

\[
A = T^{-1}WT =
\begin{bmatrix}
k_1 & \omega_1 & 0 & 0 \\
-\omega_1 & k_1 & 0 & 0 \\
0 & 0 & \text{Re}(\lambda_3) & \text{Im}(\lambda_3) \\
0 & 0 & \text{Im}(\lambda_4) & \text{Re}(\lambda_4)
\end{bmatrix}
\]

The linear part of equation (4.4) is therefore decoupled.

Variables related to the bifurcation are separated from the other variables using centre manifold reduction [35]. Relations \( y_3 = h_3(y_1, y_2) \) and \( y_4 = h_4(y_1, y_2) \) are used to reduce the dimension of the system to two, while keeping the local dynamics intact. However, because the system has only cubic nonlinearities, \( h_3 \) and \( h_4 \) have only cubic and higher-order terms. Thus, for
our purpose, they can be considered identically equal to zero, i.e. \( y_3 \approx 0 \) and \( y_4 \approx 0 \). System (4.4) reduces to

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
k_1 & \omega_1 \\
-\omega_1 & k_1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
d_{130}y_1^3 + d_{121}y_1^2y_2 + d_{112}y_1y_2^2 + d_{103}y_2^3 \\
d_{230}y_1^3 + d_{221}y_1^2y_2 + d_{212}y_1y_2^2 + d_{203}y_2^3
\end{bmatrix}.
\]

(4.6)

Performing several transformations, namely transformation in complex form, near-identity transformation and transformation in polar coordinates, the bifurcation can be characterized through its normal form

\[
r = k_1r + \delta r^3,
\]

(4.7)

where \( \delta = (3d_{130} + d_{112} + d_{221} + 3d_{203})/8. \) Details of this standard procedure can be found in [38]. Equation (4.7) has solutions \( r_0 = 0 \) and \( r^* = \sqrt{-k_1/\delta}. \) The coefficient \( \delta \) can be expressed as a linear function of the nonlinear coefficients \( \alpha_3 \) and \( \beta_3 \)

\[
\delta = \delta_0(\epsilon, \gamma, \mu_1, \mu_2) + \delta_\alpha(\epsilon, \gamma, \mu_1, \mu_2)\alpha_3 + \delta_\beta(\epsilon, \gamma, \mu_1, \mu_2)\beta_3,
\]

(4.8)

where

\[
\delta_0 = -\frac{1}{2} \mu_1(\hat{t}_{12}(3t_{11}^2t_{21} + 2t_{11}t_{12}t_{22} + t_{12}^2t_{21}) + \hat{t}_{14}(3t_{11}^2t_{21} + 2t_{11}t_{12}t_{22} + t_{12}^2t_{21}))
\]

(4.9)

\[
\delta_\alpha = -\frac{3}{8}(t_{12}^2 + t_{12}^3)(\hat{t}_{12}t_{11} + \hat{t}_{14}t_{11} + t_{12}(\hat{t}_{22} + \hat{t}_{24}))
\]

(4.10)

and

\[
\delta_\beta = -\frac{3}{8}(t_{31}^2 + t_{32}^3)(\epsilon(\hat{t}_{12}t_{31} + \hat{t}_{14}t_{31} + t_{32}(\hat{t}_{22} + \hat{t}_{24})) + \hat{t}_{14}t_{31} + \hat{t}_{24}t_{32}).
\]

(4.11)

where \( t_{ij} \) and \( \hat{t}_{ij} \) are the elements of \( T \) and of its inverse, respectively.

From equation (4.7), we see that the bifurcation is supercritical if \( \delta < 0 \) and subcritical if \( \delta > 0 \). Our objective should therefore be to design the nonlinear spring \( \beta_3 \) of the NLTVA to impose negative values of \( \delta \). Ideally, this nonlinear tuning should be carried out in the vicinity of \( y_{opt} \) and \( \mu_{2opt} \), so as to maintain LCO onset at large values of \( \mu_1 \) (see §3). Figure 4 displays the values of the coefficients \( \delta_0, \delta_\alpha \) and \( \delta_\beta \) along the stability boundary in the \( \mu_1, \mu_2, \gamma \) space. Figure 5 considers a section of these plots for \( \mu_2 = 0.12 \) (\( \mu_2 = 1.1\mu_{2opt} \)) for which stability is lost through a single Hopf bifurcation.

For a system with no structural nonlinearities (\( \alpha_3 = \beta_3 = 0 \)), i.e. a LTVA connected to a classical VdP oscillator, \( \delta_0 \) is negative, and the system is always supercritical. We now consider a LTVA connected to a VdPD oscillator (\( \beta_3 = 0 \)), figures 4b and 5b evidence a symmetric behaviour for \( \delta_\alpha \), i.e. it is positive (negative) below (above) \( \gamma \approx y_{opt} \). This uncertainty on the sign of \( \delta_\alpha \) in the region of optimum tuning poses an important practical difficulty, because a supercritical bifurcation cannot confidently be enforced in this region. For positive values of \( \alpha_3 \), the solution to avoid a catastrophic bifurcation scenario is to detune the LTVA towards greater values of \( \gamma \), which guarantees negative values of \( \delta_\alpha \). However, figure 3c indicates that this detuning is associated with a significant decrease in the value of \( \mu_{1max} \). For instance, considering \( \gamma = 1 \) decreases \( \mu_{1max} \) by approximately 30%. Following a similar reasoning, we conclude that the LTVA should be detuned towards smaller values of \( \gamma \) for negative values of \( \alpha_3 \).

The NLTVA presents increased flexibility with respect to the LTVA, because \( \beta_3 \) represents an additional tuning parameter. However, figures 4c and 5c show that the sign of \( \delta_\beta \) in the optimal tuning region is as difficult to predict as for \( \delta_\alpha \). Interestingly, \( \delta_\alpha \) and \( \delta_\beta \) have consistently an opposite sign. It can be verified that \( \delta_\alpha/\delta_\beta \approx -0.05 \) close to \( y_{opt} \), thus, \( \delta_\alpha \alpha_3 + \delta_\beta \beta_3 \approx 0 \) if \( \beta_3 \approx 0.05 \alpha_3 \). In other words, the potentially detrimental effect of the structural nonlinearity of the VdPD on the bifurcation behaviour can be compensated through a proper design of the NLTVA’s nonlinearity. Unlike the LTVA, the NLTVA can therefore be designed to enforce supercritical bifurcations in the optimal tuning region.
Figure 4. Values of (a) $\delta_0$, (b) $\delta_\alpha$ and (c) $\delta_\beta$ along the stability boundary in the $\mu_1, \mu_2, \gamma$ space for $\epsilon = 0.05$. The colour indicates the value of the corresponding coefficient. To facilitate the visualization, two different views of the same surface are given (top and bottom plots). (Online version in colour.)

Figure 5. Values of (a) $\delta_0$, (b) $\delta_\alpha$ and (c) $\delta_\beta$ for $\mu_2 = 0.12$ and $\epsilon = 0.05$.

(b) Two intersecting single Hopf bifurcations

Although the investigation of the double Hopf bifurcation that occurs along the black line in figure 2 is beyond the scope of this paper, the separate analysis of the two intersecting single Hopf bifurcations gives already some insights into the dynamics. The eigenvalues of $W$ at point C are $\lambda_{1,2} = \pm j$ and $\lambda_{3,4} = \pm j/\sqrt{1 + \epsilon}$. By performing the analysis outlined in the previous section, first considering $\lambda_{1,2}$ as the critical eigenvalues and then $\lambda_{3,4}$, we obtain respectively

$$\delta_{1,2} = \frac{1}{8} \left( -\frac{\epsilon \sqrt{\epsilon}}{1 + \epsilon} + \frac{3 \sqrt{\epsilon}}{1 + \epsilon} \alpha_3 - \frac{3 (1 + \epsilon)}{\sqrt{\epsilon}} \beta_3 \right)$$

and

$$\delta_{3,4} = \frac{3}{8} \left( -\sqrt{\epsilon} \alpha_3 + \frac{(1 + \epsilon)^2}{\sqrt{\epsilon}} \beta_3 \right).$$
Figure 6. Maximal value of $\alpha_3$ below which supercritical bifurcations for the LTVA are guaranteed ($\varepsilon = 0.05$, $\beta_3 = 0$). (a) Positive values of $\alpha_3$; (b) negative values of $\alpha_3$. The dashed line corresponds to $\gamma = \gamma_{\text{opt}}$ and the dot correspond to $\mu_2 = \mu_{2\text{opt}}$. (Online version in colour.)

The remarkable feature of equations (4.12) and (4.13) is that the ratio between $\delta_\alpha$ and $\delta_\beta$ is constant

$$\frac{\delta_\alpha}{\delta_\beta} = -\frac{\varepsilon}{(1 + \varepsilon)^2}.$$ (4.14)

The important practical consequence of this result is that the effect of $\alpha_3$ can be locally entirely compensated by $\beta_3 = \varepsilon/(1 + \varepsilon)^2 \alpha_3$. It can be shown in fact that equation (4.14) is valid along the whole line of double Hopf bifurcations. We also note that equation (4.14) is in excellent agreement with the expression $\delta_\alpha/\delta_\beta \approx -0.05$ obtained in the single Hopf case for $\varepsilon = 0.05$.

Although equation (4.14) is exactly valid for the double Hopf bifurcations, it is only approximate in other cases. Thus, choosing $\beta_3 = \varepsilon/(1 + \varepsilon)^2 \alpha_3$ does not necessarily guarantee that the system will lose stability through a supercritical Hopf bifurcation. Considering that $\delta = \delta_0 + \delta_\alpha \alpha_3 + \delta_\beta \beta_3$ and imposing $\beta_3 = \varepsilon/(1 + \varepsilon)^2 \alpha_3$, it is possible to calculate the value of $\alpha_3$ such that $\delta$ becomes positive, i.e. the critical value of $\alpha_3$ for which the absorber fails to enforce supercriticality. This value is given by $\alpha_{3\text{cr}} = -\delta_0/\delta_\alpha$ in the case of a LTVA ($\beta_3 = 0$) and by $\alpha_{3\text{cr}} = -\delta_0/\delta_\alpha + \delta_\beta \varepsilon/(1 + \varepsilon)^2$ for an NLTVA with the proposed tuning rule (4.14). The critical values of $\alpha_3$ calculated in both cases are illustrated in figures 6 and 7, respectively. To avoid very large values in the vicinity of double Hopf bifurcations (where $\alpha_{3\text{cr}} \to \infty$), the colour maps were trimmed at 1.

Considering the LTVA (figure 6), it is seen that the point C of optimal tuning of $\gamma$ and $\mu_2$ lies at the boundary between 0 and 1, resulting in a design with virtually zero robustness. For positive (negative) values of $\alpha_3$, the solution for a robust absorber is to increase either $\gamma$ or $\mu_2$ (decrease $\gamma$), which necessarily results in an earlier LCO onset, i.e. $\mu_{1\text{max}} < \sqrt{\varepsilon}/2$. If an NLTVA is adopted (figure 7), for positive values of $\alpha_3$, point C lies well inside the region where supercriticality is guaranteed, which clearly highlights the benefit of the NLTVA. For negative values of $\alpha_3$, the optimal point lies close to the boundary between 0 and 1, which means that there is much less margin for a robust design than for positive $\alpha_3$. However, compared with the LTVA with negative $\alpha_3$, the NLTVA still possesses a larger region of supercritical behaviour. Specifically, there is a new region $\gamma \approx \gamma_{\text{opt}}$ and $\mu_2 < \mu_{2\text{opt}}$ in which supercriticality can be guaranteed.

Finally, figure 8 presents the probability to have a supercritical bifurcation as a function of $\alpha_3$ (in absolute value). To reflect a realistic design scenario, uncertainty of $\pm 1\%$ and $\pm 5\%$ on the
Figure 7. Maximal value of $\alpha_3$ below which supercritical bifurcations for the NLTVA are guaranteed ($\varepsilon = 0.05$, $\beta_3 = \alpha_3\varepsilon/(1 + \varepsilon/2)$). (a) Positive values of $\alpha_3$; (b) negative values of $\alpha_3$. The dashed line corresponds to $\gamma = \gamma_{opt}$ and the dot to $\mu_2 = \mu_{2opt}$. (Online version in colour.)

Figure 8. Probability to have a supercritical bifurcation for different values of $\alpha_3$ (in absolute value). (a) LTV A; (b) NLTVA. $\gamma$ and $\mu_2$ are within ±1% and ±5% of the corresponding optimum value, respectively.

values of $\gamma$ and $\mu_2$ around point C, respectively, are considered. Again, the superiority of the NLTVA over the LTV A is evident in these plots.

5. Reduction of the amplitude of limit cycle oscillations

At this stage, the linear and nonlinear parameters of the NLTVA have been designed through stability (§3) and bifurcation (§4) analyses, respectively. There is, therefore, not much freedom left to mitigate the amplitudes of the LCOs in the post-bifurcation regime. Nevertheless, their amplitude in the vicinity of the stability boundary can be investigated through the adopted local analysis.
Figure 9. LCO iso-amplitude curves along the stability boundary for $\alpha_3 = 0.08, \varepsilon = 0.05, \beta_3 = \varepsilon/(1 + \varepsilon)^2 \alpha_3 = 0.0036$ and $\mu_1 - \mu_{1cr} = 0.01$.

Considering the normal form of a Hopf bifurcation, the amplitude of the generated LCOs is proportional to $\sqrt{-\frac{k_1}{\delta}}$, where $k_1$ is the real part of the eigenvalue related to the bifurcation (see §4a). Because $k_1 = 0$ at the loss of stability, we consider its linear approximation, i.e. $k_1 \approx (\frac{dk_1}{d\mu_1}|_{\mu_1=\mu_{1cr}})(\mu_1 - \mu_{1cr})$. The LCO amplitude in the vicinity of the loss of stability is therefore

$$r \approx \sqrt{-\frac{\frac{dk_1}{d\mu_1}|_{\mu_1=\mu_{1cr}}}{\delta}} \mu_1 - \mu_{1cr}.$$ (5.1)

The maximal value of the LCO in physical space is computed by considering $\varphi = \tan^{-1}(\frac{t_{12}}{t_{11}})$ in $q_1 \approx t_{11}r \cos(\varphi) + t_{12}r \sin(\varphi)$, where $t_{11}$ and $t_{12}$ are the coefficients of $T$ as expressed in equation (4.3) ($t_{13}$ and $t_{14}$ do not need to be taken into account, because $y_3 \approx y_4 \approx 0$). Iso-amplitude curves of the LCOs along the stability boundary are represented in figure 9. It can be observed that the amplitude of the LCOs is minimized close to the optimal tuning region, which signifies that the design of the previous sections is also relevant for LCO mitigation.

6. Numerical validation of the analytical developments

(a) Proposed tuning rules for the nonlinear tuned vibration absorber

In the previous sections, a procedure to optimize the parameters of the NLTVA was proposed, namely

$$\gamma_{opt} = \frac{1}{\sqrt{1 + \varepsilon}}, \quad \mu_{2opt} = \frac{1}{2\sqrt{1 + \varepsilon}} \quad \text{and} \quad \beta_3 = \frac{\varepsilon}{(1 + \varepsilon)^2} \alpha_3.$$ (6.1)

We recall that the ‘principle of similarity’ was adopted for selecting the mathematical form of the NLTVA nonlinearity. For these parameters, the system will lose stability at $\mu_1 = \sqrt{\varepsilon}/2$ through a supercritical Hopf bifurcation. Because the double Hopf scenario was not fully analysed and because this scenario would probably result in more involved dynamics, it is probably preferable, in practice, to detune $\mu_2$ towards values slightly greater than $\mu_{2opt}$, in order to ensure a supercritical single Hopf bifurcation. Furthermore, figure 9 shows that the LCOs have smaller amplitude for $\mu_2 > \mu_{2cr}$ rather than for $\mu_2 < \mu_{2cr}$. 


(b) Local and global bifurcation analysis

Bifurcation diagrams predicted using the analytical developments of §4a and the numerical continuation software MATCONT [39] are depicted in figure 10. Slightly detuned linear parameters, i.e. \( \mu_2 = 0.12, \gamma = 0.97/0.985 \), are considered to show the robustness of our findings. Loss of stability occurs through a single Hopf bifurcation for the two parameter sets. Figure 10 presents an excellent qualitative agreement between the analytical and numerical curves; the quantitative differences observed at higher values of \( q_1 \) are due to the fact that the analytical results are only valid locally. When there is no structural nonlinearity (\( \alpha_3 = 0 \) and \( \beta_3 = 0 \)), the bifurcation remains supercritical, and the LTVA works effectively on the classical VdP oscillator. The introduction of the structural nonlinearity (\( \alpha_3 = 0.3 \)) in the VdP oscillator gives rise to a subcritical or supercritical bifurcation in figure 10a and b, respectively. This result confirms the difficulty to predict the bifurcation behaviour of the coupled VdPD and LTVA system in the optimal region; it also highlights the detrimental role played by the structural nonlinearity of the VdPD oscillator. Conversely, the introduction of nonlinearity in the absorber (\( \beta_3 = \varepsilon/(1 + \varepsilon)^2 \alpha_3 = 0.0136 \)) allows to guarantee a supercritical bifurcation, as for the system without nonlinearity. The compensation effect brought by the NLTVA is therefore clearly demonstrated.

The analytical developments are valid only in the neighbourhood of the bifurcation leading to LCO onset. The MATCONT software [39] is now used to investigate large-amplitude LCOs. Figure 11 plots bifurcation diagrams for the case of a single Hopf bifurcation at the loss of stability. The same parameter values as those in figure 10 are used, but greater values of \( q_1 \) are investigated. A major difference with the local analysis is that fold bifurcations that can turn supercritical behaviour into subcritical behaviour are now encountered, which confirms the importance of global analysis. If the pair of folds that appears for the NLTVA in figure 11a cannot be considered as particularly detrimental, this is not the case for the LTVA in figure 11b, where bistability in a significant portion of the stable region compromises the robustness of the linear absorber.

Figure 12 represents the same results for a lower value of \( \mu_2 \) for which a double Hopf bifurcation is expected. The bifurcation diagrams are more complex with secondary Hopf (or Neimark–Sacker) bifurcations observed both for the LTVA and NLTVA; their analysis is beyond the scope of this paper. Apart from these new bifurcations, we note that the general trend of the curves is similar to that in figure 11, demonstrating a certain robustness of the absorbers with respect to parameter variations.
Figure 11. Bifurcation diagrams for $\mu_2 = 0.12$, $\alpha_3 = 0.3$ and $\varepsilon = 0.05$. (a) $\gamma = 0.970$, (b) $\gamma = 0.985$. VdPD + LTVA: $\beta_3 = 0$; VdPD + NLTVA: $\beta_3 = 0.0136$. Squares: fold bifurcations. (Online version in colour.)

Figure 12. Bifurcation diagrams for $\mu_2 = 0.097$, $\alpha_3 = 0.3$ and $\varepsilon = 0.05$. (a) $\gamma = 0.970$, (b) $\gamma = 0.985$. VdPD + LTVA: $\beta_3 = 0$; VdPD + NLTVA: $\beta_3 = 0.0136$. Squares: fold bifurcations; circles: secondary Hopf bifurcations. (Online version in colour.)

Figure 13 considers again the single Hopf case, but with a slightly greater value of the nonlinear coefficient of the NLTVA, i.e. $\beta_3 = 0.018$ instead of 0.0136. The NLTVA clearly outperforms the LTVA: not only LCO amplitudes are significantly smaller, but there is also the complete absence of dangerous bistable regions. Another important result is the strong resemblance between the behaviours of the VdP+LTVA and VdPD+NLTVA systems. This suggests that the compensation of the nonlinearity of the VdPD by the nonlinearity of the NLTVA, which was observed in previous sections, is also valid at larger amplitudes.

(c) Validation of the principle of similarity

In §4, the nonlinear spring of the NLTVA was chosen to be cubic according to the ‘principle of similarity’. Absorbers with quadratic ($\beta_2$) and quintic ($\beta_5$) nonlinearities are also considered
in this section. For $\gamma = 0.97$ in figure 14a, the NLTVA is the only absorber which loses stability through a supercritical bifurcation. For greater amplitudes, unlike the other absorbers, the NLTVA does not exhibit any bistability in the region where the trivial solution is stable. However, besides the NLTVA, the absorber with quintic nonlinearity also exhibits a beneficial effect with respect to LCOs amplitude. For $\gamma = 0.985$ in figure 14b, all absorbers lose stability through a supercritical bifurcation, but the NLTVA is the only absorber that does not possess a fold bifurcation that generates subcriticalities at greater amplitudes. The remaining LCOs for the NLTVA have also a much smaller amplitudes. All the results clearly validate the principle of similarity for the passive control of self-excited oscillations.
Figure 15. Time series for $\mu_1 = 0.025$ and $\alpha_3 = 4/3$. (a) No absorber; (b) NES with $\Lambda = 1$ and a cubic spring $\beta_3 = 0.5333$; (c) NLTVA with optimal linear parameters and a cubic spring $\beta_3 = 0.0605$.

Figure 16. Bifurcation diagrams of the VdPD oscillator with an attached NES for $\alpha_3 = 0.3$ and $\varepsilon = 0.05$. (a) $c_2 = 0.05$ ($\Lambda = 1$); (b) $k_{nl2} = 0.01$. Squares, fold bifurcations; circles, secondary Hopf bifurcations. (Online version in colour.)

(d) Comparison with nonlinear energy sink

As stated in the Introduction, the NES has been widely used in the past decade for the passive absorption of forced and self-excited oscillations. The main difference between the NES and the NLTVA is that the former absorber possesses an essentially nonlinear spring, i.e. without a linear component. Domany & Gendelman [40] showed that the stability boundary of a VdPD oscillator with an attached NES can be approximated by the curve $2\mu_1/\varepsilon = \Lambda/(\Lambda^2 + 1)$, where $\Lambda = c_2/(m_2\omega_{n1})$. Thus, the maximal value of $\mu_1$ in order to have stability is obtained for $\Lambda \approx 1$ and it is $\mu_{1\text{max, NES}} \approx \varepsilon/4$, which is the square of the value obtainable with the NLTVA ($\mu_{1\text{max}} = \sqrt{\varepsilon}/2$). For $\varepsilon = 0.05$, $\mu_{1\text{max}}$ is therefore approximately 10 times greater for the NLTVA than for the NES. This demonstrates that the presence of a linear spring in the absorber is critical for the enlargement of the stable region of the trivial solution.

For illustrating the respective performance, a VdPD oscillator with $\mu_1 = 0.025$ and $\alpha_3 = 4/3$ is considered. Figure 15a depicts its LCO when no absorber is attached. Figure 15b,c show that the NES can only reduce the LCO amplitude for these parameters, whereas the NLTVA can completely eliminate the limit cycle and make the coupled system converge towards the trivial solution.
The performance of an NES attached to the VdPD oscillator was also investigated using the MATCONT software. In figure 16a, the NES damper coefficient $c_2$ is optimized with respect to stability and $k_{nl2}$ varies, whereas $k_{nl2} = 0.01$ and $c_2$ vary in figure 16b. Although a detailed performance analysis is beyond the scope of this paper, variation of $k_{nl2}$ and $c_2$ are unable to improve significantly the NES performance. The comparison of figures 13 and 16 provides evidence that, besides a much larger stable region, the NLTVA generates smaller LCO amplitudes than the NES.

7. Conclusion

The purpose of this paper was to investigate the performance of the NLTVA for suppression of self-excited oscillations of mechanical systems. A distinct feature of this absorber is that the mathematical form of its nonlinearity is selected according to a principle of similarity with the nonlinear primary system. Thanks to detailed stability and bifurcation analyses, a complete analytical design of the NLTVA was obtained, which was validated by detailed numerical calculations in the MATCONT software.

Thanks to the complementary roles played by the linear and nonlinear springs of the NLTVA, this absorber was shown to be effective for LCO suppression and mitigation, as it maximizes the stability of the trivial equilibrium point, guarantees supercritical bifurcations and reduces the amplitude of the remaining LCOs.

Data accessibility. There are no data related to this research.

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